Boundary value problems for second-order elliptic equations and related topics

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Dedication

Para el presidente Rafael Correa,

y para los socialistas de la Revolución Ciudadana en Ecuador,

una gran parte de este trabajo, lo hice inspirado por ustedes y por sus legítimas luchas.

Abstract

We study perturbation results for boundary value problems for second-order elliptic partial differential equations, and the exponential decay of solutions to generalized Schrödinger operators. First, through the use of sawtooth domains and the extrapolation technique of Carleson measures, we show the stability of the solvability of the Dirichlet problem for (additive) Carleson perturbations of certain degenerate elliptic operators $-\operatorname{div} A\nabla$ on domains with low dimensional boundaries (joint work with S. Mayboroda). Then, with a different method of proof, we expand these perturbation results to more abstract domains (including some domains with mixed-dimensional boundaries) and a broader type of Carleson perturbation, yielding some new applications (including to free boundary problems) (joint work with J. Feneuil). Next, together with S. Bortz, S. Hofmann, J.L. Luna García, and S. Mayboroda, we consider the uniformly elliptic operators $L = -\operatorname{div}(A\nabla + B_1) + B_2\nabla + V$ on the upper half space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{t > 0\}$, $n \geq 3$, with $t-$independent coefficients, and we prove the L^2 solvability of the Dirichlet, Neumann and Regularity problems under the condition that $|B_1|, |B_2|, |V|^2$ have small $L^n(\mathbb{R}^n)$ norm. Finally, we show that for generalized magnetic Schrödinger operators $-(\nabla - i\mathbf{a})^T A(\nabla - i\mathbf{a}) + V$, with certain conditions providing an uncertainty principle, resolvents and Lax-Milgram solutions exhibit exponential decay (in an L^2-sense), and we improve these estimates to upper pointwise exponential decay for the magnetic Schrödinger operator $-(\nabla - i\mathbf{a})^2$, and to sharp (that is, upper and lower) pointwise exponential decay for the Schrödinger operator on a non-homogeneous medium $-\operatorname{div} A\nabla + V$ (joint work with S. Mayboroda).

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Chapter 1

Introduction

In this dissertation, we present a body of work regarding boundary value problems for second-order elliptic partial differential equations and fine properties of their solutions and their spectra. First, we will formally define our objects of study and give a unifying framework for our results in Section 1.1, then in Section 1.2 we will present the main theorems of this thesis, as well as provide a small bite of each chapter. Section 1.3 provides a historical survey of the literature adjacent to our results, and finally, Section 1.4 gives some universal notation and conventions in this thesis that will be helpful in reading the rest of the manuscript.

1.1 Our objects of study

Let $N \in \mathbb{N}, N \geq 2$, and $\Omega \subseteq \mathbb{R}^N$ be an open set (we shall later put more assumptions on $\partial\Omega$), and define the operator L acting on functions $u : \Omega \to \mathbb{K}$ (where the field $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$) by the formula

$$\begin{aligned} Lu &= -\mathrm{div}(A\nabla u + B_1 \cdot u) + B_2 \cdot \nabla u + Vu \\ &:= -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(A_{i,j}\frac{\partial u}{\partial x_j} + B_{1i} \cdot u\Big) + \sum_{i=1}^{N} B_{2i}\frac{\partial u}{\partial x_i} + Vu. \end{aligned} \tag{1.1.1}$$

Here, $A = (A_{i,j})$ is a matrix function mapping Ω into the space of $N \times N$ matrices with components in the field $\mathbb{K}$; we denote this latter space by $\mathbb{M}_{N\times N}$. Furthermore,

$B_1 : \Omega \to \mathbb{K}^N, B_2 : \Omega \to \mathbb{K}^N$, $V : \Omega \to \mathbb{K}$ are $L^1_{\text{loc}}(\Omega)$ functions. Given a positive function $w \in L^1_{\text{loc}}(\Omega)$, we say that L is *w-elliptic* if there exists a positive constant C_A such that

$$\frac{1}{C_A} w(X)|\xi|^2 \leq \Re e \, \langle A(X)\xi, \xi \rangle \equiv \Re e \sum_{i,j=1}^{n} A_{ij}(X)\xi_j \bar{\xi}_i \tag{1.1.2}$$

and

$$|\langle A(X)\xi, \zeta \rangle| \leq C_A w(X), \qquad \text{for every } X \in \Omega. \tag{1.1.3}$$

If L is 1-elliptic, then we call L *uniformly elliptic* (also known as *strongly elliptic*, or simply, *elliptic*). Thus, the uniformly elliptic operators satisfy the estimates

$$\frac{1}{C_A}|\xi|^2 \leq \Re e \, \langle A(X)\xi, \xi \rangle \qquad \text{and} \qquad |\langle A(X)\xi, \zeta \rangle| \leq C_A, \qquad \text{for every } X \in \Omega. \tag{1.1.4}$$

When the weight w is understood from context, we will usually omit it in this manuscript, and refer to w-elliptic operators as *elliptic* operators, parting slightly from conventions elsewhere.

A large portion of the related literature concerns the *homogeneous second-order* uniformly elliptic operators

$$L = -\operatorname{div} A\nabla \tag{1.1.5}$$

for which all of the lower-order terms B_1, B_2, and V are zero. When A is the $N \times N$ identity matrix, then $L = -\Delta$ is known as the *Laplace operator*. The terms B_1, B_2 are usually called *drift* terms, or *first-order* terms, while V is called the *electric potential*, *potential*, or *zeroth-order* term. When $B_1 \equiv B_2 \equiv 0$ and $V > 0$ almost everywhere (with respect to the $N-$dimensional Lebesgue measure) in Ω, we call

$$L_E = -\operatorname{div} A\nabla + V \tag{1.1.6}$$

a *Schrödinger operator*. If $\mathbb{K} = \mathbb{C}$ and $\mathbf{a} : \Omega \to \mathbb{R}^N$ is a vector function, then

$$L_M = -(\nabla - i\mathbf{a})^2 + V \tag{1.1.7}$$

is the *magnetic Schrödinger operator*. Finally if L is of the form

$$L = -(\nabla - i\mathbf{a})^T A(\nabla - i\mathbf{a}) + V, \tag{1.1.8}$$

then L is a *generalized Schrödinger operator*.

The partial differential equations that we consider formally take the form $Lu = g$ in Ω, for $g : \Omega \to \mathbb{K}$ a function defined on Ω. We say that u is a *solution* for L if $Lu = 0$. The solutions for the Laplacian (that is, those functions u which verify $-\Delta u = 0$) are called *harmonic* functions. If the coefficients of L are rough, it may not be possible to find a smooth enough solution u for which the formal expression (1.1.1) is well-defined; for this reason we will always interpret the equation $Lu = g$ through a weaker integral formulation: we say that $Lu = g$ in Ω *in the weak sense* if the identity

$$\int_\Omega \Big[A\nabla u \cdot \overline{\nabla \phi} + uB_1 \cdot \overline{\nabla \phi} + \overline{\phi} B_2 \cdot \nabla u + V u \overline{\phi} \Big] \, dX = \int_\Omega g \overline{\varphi} \, dX \tag{1.1.9}$$

holds for each $\mathbb{K}$-valued compactly supported smooth function φ in Ω.

A classical problem for partial differential equations with origins in physical phenomena is the question of determining a solution for L in the domain Ω that verifies certain prescribed conditions at the boundary $\partial\Omega$. These problems are known as *boundary value problems*. The most basic type of boundary value problem is the *Dirichlet problem*: given a function $f : \partial\Omega \to \mathbb{K}$, the Dirichlet problem (formally) consists of finding a function u verifying the identities

$$\begin{cases} Lu = 0, & \text{in } \Omega, \quad \text{and} \\ \quad u = f, & \text{on } \partial\Omega. \end{cases} \tag{1.1.10}$$

Thus the Dirichlet problem asks for a solution of the partial differential equation in the domain whose values are fixed to a predetermined data on the boundary of the domain. When $\partial\Omega$ is smooth, and $f : \partial\Omega \to \mathbb{K}$ is a continuous real function, and $L = -\Delta$, then each identity in (1.1.10) can be rigorously interpreted as written, in the pointwise sense. However, in light of our weak formulation (1.1.9) of the elliptic partial differential equation, and the fact that the coefficients of L may be rough, the functions u that we deal with will only seldom be properly pointwise defined on $\partial\Omega$. For this reason, we must also reinterpret the expression $u = f$ on $\partial\Omega$. One common and useful way to reinterpret this identity arises from classical boundary value problems for holomorphic functions in the plane: we will replace "$u = f$" with

$$u \longrightarrow f \quad \textit{non-tangentially},$$

which means that for some aperture $\alpha > 0$, we have that

$$\lim_{\gamma_\alpha(x)} u(X) = f(x), \qquad \text{for each } x \in \partial\Omega,$$

where $\gamma_\alpha(x)$ is the *non-tangential cone* of aperture α centered at x:

$$\gamma_\alpha(x) := \{X \in \Omega \,:\, |X - x| < (1+\alpha)\,\mathrm{dist}(X, \partial\Omega)\}. \tag{1.1.11}$$

The existence of such cones, or more general similar non-tangential approach regions, by itself puts some restrictions on the boundary, and we will make those precise later.

Throughout this thesis, we will be interested in solving boundary value problems for a whole class of prescribed boundary data, rather than only some specific function f. In connection with this goal, we will seek certain *uniform estimates* that control (some aspect of) the solutions in terms of the class of functions considered. The point of such a uniform estimate is to prevent a “butterfly effect” situation where very small perturbations in the boundary data lead to drastic changes on the solution. A boundary value problem (with boundary data in some prescribed class) is called *well-posed* if for each admissible boundary data, there exists a unique solution, and this solution satisfies the uniform estimates.

Let us give two examples which are of immediate concern to us. Given a smooth bounded domain Ω, the first example is the Dirichlet problem *with continuous data* on Ω, which consists of (uniquely) solving the Dirichlet problem for each $f \in C(\partial\Omega)$. It is classical that this problem is solvable for many elliptic operators that we consider, and in particular, for the Laplacian $-\Delta$. A basic but extremely important estimate tied to the Dirichlet problem with continuous data is the *maximum principle*: If u verifies $-\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$, then

$$\sup_{X\in\Omega} |u(X)| \leq \max_{x\in\partial\Omega} |f(x)| = \|f\|_{C(\partial\Omega)}. \tag{1.1.12}$$

This estimate immediately gives the uniqueness of the solutions to the Dirichlet problem with data $f \in C(\partial\Omega)$, but it has many more uses; we will see another such application further below. Our second example is the Dirichlet problem *with data in* L^p for some $p \in (1, \infty)$, denoted $(\mathrm{D})_p$, which we state for the Laplacian $L = -\Delta$ on a smooth bounded domain Ω. In addition to finding a unique solution to the Dirichlet problem for each $f \in L^p(\partial\Omega)$, we also ask that the following well-posedness condition holds: there exists a constant $C > 0$ such that for each $f \in L^p(\partial\Omega)$, we have that

$$\|Nu\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}, \tag{1.1.13}$$

where u is the solution to (1.1.10), σ is the surface measure (i.e. the restriction of the Hausdorff $(N-1)-$dimensional measure) on $\partial\Omega$, and N is an operator known as the *non-tangential maximal function*. The latter takes functions in the domain Ω to functions on the boundary $\partial\Omega$, and is defined as

$$(Nu)(x) = (N_\alpha u)(x) := \operatorname*{ess\,sup}_{X\in\gamma_\alpha(x)} |u(X)|, \qquad x \in \partial\Omega, \quad \alpha > 0. \tag{1.1.14}$$

When α is understood and fixed from context, we will usually drop it in the subscript for the non-tangential maximal function (this is also justified by the fact that, because of Fubini's theorem, the L^q norms of $N_\alpha u$ and $N_\beta u$ are equivalent, for any admissible α, β, q, u). Moreover, this definition of N requires that u is a locally essentially bounded function, which is true for the harmonic functions and for solutions to many other elliptic operators, such as the Schrödinger operator (with positive electric potential V) or the homogeneous second-order operators $-\operatorname{div} A\nabla$. However, the solutions to the general second order elliptic operators with complex coefficients considered in Chapters 4 and 5 may fail to be locally essentially bounded. For this reason, in these two chapters we consider an alternate but more appropriate definition of the non-tangential maximal function; for details, see (1.2.12) and the introduction to Chapter 4.

Let us summarize the Dirichlet problem with L^p data: we say that $(\mathrm{D})_p$ is *solvable* if there exists a positive constant C and an aperture $\alpha > 0$ such that for each $f \in L^p(\partial\Omega, \sigma)$, there exists a function u verifying the conditions

$$(\mathrm{D})_p \begin{cases} Lu = 0 & \text{in } \Omega, \\ u \longrightarrow f & \text{non-tangentially}, \\ \|Nu\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}. \end{cases} \tag{1.1.15}$$

When $L = -\operatorname{div} A\nabla$, the solvability of $(\mathrm{D})_p$ is strongly tied to the absolute continuity of the *elliptic measure* with respect to the surface measure. The elliptic measure is a family of probability measures $\{\omega^X\}_{X\in\Omega}$ on $\partial\Omega$, afforded by the Riesz Representation and the maximum principle, which allows us to write the solution u to the Dirichlet problem with continuous data $f \in C_c(\Gamma)$ as

$$u(X) = \int_\Gamma f \, d\omega^X. \tag{1.1.16}$$

The elliptic measure is known as *harmonic measure* when $L = -\Delta$, and in this case it is also a subject of intense study in geometry, and in probability as well, through a connection to Brownian motion. It turns out that, in a robust sense[1], $(\mathrm{D})_p$ is solvable for some p if and only if the elliptic measure is quantifiably absolutely continuous with respect to the surface measure; more precisely, $\omega \in A_\infty(\sigma)$, or $\omega \ll \sigma$ and the Radon-Nikodym derivative $\frac{d\omega}{d\sigma}$, known as the *Poisson kernel*, satisfies scale-invariant Reverse Hölder inequalities; see Definition 2.7.28 for the context within Chapter 2.

Another archetypal boundary value problem is the *Neumann problem*. This time we are given a function $g : \partial\Omega \to \mathbb{K}$ and search for a (formal) solution of L whose flux through the boundary is predetermined by g:

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g, & \text{on } \partial\Omega. \end{cases} \tag{1.1.17}$$

The Neumann problem with data in L^2 is defined in (1.2.14). Of course, as above, the issue arises of rigorously defining the normal derivative $\partial u/\partial\nu$ for the weak solutions. This is in general more delicate than for the Dirichlet boundary condition, and we postpone its discussion for now, to be revisited in Chapters 4 and 5 where the Neumann problem is dealt with.

There are many other types of boundary value problems, such as Robin-type boundary conditions, or mixed Dirichlet and Neumann boundary conditions, but in this manuscript we will mainly focus on the Dirichlet and Neumann problems. In fact, in chapters 2 and 3, we will only consider the Dirichlet problem, while in chapters 4 and 5 we will tackle both the Dirichlet problem and the Neumann problem, as well as a variant of the Dirichlet problem, known as the *Regularity problem* (see Section 1.2.3 and (1.2.15)).

The question of whether a boundary value problem is solvable depends on several

[1]for more details, see Theorem 2.7.40 in Chapter 2

factors and their combinations, some of which we now list.

- The roughness of the prescribed boundary condition. For instance, when Ω is the half-space $\mathbb{R}^{n+1}_+$ and $L = -\operatorname{div} A\nabla$ is uniformly elliptic, can we solve $(\mathrm{D})_p$ on the half-space for some $p \in (1, \infty)$? The answer depends on the properties of A and the value of p.
- The geometry of Ω and its boundary. For example, can we find geometric and topological conditions on Ω and $\partial\Omega$ that guarantee the solvability of $(\mathrm{D})_p$ for the Laplacian $L = -\Delta$, for some $p \in (1, \infty)$? The answer is yes, and the sharpness of these conditions has been heavily studied in the past decade.
- The coefficients of L. How well-behaved should the coefficients of L be to guarantee well-posedness? Are A, B_1, B_2, V complex-valued or real-valued? What is the structure of the matrix function A? What kind of degeneracy in the ellipticity of the matrix function is admissible? The case of the homogeneous second-order elliptic operators (1.1.5) has alone attracted considerable attention in the last fifty years. The systematic study of the general second-order operators (1.1.1) is more recent, while certain degenerate elliptic operators have been studied in the past few years with a focus on the link between PDEs and geometry.

Our work hereby presented addresses, at least partially, some combinations of each of the above three factors. Pointedly, in Chapter 2, we are mainly interested in certain real degenerate elliptic operators $-\operatorname{div} A\nabla$ on domains with rough low dimensional boundaries (meaning that the boundary has codimension strictly larger than 1); in Chapter 3 we significantly generalize the results of Chapter 2 to deal with other conditions on A and different types of boundaries (including certain mixed-dimensional boundaries); and in Chapters 4 and 5 we consider only L^2 solvability on the half-space, but we bring in lower order terms B_1, B_2, V, and we allow for complex-valued coefficients.

In Chapter 6, we undertake a slightly different but still heavily related line of study, where we seek to understand how certain conditions on the structure of L yield some quantitative effects on the solutions to Schrödinger operators. Specifically, we consider the *fundamental solution* Γ to Schrödinger and magnetic Schrödinger operators, which formally solves $L\Gamma = \delta$ where δ is the Dirac mass centered at 0, and we prove how the electric and magnetic potentials induce exponential decay on Γ.

1.2 Main results and chapter presentations

We are now appropriately positioned to give a small introductory passage, as well as to present the main results, in each of the specific research directions that we have undertaken in this thesis.

1.2.1 Chapter 2: Carleson perturbations of elliptic operators on domains with low dimensional boundaries

The research in this chapter was done in collaboration with S. Mayboroda.

In [DFM19b], G. David, J. Feneuil and S. Mayboroda started a program to characterize the geometry of boundaries of high co-dimension via the theory of certain degenerate elliptic operators, crafted to overcome the fundamental myopia of uniformly elliptic operators vis-à-vis the geometry of low dimensional boundaries. Let $d \leq n-1$ be the Hausdorff dimension of the closed set $\Gamma \subset \mathbb{R}^n$. Furthermore, suppose that Γ is quantifiably $d-$dimensional; more precisely, suppose that Γ is $d-$Ahlfors-David regular (see Definition 2.2.1). Formally, consider

$$(Lu)(X) = -\operatorname{div}\Big(\frac{\mathcal{A}(X)}{\operatorname{dist}(X,\Gamma)^{n-d-1}}\nabla u(X)\Big), \qquad X \in \Omega := \mathbb{R}^n \backslash \Gamma, \tag{1.2.1}$$

where $\mathcal{A}$ is a uniformly elliptic matrix. Observe that in compactly contained subsets of Ω, the operator L is strongly elliptic, but not uniformly so up to the boundary Γ, unless $d = n-1$. Instead, the operator must degenerate at a fixed rate which morally forces harmonic functions to respect the high co-dimension sets. It turns out that for these operators, one can recover an elliptic theory [DFM19b], and in particular an analogue of elliptic measure can be devised.

In Chapter 2, we present results concerning the perturbations of solvability of the Dirichlet problem with L^p data, $(\mathrm{D})_p$, for the operators as in (1.2.1). More precisely, we extend to the higher-codimensional case the perturbation theory of Fefferman-Kenig-Pipher [FKP91], initially considered on Lipschitz domains but which has been extended to more general situations in a co-dimension 1 (or close) setting [MPT13, HM12, CHMT20, AHMT]. The following theorem relies on the extrapolation of Carleson measures [LM95] [HL01a]. A particularly delicate technical aspect of the proof was our construction of mixed-dimensional sawtooth domains which admit an elliptic PDE theory. This is joint work with S. Mayboroda.

Theorem 1.2.2 (Carleson perturbation preserves Dirichlet problem solvability, [MP21])**.** *Suppose that Γ is d-ADR with $d \in [1, n-1)$, $n \geq 3$. Let two operators L_0 and L be given as in (1.2.1) with associated bounded and uniformly elliptic real (not necessarily symmetric) matrices $\mathcal{A}_0$ and $\mathcal{A}$. We define the disagreement of the matrices $\mathcal{A}$, $\mathcal{A}_0$ as*

$$\mathfrak{a}(X) := \sup_{Y \in B(X, \delta(X)/2)} |\mathfrak{E}(Y)|, \quad \mathfrak{E}(Y) := \mathcal{A}(Y) - \mathcal{A}_0(Y), \qquad X \in \Omega. \tag{1.2.3}$$

Assume that $\delta(X)^{d-n} \mathfrak{a}^2 \, dX$ is a Carleson measure; that is, assume that there exists a constant $C \geq 1$ such that for each surface ball $\Delta = B(x,r) \cap \Gamma$, the estimate

$$\iint_{T(\Delta)} \frac{\mathfrak{a}(X)^2}{\delta(X)^{n-d}} \, dX \leq C\sigma(\Delta) \tag{1.2.4}$$

holds, where $T(\Delta) = B(x,r) \cap \Omega$. Then, if $(\mathrm{D})_{p'}$ is solvable for L_0 for some $p' \in (1,\infty)$, then there exists $q' \in (1, \infty)$ such that $(\mathrm{D})_{q'}$ is solvable for L.

We also have the following "small constant" variant of the previous result, which does allow the preservation of the solvability in the same L^p space.

Theorem 1.2.5 (Small Carleson perturbation preserves $(\mathrm{D})_p$, [MP21])**.** *Suppose that Γ is d-ADR with $d \in [1, n-1)$, $n \geq 3$. Let two operators L_0 and L be given as in (1.2.1) with associated uniformly elliptic real (not necessarily symmetric) matrices $\mathcal{A}_0$ and $\mathcal{A}$. Let ω_0, ω denote the respective elliptic measures. Moreover, suppose that there exists $p' \in (1, \infty)$ such that $(\mathrm{D})_{p'}$ is solvable for L_0. Define $\mathfrak{a}$ as in (1.2.3), and assume that there exists $\varepsilon_0 > 0$ so that*

$$\iint_{T(\Delta)} \frac{\mathfrak{a}(X)^2}{\delta(X)^{n-d}} \, dX \leq \varepsilon_0 \sigma(\Delta), \qquad \textit{for all } \Delta \subset \Gamma. \tag{1.2.6}$$

Then for all ε_0 small enough, depending only on n, d, the d-ADR constant of Γ, the ellipticity of L_0 and L, and the RH_p characteristic of the Poisson kernel $\frac{d\omega_{L_0}}{d\sigma}$, we have that $(\mathrm{D})_{p'}$ is solvable for L as well, and the RH_p characteristic of $\frac{d\omega_{L_1}}{d\sigma}$ depends only on the same parameters as does ε_0.

Our theorems imply robustness of the solvability of the Dirichlet problem on domains with uniformly rectifiable low dimensional boundaries (see Corollary 2.1.7) . In [DMb] and [Fen], the elliptic measure was seen to be quantifiably absolutely continuous with respect to the surface measure for the operators $L_\alpha = -\operatorname{div} D_\alpha^{d+1-n}\nabla$, where D_α is the *regularized distance*

$$D_{\mu,\alpha}(X) := \Big(\int_\Gamma |X-y|^{-d-\alpha}\, d\mu(y) \Big)^{1/\alpha}, \qquad \alpha > 0,\ X \in \mathbb{R}^n \backslash \Gamma. \tag{1.2.7}$$

The operators L_α are thought to be the analogues of the Laplacian in the low dimension setting. Our theorems allow us extend the $\omega \ll \sigma$ conclusion to many other operators not of the special form L_α. In addition, Theorem 1.2.2 can also be used along with a special construction of [DEM] to obtain for arbitrary d-ADR Γ with $d < n-2$, uncountably many operators L close to L_{α_0} for some $\alpha_0 > 0$ (called the "magic α") such that $\omega_L \ll \sigma$ (see Corollary 2.1.8).

1.2.2 Chapter 3: Generalized Carleson perturbations and applications

The research in this chapter was done in collaboration with J. Feneuil.

We have been able to show that Theorem 1.2.2 holds under a different type of Carleson perturbation.

Theorem 1.2.8 (Scalar-multiplicative Carleson perturbations preserves Dirichlet problem solvability, [FP])**.** *Suppose that* Γ *is* d-ADR *with* $d \in [1, n-1)$, $n \geq 3$. *Let* L_0 *be given as in (1.2.1) with associated bounded and uniformly elliptic real (not necessarily symmetric) matrix* $\mathcal{A}_0$. *Assume that* L *is of the form*

$$(Lu)(X) = -\operatorname{div}\Big(b(X)\frac{\mathcal{A}_0(X)}{\operatorname{dist}(X,\Gamma)^{n-d-1}}\nabla u(X)\Big), \qquad X \in \Omega := \mathbb{R}^n\backslash\Gamma, \tag{1.2.9}$$

where b *is a scalar function on* Ω *such that* $\frac{1}{C} \leq b \leq C$ *and* $\operatorname{dist}(\cdot,\Gamma)\nabla b$ *is a Carleson measure. Then, if* $(\mathrm{D})_{p'}$ *is solvable for* L_0 *for some* $p' \in (1,\infty)$, *then there exists* $q' \in (1,\infty)$ *such that* $(\mathrm{D})_{q'}$ *is solvable for* L.

The above theorem is one of several consequences of the main results in Chapter 3 (see Section 3.1.1). The gist of it is that the Carleson perturbation is now of a scalar-multiplicative form, rather than an additive form as in Theorem 1.2.2. A constraint of the additive Carleson perturbation is that it requires the two operators L and L_0 to agree at the boundary, whereas there is no such constraint for the multiplicative perturbation considered in Theorem 1.2.8. Our result holds in a great generality regarding the geometry of the boundary of the domain, allowing us to consider low-dimensional boundaries, co-dimension 1 boundaries, and even some boundaries of mixed dimension (see Definition 3.2.10 for the generality of the domains that we consider); while some implications of our results are new even for the 1-sided non-tangentially accessible domains (see Section 3.2 for more details and definitions). We are also able to consider Carleson perturbations by an antisymmetric matrix (see (3.1.16) and Theorem 3.1.19). Applications of our results and our methods include providing alternate proofs of solvability for the scalar case of the Dahlberg-Kenig-Pipher operators (because they can be seen as Carleson perturbations from the Laplacian; see Corollary 3.1.28 and the surrounding discussion), solvability results for equations with drift terms, a new characterization of A_∞ among elliptic measures (Corollary 3.1.25), and insights regarding the coefficients of certain purely degenerate elliptic operators (see the last two paragraphs of Section 3.1.2).

1.2.3 Chapters 4 and 5: Critical perturbations of elliptic operators with lower order terms

The research in these chapters was done in collaboration with S. Bortz, S. Hofmann, J. L. Luna García, and S. Mayboroda. The results and proofs in these chapters will also appear in the doctoral thesis of J. L. Luna García.

In this pair of chapters, we have tackled Dirichlet, Neumann, and Regularity problems for the (complex) second-order elliptic operators with lower order terms, under a smallness condition on scale-invariant norms of the lower order terms. The operator

$$\mathcal{L} := -\mathrm{div}\,(A\nabla + B_1) + B_2 \cdot \nabla + V \tag{1.2.10}$$

is defined on $\mathbb{R}^n \times \mathbb{R} = \{(x,t)\}$, $n \geq 3$, where $A = A(x)$ is an $(n+1)\times(n+1)$ matrix of L^∞ complex coefficients, defined on $\mathbb{R}^n$ (independent of t) and satisfying a uniform ellipticity condition (1.1.4). The first order complex coefficients $B_1 = B_1(x), B_2 = B_2(x) \in \left(L^n(\mathbb{R}^n)\right)^{n+1}$ (independent of t) and the complex potential $V = V(x) \in L^{\frac{n}{2}}(\mathbb{R}^n)$ (again independent of t) are such that

$$\max\left\{\|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\} \leq \rho, \tag{1.2.11}$$

where ρ will be taken small enough.

To state the boundary value problems for which we consider well-posedness in the L^2 sense, we ought to recall the definition of the (L^2-averaged) *non-tangential maximal operator* $\tilde{\mathcal{N}}_2$. Given $x_0 \in \mathbb{R}^n$, write $\gamma(x_0) = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x - x_0| < t\}$, and note that this cone coincides with the non-tangential cone $\gamma_{\sqrt{2}-1}(x_0)$ defined in (1.1.11). Then, for $u : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ we write

$$\tilde{\mathcal{N}}_2 u(x_0) := \sup_{(x,t)\in\gamma(x_0)} \Big(\fint\!\!\!\fint_{|x-y|<t,\, |s-t|<\frac{t}{2}} |u(y,s)|^2 \, dy \, ds \Big)^{\frac{1}{2}}. \tag{1.2.12}$$

We consider the Dirichlet problem

$$(\mathrm{D})_2 \begin{cases} \mathcal{L}u = 0 \qquad \text{in } \mathbb{R}^{n+1}_+, \\ \lim_{t\to 0} u(\cdot,t) = f \text{ strongly in } L^2(\mathbb{R}^n) \text{ and } u \to f \text{ non-tangentially}^2, \\ \tilde{\mathcal{N}}_2 u \in L^2(\mathbb{R}^n), \\ \iint_{\mathbb{R}^{n+1}_+} t|\nabla u(x,t)|^2 \, dx \, dt < \infty, \\ \lim_{t\to\infty} u(\cdot,t) = 0 \quad \text{ in the sense of distributions,} \end{cases} \tag{1.2.13}$$

the Neumann problem

$$(\mathrm{N})_2 \begin{cases} \mathcal{L}u = 0 \qquad \text{in } \mathbb{R}^{n+1}_+, \\ \frac{\partial u}{\partial \nu^{\mathcal{L}}} := -e_{n+1}(A\nabla u + B_1 u)(\cdot,0) = g \in L^2(\mathbb{R}^n), ^3 \\ \tilde{\mathcal{N}}_2(\nabla u) \in L^2(\mathbb{R}^n), \\ \iint_{\mathbb{R}^{n+1}_+} t|\partial_t \nabla u(x,t)|^2 \, dx \, dt < \infty, \\ \lim_{t\to\infty} \nabla u(\cdot,t) = 0 \text{ in the sense of distributions,} \end{cases} \tag{1.2.14}$$

[2]Since the solutions u do not satisfy pointwise bounds, non-tangential convergence is also understood in an averaged sense; see Definition 5.2.5.

[3]The boundary data is achieved in the distributional sense; see Section 4.4.

and the Regularity problem

$$(\mathrm{R})_2 \begin{cases} \mathcal{L}u = 0 \qquad \text{in } \mathbb{R}^{n+1}_+, \\ u(\cdot,t) \longrightarrow f \qquad \text{weakly in } Y^{1,2}(\mathbb{R}^n)^4 \text{ and non-tangentially}, \\ \tilde{\mathcal{N}}_2(\nabla u) \in L^2(\mathbb{R}^n), \\ \iint_{\mathbb{R}^{n+1}_+} t|\partial_t \nabla u(x,t)|^2\, dx\, dt < \infty, \\ \lim_{t\to\infty} \nabla u(\cdot,t) = 0 \text{ in the sense of distributions.} \end{cases} \tag{1.2.15}$$

Our main result at the culmination of these two chapters is the following theorem.

Theorem 1.2.16 ([BHL+b], [BHL+a])**.** *Suppose that A is an elliptic matrix of complex-valued $t-$independent coefficients, and that $B_1 \in L^n(\mathbb{R}^n)$, $B_2 \in L^n(\mathbb{R}^n)$, $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ are $t-$independent complex-valued lower order terms. Furthermore, assume that A is either Hermitian, block-form, or constant. Then there exists $\rho_0 > 0$ depending only on dimension and ellipticity such that if $\mathcal{L}$ is given by (1.2.10) and*

$$\max\left\{\|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\} < \rho_0,$$

then $(\mathrm{D})_2, (\mathrm{N})_2, (\mathrm{R})_2$ are uniquely solvable for the operator $\mathcal{L}$.

Chapter 4 gives the first part of the proof, setting up the preliminaries and obtaining the following square function estimate, interesting in its own right.

Theorem 1.2.17 (Square function bound for the single layer potential, [BHL+b])**.** *Suppose that A is an elliptic matrix of complex-valued $t-$independent coefficients, and that $B_1 \in L^n(\mathbb{R}^n)$, $B_2 \in L^n(\mathbb{R}^n)$, $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ are $t-$independent complex-valued lower order terms. Then, there exists $\varepsilon_0 > 0$, depending only on dimension and ellipticity such that if*

$$\mathcal{L} := -\operatorname{div}([A+M]\nabla + B_1) + B_2 \cdot \nabla + V,$$

[4] The space $Y^{1,2}(\mathbb{R}^n)$ is defined in (4.2.5).

with $M \in \mathcal{M}_{n+1}(\mathbb{R}^n, \mathbb{C})$*, and*

$$\|M\|_{L^\infty(\mathbb{R}^n)} + \|B_1\|_{L^n(\mathbb{R}^n)} + \|B_2\|_{L^n(\mathbb{R}^n)} + \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} < \varepsilon_0,$$

then for each $m \in \mathbb{N}$*, we have the estimate*

$$\iint_{\mathbb{R}^{n+1}_+} \left| t^m \partial_t^{m+1} \mathcal{S}_t^{\mathcal{L}} f(x) \right|^2 \frac{dx\, dt}{t} \le C \|f\|^2_{L^2(\mathbb{R}^n)},$$

where C *depends on* m*, dimension, and the ellipticity of* A*. Here,* $\mathcal{S}_t^{\mathcal{L}} f$ *is the single layer potential of* f *(see Definition 4.4.1). Under the same hypothesis, the analogous bounds hold for* $\mathcal{L}$ *replaced by* $\mathcal{L}^*$*, and for* $\mathbb{R}^{n+1}_+$ *replaced by* $\mathbb{R}^{n+1}_-$*.*

Using Theorem 1.2.17 and a uniform L^2 estimate on slices (Theorem 4.1.2) from Chapter 4, we complete the solvability of the aforementioned boundary value problems in Chapter 5 by proving the desired non-tangential maximal function estimates, as well as the existence and uniqueness properties of the problems (1.2.13)-(1.2.15).

Theorem 1.2.16 extends the L^∞ perturbation result of matrices in [AAA+11] to scale-invariant lower-order term perturbations. There are many difficulties arising from considering lower order terms (for instance, the failure of DeGiorgi-Nash-Moser estimates, and the seemingly innocuous fact that $\mathcal{L}1 \neq 0$), which necessitated the use of new techniques to solve the problem. Our method of proof thus relies on an abstract theory of layer potentials, a vector-valued Tb theorem, and weighted extrapolation theory. Applications of our results include the first solvability properties of the aforementioned boundary value problems for the time-independent magnetic Schrödinger operator in this setting of rough coefficients.

1.2.4 Chapter 6: Exponential decay estimates for fundamental solutions of operators of Schrödinger type

The research in this chapter was done in collaboration with S. Mayboroda.

The exponential decay of solutions to the Schrödinger operator in the presence of a positive potential is an important property underpinning the foundation of quantum physics. However, establishing a precise rate of decay for complicated potentials is a challenging open problem to this date. In Chapter 6, we use a generalized version of the Fefferman-Phong uncertainty principle to show that the fundamental solution of a generalized Schrödinger operator with a non-negative potential enjoys exponential decay bounds from above and below. They are governed by a certain version of the Agmon distance. For instance, if A is an elliptic matrix with real, bounded coefficients, and $V \in RH_{\frac{n}{2}}$, then (see Corollaries 6.6.16 and 6.7.35)

$$\frac{c_1 e^{-\varepsilon_1 d(X,Y,V)}}{|X-Y|^{n-2}} \leq \Gamma_E(X,Y) \leq \frac{c_2 e^{-\varepsilon_2 d(X,Y,V)}}{|X-Y|^{n-2}}, \tag{1.2.18}$$

where Γ_E is an integral kernel of the generalized electric Schrödinger operator, that is, the fundamental solution to $L_E = -\mathrm{div} A\nabla + V$, $X, Y \in \mathbb{R}^n$, interpreted in a suitable weak sense, and d is a suitable Agmon distance function depending on V (see below). In fact, we establish the upper estimates for a considerably more general class of operators, which can be formally written as $L = -(\nabla - i\mathbf{a})^T A(\nabla - i\mathbf{a}) + V$ including, in particular, the magnetic Schrödinger operator (1.1.7), under certain mild assumptions (see Theorem 6.6.7). Moreover, we note that the existence of a fundamental solution of the magnetic Schrödinger operator is new under our weak assumptions (see Theorem 6.5.35). We also obtain a novel exponential decay estimate for the resolvents: if L is as above and $f \in L^2(\mathbb{R}^n)$ with compact support, then (see Theorem 6.4.16)

$$\begin{aligned}\int\limits_{\left\{X \in \mathbb{R}^n \,|\, d(X, \operatorname{supp} f, V + |\mathbf{B}| + \frac{1}{t^2}) \geq \tilde{d}\right\}} & m\big(\cdot, V + |\mathbf{B}| + \tfrac{1}{t^2}\big)^2 \left|(1+t^2L)^{-1} f\right|^2 e^{2\varepsilon d(\cdot, \operatorname{supp} f, V + |\mathbf{B}| + \frac{1}{t^2})} \\ & \leq C \int_{\mathbb{R}^n} |f|^2 m\big(\cdot, V + |\mathbf{B}| + \tfrac{1}{t^2}\big)^2, \end{aligned}\tag{1.2.19}$$

where m is the *Fefferman-Phong-Shen maximal function*

$$\frac{1}{m(X,w)} := \sup_{r>0}\Big\{ r : \frac{1}{r^{n-2}} \int_{B(X,r)} w \leq 1 \Big\}, \tag{1.2.20}$$

the function

$$d(X,Y,w) = \inf_{\gamma} \int\limits_0^1 m(\gamma(t), w) |\gamma'(t)| \, dt, \tag{1.2.21}$$

is the associated *Agmon distance*, and $\mathbf{B}$ is the *magnetic field*,

$$\mathbf{B} = \text{ curl } \mathbf{a} = \Big(\frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}\Big)_{1\leq j,k\leq n}. \tag{1.2.22}$$

We have also shown an analogue of estimate (1.2.19) for the Lax-Milgram solutions (see Corollary 6.4.29 and (6.1.4)). Our exponential decay result for the fundamental solutions of the Schrödinger operator $-\operatorname{div} A\nabla + V$, and for the magnetic Schrödinger operator $-(\nabla - i\mathbf{a})^2$, has already been used by J. Bailey in the recent paper [Bai] to find large classes of weights w so that the Riesz transforms associated to the operators are $L^p(w)$-bounded.

1.3 Historical survey

1.3.1 Early background

To describe our results and our motivation, let us start with some fundamental results for the quintessential elliptic partial differential operator, the Laplacian, $-\Delta$. A first classical and surprising result regarding the harmonic functions is that they are C^∞ everywhere in the open domain of solvability. But what occurs at the boundary of the domain, $\partial\Omega$? As was known already in the 19th century, on certain nice domains like the half-space or the ball, it is possible to construct (via *Poisson's formulae* and *Green representation formulae*, in particular) harmonic functions which satisfy certain a priori *boundary conditions*.

Thus, for instance, given a continuous, bounded function f on $\partial\Omega$, the problem of finding $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ such that

$$\begin{cases} -\Delta u = 0, & \text{in} \quad \Omega, \\ \quad u = f, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3.1}$$

came to be known as the Dirichlet problem (D) (with continuous data). A different, yet similarly scientifically relevant problem is the Neumann problem (N), where one must determine $u \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$ verifying

$$\begin{cases} -\Delta u = 0, & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = g, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3.2}$$

where g is continuous and bounded on $\partial\Omega$, and $\frac{\partial u}{\partial \nu}$ is the *normal derivative* of u. Both (D) and (N) are examples of boundary value problems (BVPs).

On the other hand, it was not difficult to construct harmonic functions which blow up at the boundary. In the plane, this can be seen through the intrinsic link between the Laplace equation and the analysis of complex variables, as both the real and imaginary parts of holomorphic functions must be harmonic functions. Hence, in a strong sense, the study of boundary behavior of holomorphic (i.e. analytic) functions yielded immediate insight into the boundary behavior of harmonic functions in two dimensions. In this vein, there is the classical result of Fatou on the existence of non-tangential limits for harmonic functions on the disk D, where one can guarantee that the harmonic function converges non-tangentially towards the boundary of the disk ∂D provided that there is a $p \in (1, \infty]$ such that the supremum of the Lebesgue L^p norms on all concentric circles is controlled. In fact, Fatou's theorem allows one to solve the Dirichlet problem with rough data $f \in L^p(\partial D)$ on the disk, and was one of the first major applications of the then-novel Lebesgue theory of integration.

For the first half of the 20th century, and based on the groundwork laid by Fatou, there was a major interest in characterizing exactly how the harmonic (or also, holomorphic) functions achieved the boundary values on the disk. The works of F. and M. Riesz, Nevanlinna, and many others played majorly into this endeavor, with so-called Hardy spaces lying at the center of this study. On the other hand, the techniques relied heavily on the theory of complex variables, which restricted a lot of attention to the case of the plane. In parallel, the school of Zygmund and his student Calderón developed real-variable tools to study problems in harmonic analysis, which were seen to have immense ramifications to the study of the Laplace equation in higher dimensions. Landmark results here were the 1960 paper of Stein and Weiss [SW60] and the 1972 paper of Fefferman and Stein [FS72], in which Hardy spaces on higher dimensions were defined, and their properties studied. With the development of this real-variable theory, Dahlberg was able to solve (D) with data in L^2 on the domain above a Lipschitz graph [Dah77], [Dah79]. In this way, along with many essential contributions by Jerison, Kenig, Pipher, and others, the study of boundary value problems on rough domains was launched.

1.3.2 Boundary value problems for homogeneous second-order elliptic operators

In what follows, we always assume that $B_1 \equiv B_2 \equiv V \equiv 0$ unless explicitly mentioned otherwise. The history of work in solving the problems $(\mathrm{D})_2, (\mathrm{N})_2, (\mathrm{R})_2$ on rough domains can be tracked back to the celebrated paper [Dah77], where Dalhberg solved $(\mathrm{D})_2$ in the case that $L \equiv -\Delta$ on a Lipschitz domain Ω. The solvability of $(\mathrm{N})_2$ and $(\mathrm{R})_2$ in this setting was obtained by Jerison and Kenig in [JK81b]. Solvability of these same problems via the method of layer potentials is due to Verchota in [Ver84], using the L^2-boundedness of the Cauchy integral operator on a Lipschitz curve.

To extend solvability results of boundary value problems (with L^p boundary data) from the Laplacian $-\Delta$ to the homogeneous second-order elliptic operators $-\operatorname{div} A\nabla$, further assumptions need to be imposed on the matrix A. The conditions that have historically been considered can roughly be categorized into either t-independent, regularity, or perturbative assumptions.

The $t-$independence condition. This assumption on the elliptic matrix A is a natural starting place to study boundary value problems owing to two main observations. First, the $t-$independent setting is the situation that arises from the pullback of the Laplacian on a domain above a Lipschitz graph via the mapping that "flattens" the boundary. Second, in light of [CFK81], a square Dini condition on the transversal modulus of continuity of A is necessary in order to have solvability of the Dirichlet problem with rough data. More precisely, if ω is any real-valued function satisfying

$$\int_0^1 \frac{(\omega(\tau))^2}{\tau}\, d\tau = +\infty,$$

then Caffarelli, Fabes and Kenig showed that there exists a real, symmetric, elliptic matrix $A(x,t)$ whose transverse modulus of continuity (that is, the modulus of continuity in the $t-$direction) is controlled by ω, but for which the $L-$elliptic measure and the Lebesgue measure on the boundary are mutually singular, precluding the possibility that the Dirichlet problem is solvable for any $p > 1$. On the other hand, Fabes, Jerison and Kenig showed in [FJK84] that $(\mathrm{D})_2$ holds if one assumes that the transverse modulus of continuity $\omega(\tau) \equiv \sup\limits_{x\in\mathbb{R}^n, 0<t<\tau} |A(x,t) - A(x,0)|$ satisfies the square Dini condition $\int_0^1 \frac{(\omega(\tau))^2}{\tau} d\tau < \infty$ and $A(x,0)$ is sufficiently close to a constant matrix.

Now, assuming that $\Omega = \mathbb{R}^{n+1}$ and A is a real, symmetric, bounded, elliptic, $t-$independent matrix, the problem $(\mathrm{D})_2$ was solved in [JK81a] (grounded in the pioneering work of Dahlberg [Dah77, Dah79] for the Laplacian on Lipschitz domains), while $(\mathrm{N})_2$ and $(\mathrm{R})_2$ were solved by Kenig and Pipher in [KP93a]. In either case, the solution did not rest on the method of layer potentials.

In the setting where A is a complex, bounded, elliptic, $t-$independent, not necessarily self-adjoint matrix, several results in different special cases were obtained:

- If A has "block" form

$$A = \left(\begin{array}{ccc|c} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right) \tag{1.3.3}$$

  where $B = B(x)$ is an $n \times n$ matrix, then $(\mathrm{D})_2$, $(\mathrm{N})_2$, and $(\mathrm{R})_2$ were found to be solvable. Indeed, in this case $(\mathrm{D})_2$ is a consequence of the theory of semigroups, while Kenig observed in [Ken94] that $(\mathrm{R})_2$ amounted to solving the Kato square root problem for the $n-$dimensional operator

$$J = -\mathrm{div}_x B(x) \nabla_x,$$

  while $(\mathrm{N})_2$ amounted to the L^2-boundedness of the Riesz transforms $\nabla J^{-\frac{1}{2}}$ (or equivalently, to solving the Kato problem for $adj(J)$). These results were obtained in the desired generality in [AHL$^+$02b], along with a solution to the Kato problem in the setting of second-order elliptic operators. These results have been more recently generalized to include the setting of block-triangular matrices [AMM13], and even to certain degenerate elliptic matrices [ARR15].
- Fabes, Jerison and Kenig first showed that $(\mathrm{D})_2$ is solvable for small, complex perturbations of constant elliptic matrices in [FJK84]. Later, in [AAA$^+$11], the authors gave the following powerful perturbation result using the method of layer potentials: $(\mathrm{D})_2$, $(\mathrm{N})_2$, and $(\mathrm{R})_2$ are solvable for small, complex perturbations of real, symmetric, bounded, elliptic, $t-$independent matrices. They reached this result through the method of layer potentials, and used the fact that the so-called De Giorgi-Nash-Moser bounds are stable under small complex perturbations (we will describe these bounds more precisely later). Auscher, Axelsson and Hofmann gave

in [AAH08] perturbation results akin to those of [AAA+11] without the assumption of DeGNM bounds nor the use of layer potentials. They introduced a new technique which has come to be known as the first-order calculus method, whereby one exploits representations of the boundary value problems using the functional calculus for certain Dirac type operators. Auscher, Axelsson and McIntosh [AAM10] later improved the perturbation results for $(\mathrm{D})_2$ and $(\mathrm{N})_2$ by showing that the set of A for which these problems are well-posed is an open set.

- For A real but not symmetric, Kenig, Koch, Pipher and Toro obtained in [KKPT00] the solvability of $(\mathrm{D})_p$ in the case that $n = 1$, for p sufficiently large depending on L. Furthermore, they construct a family of examples in $\mathbb{R}^2_+$ satisfying that for each $p > 1$, there exists a matrix $A(x)$ which is an appropriate perturbation of the 2×2 identity matrix, for which the corresponding problem $(\mathrm{D})_p$ is not solvable. Regarding the solvability of $(\mathrm{N})_{p'}$ and $(\mathrm{R})_{p'}$ in this setting on $\mathbb{R}^2_+$, Kenig and Rule showed in [KR09] the solvability of these problems for when p' is the Hölder conjugate to the p from the solvability of $(\mathrm{D})_p$ in [KKPT00]. Later, Barton proved in [Bar13] that the solvability results mentioned here are stable under small complex perturbations.
- For A real but not symmetric and $n \geq 2$, the solvability of $(\mathrm{D})_p$ for p sufficiently large depending on L was achieved by Hofmann, Kenig, Mayboroda and Pipher in [HKMP15b]. Furthermore, they later showed in [HKMP15a] that for complex-valued, not necessarily symmetric, $t-$independent, bounded, elliptic matrices, where in addition the solutions to $Lu = 0$ in $\mathbb{R}^{n+1}_\pm$ satisfy the DeGNM bounds (and same is true for the adjoint operator L^*), the problem $(\mathrm{D})_p$ is solvable if and only if $(\mathrm{R})_{p'}$ is solvable for p' the Hölder conjugate to p, and $p \in (2-\varepsilon, \infty)$, for some small constant ε. Here, the equivalence of the problems $(\mathrm{D})_p$ and $(\mathrm{R})_{p'}$ is understood up to taking adjoints of the operators. In particular, they deduce the solvability of $(\mathrm{R})_{p'}$ when A is real, for p' close enough to 1. The analogous question for the Neumann problem remains open.

Inspired by the aforementioned literature, one may naturally ask

Question 1. *Is the well-posedness of the problems $(\mathrm{D})_2$, $(\mathrm{N})_2$, and $(\mathrm{R})_2$ stable under complex $t-$independent scale-invariant lower-order term perturbations?*

This is the problem that we tackle in Chapters 4 and 5, and we ultimately answer in the affirmative (Theorem 1.2.16). Our main results in these two chapters join a comparatively sparse but increasingly lively literature of the well-posedness of boundary value problems for second-order elliptic equations with lower order terms; we discuss this literature further below in Section 1.3.3.

The regularity assumption. This condition is borne out from a conjecture posed by Dahlberg in 1984. Dahlberg, Kenig and Stein constructed [Dah86b] a one-to-one mapping from a Lipschitz domain onto the half-plane for which the pullback of the Laplacian results in a symmetric elliptic operator $L = -\operatorname{div} A\nabla$ on the half-plane Ω satisfying (recall δ is defined in (2.2.5))

(A1) $\delta \nabla A \in L^\infty(\Omega)$, and

(A2) $\delta |\nabla A|^2 \, d\mathscr{L}^n$ is a Carleson measure on Ω; that is, there exists $C > 0$ so that for each $x \in \partial\Omega$ and $r > 0$, if $B(x,r)$ is a ball in $\mathbb{R}^n$, we have that

$$\iint_{B(x,r)} \delta(Y) |\nabla A(Y)|^2 \, dY \leq C r^{n-1}.$$

Since Dahlberg had shown in his celebrated work [Dah77, Dah79] that $(\mathrm{D})_2$ was solvable for the Laplacian on a Lipschitz domain, he reasonably conjectured that $(\mathrm{D})_2$ is solvable for any real symmetric elliptic matrix A satisfying the assumptions (A1)-(A2). This question would be resolved over a decade later by Kenig and Pipher [KP01], and the real elliptic operators whose matrices satisfy (A1)-(A2) have since come to be known as the Dahlberg-Kenig-Pipher (DKP) operators. These regularity assumptions are close to optimal (see [FKP91, Theorem 4.11], [Pog], and [HMM$^+$b, Corollary 6.3]). We mention that, by assuming some smallness of the Carleson measure in (A2), Dindos-Petermichl-Pipher [DPP07] have obtained the solvability of $(\mathrm{D})_p$ for $p \in (1, \infty)$.

The Carleson perturbation condition. Other than the $t-$independent and regularity conditions, it is natural to ask whether the solvability of boundary value problems should be stable under some perturbations of the matrices, although this raises the question of what type of perturbation to consider. Let us be more precise: suppose that L_0 and L are two real elliptic second-order divergence form operators on Ω, with associated matrices A_0 and A, and elliptic measures ω_0 and ω, respectively. Recall that the solvability of the Dirichlet problem $(\mathrm{D})_q$ for some $q > 0$ is equivalent to the quantitative absolute continuity (the A_∞ property) of the elliptic measure with respect to the surface measure σ on the boundary of the domain.

Question 2. *What conditions may we ask of the pair* (A, A_0) *so that if* $\omega_0 \in A_\infty(\sigma)$*, then* $\omega \in A_\infty(\sigma)$*?*

The classical answer to this question is the well-known *Carleson perturbation condition*: in the setting of the upper-half space, Fefferman, Kenig and Pipher [FKP91] showed that $\omega \in A_\infty(\sigma)$ provided that $\omega_0 \in A_\infty(\sigma)$ and that the measure

$$d\mu(x,t) := \Big(\sup_{W(x,t)} |A - A_0|\Big)^2 \frac{dxdt}{t} \tag{1.3.4}$$

is a *Carleson measure*, where $W(x,t)$ is a "Whitney box" centered at (x,t) (this means that its sidelength is proportional to its distance to the boundary $\mathbb{R}^n$). We briefly postpone the literature review of Carleson perturbations until Section 1.3.5. Question 2 is also a main object of study in Chapters 2 and 3, but we consider operators with degenerate coefficients and domains with low-dimensional boundaries (in fact, in Chapter 3 we greatly generalize the domains considered to axiomatic *PDE friendly domains*, see Definition 3.2.10).

We end this review of the history of the work on homogeneous strongly elliptic operators by remarking that the *a priori* connections between the different problems $(\mathrm{D})_p$, $(\mathrm{N})_{p'}$ and $(\mathrm{R})_{p'}$ have also been of great interest. In some instances (say, A is real, $t-$independent), one has that the solvability of $(\mathrm{R})_p$ for L implies solvability of $(\mathrm{D})_{p'}$ for the adjoint operator L^*, and viceversa (where p' is the Hölder conjugate to p) (see [Ken94]), but Mayboroda found in [May10] that such implications need not hold in the general setting of complex coefficients, even for $t-$independent matrices. We refer to [May10] for a more systematic review of these connections.

1.3.3 Boundary value problems with lower-order terms

The literature in the setting with lower order terms present (that is, not all of B_1, B_2, V are identically 0) is much more sparse. In [HL01b], *parabolic* operators with singular drift terms B_2 were studied, and their results would later be applied toward (D_p) for elliptic operators with singular drift terms B_2 in [KP01] and [DPP07]. When $A \equiv I$, $B_1 \equiv B_2 \equiv 0$ and $V > 0$ satisfies certain conditions, Shen proved the solvability of (N_p) on Lipschitz domains in [She94]. His results were later extended in [Tao12, TW01] to (R_p) and under weaker assumptions on the potential V. It is a critical element of the proof that the leading term of $L \equiv -\Delta + V$ is the Laplacian, and the question of

$(\mathrm{N})_p$ $-$solvability for Schrödinger operators on rough domains in the case that $A \neq I$ remain open, even under generous assumptions on V.

More recently, the problems $(\mathrm{D})_2$ and $(\mathrm{R})_2$ for equations with lower order terms have been considered in [Sak19] in bounded Lipschitz domains, under some continuity and sign assumptions on the coefficients. Solvability results for the variational Dirichlet problem of equations with lower order terms on unbounded domains have been obtained in [Mou]. Finally, we bring attention to [MT], where, through the development of a holomorphic functional calculus, the authors proved the L^2 well-posedness of the Dirichlet, Neumann, and regularity problems in the $t-$independent half-space setting for the Schrödinger operator $-\operatorname{div} A\nabla + V$ with Hermitian A and potential V in the reverse Hölder class $RH^{\frac{n}{2}}$.

1.3.4 Degenerate elliptic operators and lower dimensional boundaries

In a robust way, quantifiable well-posedness of the Dirichlet problem is equivalent to quantifiable absolute continuity of the elliptic measure with respect to the boundary surface measure, whenever the latter makes sense [FKP91, DKP11, DJK84, Zha18]. In turn, this property of quantifiable absolute continuity of elliptic measure has been successfully tied to quantifiable geometric and topological properties of the boundary of the domain, when the boundary has co-dimension 1. We do not attempt to comprehensively review the literature in this area, but let us mention that considerable attention has been devoted to studying the geometric assumptions on co-dimension 1 boundaries for which the elliptic measure is absolutely continuous with respect to the surface measure [Dah77, DJ90, Sem89, BL04, HM14], as well as the converse so-called free boundary problems, where geometric information of the boundary is deduced from a priori solvability properties [Azz, HMU14, AHM$^+$17, AHM$^+$16], culminating in the recent results of [AHM$^+$], which gives a complete picture of the relationship between absolute continuity of elliptic measure with respect to surface measure on the one hand, and uniform rectifiability plus a quantitative connectivity property on the other hand, for boundaries of co-dimension 1.

However, when the boundary has co-dimension larger than 1, the correspondence between geometry and the theory of the Dirichlet problem for uniformly elliptic operators is severed, essentially owing to deep (and, as of yet, not completely understood) dimensional constraints on the support of the harmonic measure [BJ90, Bou87, Wol95]. Indeed, if we attempted to solve the Laplace equation outside of a boundary of high enough co-dimension, the equation does not "see" the boundary, and thus the uniformly elliptic PDE theory is not a correct lens by which to characterize the geometry of such boundaries.

This is where the recent program of David, Feneuil, and Mayboroda [DFM19b], briefly described in Section 1.2.1, comes into play. They consider the degenerate elliptic operators (1.2.1) on the domains with rough low dimensional boundaries, and in several papers they have established a robust elliptic theory whose connections to low-dimensional boundary geometry is still a matter of ongoing research. Degenerate operators have been previously considered in many previous works, for instance, [FKS82], [FJK82], and [ARR15], but the solvability of boundary value problems on domains with rough boundaries (let alone low dimensional boundaries) had not been studied.

Let us review a bit of the theory developed so far for the operators of the form (1.2.1). In [DFM19a] the authors provided an analogue of Dahlberg's result [Dah77] which holds for their weighted elliptic operators. More precisely, for a $d-$dimensional Lipschitz graph Γ with small Lipschitz constant, David, Feneuil and Mayboroda constructed a weighted elliptic operator of the form (1.2.1) so that the elliptic measure is absolutely continuous with respect to the surface measure on Γ. The equivalence of quantitative well-posedness of the Dirichlet problem and quantitative absolute continuity of elliptic measure was considered by Mayboroda and Zhao in [MZ19], so that from the two works [DFM19a], [MZ19] we see the first solvability results of the Dirichlet problem for the operators defined in [DFM19b]. More recently, the Dirichlet problem $(\mathrm{D})_p$ was tackled by Feneuil, Mayboroda, and Zhao in [FMZ] under some small Carleson norm assumptions on the coefficient matrix $\mathcal{A}$, extending results of [DP19] to this setting (see also [DPP07]).

Regarding solvability of the Dirichlet problem on domains with uniformly rectifiable low dimensional boundaries, David and Mayboroda [DMb] have shown that for a suitable substitute of the Laplacian in the low dimensional setting, the elliptic measure is absolutely continuous with respect to the surface measure on $d-$dimensional uniformly rectifiable boundaries, with $d \leq n-2$ an integer (see also Feneuil [Fen] for a different proof). On the other hand, in [DEM], David, Engelstein, and Mayboroda manufactured an example which shows that for *any* $d-$Ahlfors-David regular set Γ with $d < n-2$, there exists a special operator L_{DEM} formally belonging to the class (1.2.1) whose elliptic measure is absolutely continuous with respect to the surface measure. The latter result lies in sharp contrast to the landmark free-boundary result in the co-dimension 1 case [AHM$^+$]. Indeed, it implies that for d-ADR sets with $d < n-2$ an integer (recall that the complements of these sets always verify the interior Corkscrew and Harnack Chain properties), uniform rectifiability alone cannot possibly characterize the A_∞ property of the elliptic measure for all operators in the class (1.2.1). It is possible that the case of L_{DEM} is a miraculous arithmetic cancellation, and the free boundary result is still valid in some capacity. We show in Chapter 2, however, that even if so, this arithmetic cancellation produces an entire family of counterexamples - see Corollary 2.1.8 below (this result is also extended in Chapter 3 via the generalized Carleson perturbations).

In Chapter 2, we aim to further develop the theory of these degenerate elliptic operators by showing that quantifiable well-posedness of the Dirichlet problem is an open property. A bit more precisely, we show that the Fefferman-Kenig-Pipher theory of Carleson perturbations (see Section 1.3.5 below for the literature review) has a natural analogue in the setting of the degenerate elliptic operators (1.2.1). As mentioned in the last paragraph, this Carleson perturbation result has important implications in the theory of these operators.

We also mention briefly that an axiomatic elliptic PDE theory for domains with mixed-dimensional boundaries is realized in [DFM]. In Chapter 3, we have extended Carleson perturbation results to a large class of domains which contains in particular the domains with mixed-dimensional boundaries of [DFM].

1.3.5 History of Carleson perturbations

The first results in this direction are found in [FJK84, Dah86a]. In the setting where Ω is the unit ball in $\mathbb{R}^n$, the condition that Dahlberg asked of the pair (A, A_0) of symmetric operators is that the *disagreement* $\rho(A, A_0)$ defined as

$$\rho(A, A_0)(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|, \qquad X \in \Omega,$$

satisfies the following *vanishing Carleson measure condition*

$$\lim_{r\searrow 0}\sup_{x\in\partial\Omega} h(x,r) = \lim_{r\searrow 0}\sup_{x\in\partial\Omega}\Big(\frac{1}{\sigma(B(x,r)\cap\partial\Omega)}\iint_{B(x,r)\cap\Omega}\frac{\rho(A,A_0)^2(X)}{\delta(X)}\,dX\Big)^{\frac{1}{2}} = 0, \tag{1.3.5}$$

where σ is the Hausdorff $(n-1)-$dimensional measure on the unit sphere $\partial\Omega$. In this case, if $\omega_0 \in A_\infty(\sigma)$ and its Poisson kernel $k_0 = d\omega_0/d\sigma \in RH_p$ (see Proposition 3.3.28), then $\omega \ll \sigma$ and $k = d\omega/d\sigma \in RH_p$, so that the solvability of $(\mathrm{D})_{p'}$ is stable (with the same p') under the condition (1.3.5). The fact that the reverse Hölder exponent is preserved by (1.3.5) suggests that there might be a weaker condition than (1.3.5) which preserves the A_∞ membership but not the RH exponent. Fefferman [Fef89] thus showed a few years later that, again in the context of symmetric operators on the unit ball, if $\omega_0 \in A_\infty(\sigma)$, and if the *area integral*

$$\mathscr{A}(\rho(A,A_0))(x) := \Big(\iint_{\gamma(x)}(\rho(A,A_0)(X))^2\frac{dX}{|B(X,\delta(X)/2)|}\Big)^{1/2}, \qquad x\in\partial\Omega,$$

satisfies

$$\mathscr{A}(\rho(A,A_0)) \in L^\infty(\partial\Omega,\sigma), \tag{1.3.6}$$

then $\omega \in A_\infty(\sigma)$. It is clear that (1.3.6) is not a vanishing condition; moreover, via Fubini's theorem, one can see that (in the case of the unit ball)

$$h(x,r) \lesssim \Big(\frac{1}{\sigma(B(x,Cr))\cap\partial\Omega)}\iint_{B(x,Cr)\cap\partial\Omega}\mathscr{A}(\rho(A,A_0))(x)^2\,d\sigma\Big)^{1/2}$$
$$\leq \|\mathscr{A}(\rho(A,A_0))\|_{L^\infty(\partial\Omega,\sigma)},$$

and it would be shown in [FKP91] that (1.3.6) does not preserve the RH exponent. Next, one may wonder whether (1.3.6) is an optimal condition on $\rho(A,A_0)$ that guarantees the stability of the A_∞ property. But the answer to this question is *no*: Fefferman-Kenig-Pipher [FKP91] showed that the optimal assumption (at least in the cases of the unit ball or half-plane) which preserves the A_∞ property is that $\rho(A,A_0)^2\delta^{-1}$ is the density of a Carleson measure; in other words, the optimal condition is that

$$\sup_{r\in(0,\mathrm{diam}(\partial\Omega))}\sup_{x\in\partial\Omega} h(x,r) < +\infty. \tag{1.3.7}$$

With the landmark paper of [FKP91], one could say that the contemporary era of the perturbation results was launched: since then, the perturbation results have often assumed variants of the Carleson measure hypothesis (1.3.7), at least away from the realm of t-independent operators. The proof of optimality of the aforementioned [FKP91] result relied on a newfound characterization of A_∞ on $\mathbb{R}^n$ via a Carleson measure property. The Fefferman-Kenig-Pipher characterization can be formulated as follows [Ste93a, Page 225]: Suppose that $w \in L^1_{\rm loc}(\mathbb{R}^n)$ is a non-negative function such that the measure $w\,dx$ is doubling on $\mathbb{R}^n$, and that Φ is a non-negative Schwartz function with $\int_{\mathbb{R}^n} \Phi\,dx = 1$. Then $w \in A_\infty$ if and only if

$$d\mu := \frac{|\nabla_x (w * \Phi_t)|^2}{|w * \Phi_t|^2} t\,dx\,dt \tag{1.3.8}$$

is a Carleson measure in $\mathbb{R}^{n+1}_+$ (here, $\partial\mathbb{R}^{n+1}_+$ is endowed with the $n-$dimensional Hausdorff measure). They go on to show characterizations of A_p and RH_p through similar Carleson measure conditions [FKP91, Theorem 3.3]. This characterization of A_∞ via Carleson measures is not *too* surprising, owing to the classical results that $w \in A_\infty$ implies $\log w \in BMO$, and the square-function characterization of BMO [Ste93a]. The FKP result also fits as a multiplicative analogue of the classical theory of differentiation and a condition of Zygmund; see [FKP91] for further discussion.

The FKP perturbation survives in rough domains and for degenerate elliptic operators

In the past few decades there has been a lot of interest in the Dirichlet problem on domains satisfying weak topological and geometric assumptions. A thorough review of this area is outside our scope, but some highlighted works include [JK81a, Sem89, DJ90, BL04, HM14]. Of course, one immediately wonders whether the FKP perturbation theory holds in these more general domains. Along these lines, Milakis-Pipher-Toro [MPT14] obtained the analogue of the FKP perturbation result for the bounded chord-arc domains (these have quantitative openness both in the interior and exterior of the domain, as well as quantitative path-connectedness, and their boundary is $(n-1)-$Ahlfors-David regular; see Section 3.2 for precise statements). They also obtained the stability of the RH_p condition if the measure on Ω with density $\rho(A, A_0)^2\delta^{-1}$ is a Carleson measure with small enough norm (depending on p and A_0) (see also [Esc96] and [MT10]).

It is known that the bounded chord-arc domains have uniformly rectifiable boundaries [DJ90, HMU14]. Cavero-Hofmann-Martell [CHM19] have proved that the FKP perturbation theory holds also for real symmetric operators on the more general 1-sided chord-arc domains (quantitative openness and quantitative path-connectedness inside the domain and $(n-1)-\mathrm{ADR}$ boundaries) (see Section 3.2). Their method relies on an extrapolation of Carleson measures technique developed by Lewis and Murray [LM95], which was first used to give an alternate proof of the FKP perturbation result by Hofmann and Martell [HM12]. The technique makes heavy use of sawtooth domains and a Dahlberg-Jerison-Kenig projection lemma which allows one to compare measures on the sawtooth domain to their projections on the original boundary. A year later, Cavero-Hofmann-Martell-Toro [CHMT20] devised a different method of proof for the FKP perturbation result in the same setting of 1-sided CAD (and extending to the non-symmetric case), using a generalization of a result of Kenig-Kirchheim-Pipher-Toro [KKPT16] that weak-BMO solvability of L implies the A_∞ property for the elliptic measure of L (in fact, this is a characterization [CHMT20, HMT]).

The state-of-the-art for the $1-$elliptic operators (1.1.4) lies in the article of Akman-Hofmann-Martell-Toro [AHMT], where they generalize the FKP perturbation theory to the situation of uniform domains satisfying a capacity density condition. Since the $(n-1)-$dimensional Hausdorff measure of the boundary of the domain need not be ADR (indeed, it could potentially be locally infinite), their perturbation result is stated among the elliptic measures only, with no reference to an underlying surface measure. Thus, a main result of theirs reads as follows: Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded (for simplicity, but they consider unbounded domains too) uniform domain satisfying the CDC and fix X_0 in the "center" of Ω (for instance, X_0 can be any Corkscrew point of a ball with radius $\mathrm{diam}(\partial\Omega)/2$). Let L_0, L be two elliptic operators with associated matrices A_0, A, associated elliptic measures ω_0, ω, and associated Green's functions G_0, G, respectively. If

$$\begin{aligned}&\sup_{r\in(0,\mathrm{diam}(\partial\Omega))}\sup_{x\in\partial\Omega} g(x,r)\\&=\sup_{r\in(0,\mathrm{diam}(\partial\Omega))}\sup_{x\in\partial\Omega}\frac{1}{\omega_0^{X_0}(B(x,r)\cap\partial\Omega)}\iint_{B(x,r)\cap\Omega}\rho(A,A_0)(Y)^2\frac{G_0(X_0,Y)}{\delta(Y)^2}\,dY<+\infty,\end{aligned} \tag{1.3.9}$$

then $\omega \in A_\infty(\omega_0)$ (see Definition 3.1.11). They consider the expression $g(x,r)$ based on an analogous one used as an intermediate step in [FKP91]. Furthermore, it is shown that if there exists a doubling measure σ on $\partial\Omega$ such that $\omega_0 \in A_\infty(\sigma)$ and $\rho(A, A_0)$ satisfies (1.3.9), then $\rho(A, A_0)$ also satisfies (1.3.7) (with $h(x,r)$ defined using σ), and $\omega \in A_\infty(\sigma)$. In this way, we see that the results of [AHMT] do properly generalize the perturbation results of [FKP91]. They are also able to generalize the area integral condition of Fefferman [Fef89], so that if $\mathscr{A}(\rho(A, A_0)) \in L^\infty(\partial\Omega, \omega_0)$, then $\omega \in A_\infty(\omega_0)$. They also show that "small constant" assumptions lead to $\omega \ll \omega_0$ and $\frac{d\omega}{d\omega_0} \in RH_p(\omega_0)$.

Lastly, let us mention that these Carleson perturbation results have been partially extended to the complex-coefficient setting (and even to elliptic systems) by Auscher and Axelsson in [AA11] under the assumption that A_0 is $t-$independent, that L_0 is L^2 solvable, and that the Carleson norm of the measure in (1.3.4) is small enough. They then obtain L^2 solvability for L_1. Analogous L^p solvability results were obtained by Hofmann, Mayboroda and Mourgoglou in [HMM15a].

1.3.6 Exponential decay of fundamental solutions to Schrödinger operators

Let us now discuss the adjacent literature to the results mentioned in Section 1.2.4.

The first results expressing upper estimates on the solutions in terms of a certain distance associated to the potential V go back to Agmon [Agm82]. He has introduced a distance function which now bears his name and which we will discuss below, and showed that the solution decays exponentially in the region where $V \geq 0$. Agmon's estimates, however, are clearly non-sharp for most non-trivial potentials, for a simple reason that solutions carry some amount of regularity and low values of V in small regions should not drastically affect their decay properties. This vague statement is very hard to quantify; in many situations, however, the behavior of solutions, notably of the eigenfunctions, is rather precisely governed by the uncertainty principle. The latter has a few manifestations. In particular, the most recent one in [ADF$^+$] yielded astonishingly accurate estimates on eigenfunctions even for the prototype of the Anderson model based on disordered potentials. Here, however, we will use a much earlier result due to Fefferman and Phong which has been later extended, e.g., to the context of the magnetic Schrödinger operator - one of our main objects of study in Chapter 6. This extension unfortunately currently seems beyond the reach of the methods in [ADF$^+$].

Needless to say, (1.2.18) underlines sharpness of the emerging estimates. The only context in which (1.2.18) have been proved before is that of the classical Schrödinger operator $-\Delta + V$ [She99], and we, of course, build on the ideas from [She99]. As we mentioned before, to the best of our knowledge, no sharp results on the exponential decay of the kernels to the magnetic Schrödinger operator or generalized Schrödinger operator existed in the literature. In fact, even the existence of the fundamental solution to the magnetic Schrödinger operator for $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$, subject to the usual bound by a multiple of $|X-Y|^{2-n}$ (Theorem 6.5.35), seemed to be out of reach, as previous treatises normally imposed somewhat ad hoc conditions of smoothness for the magnetic field $\mathbf{a} \in C^2$ or at least $\mathbf{a} \in L^4_{\text{loc}}(\mathbb{R}^n)$, div $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ [KS00, BA10]. As we will see in Section 6.2, both situations are considerably simpler than ours but not completely natural, for $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ are the minimal restrictions allowing one to make sense of the bilinear form associated to L_M in the weak sense.

Furthermore, certain polynomial upper estimates on the fundamental solutions in terms of m have been established in a variety of contexts, in particular, in [She95] for $-\Delta + V$, and in [She96b] for the magnetic Schrödinger operator, under assumptions similar to ours. Polynomial decay is sufficient for establishing key properties of the associated Riesz transforms and similar operators - the main goal of the majority of these papers, but is obviously not sharp. An attempt to get exponential decay has been made at [Kur00]. In this paper the author treated the heat kernel estimates for $-\text{div}A\nabla + V$ (A real and symmetric) and $-(\nabla - i\mathbf{a})^2 + V$ and integrating them obtained for these two operators

$$|\Gamma(X,Y)| \le \frac{Ce^{-\varepsilon(1+m(X,V+|\mathbf{B}|)|X-Y|)^{\frac{2}{2k_0+3}}}}{|X-Y|^{n-2}} \quad \text{for a.e. } X,Y \in \mathbb{R}^n,$$

for some $k_0 > 0$, which is, once again, not sharp, as can be seen from (1.2.18) and (6.1.7). Finally, without an attempt of a comprehensive review of the theory, we mention that resolvent estimates are routinely used in many aspects of semiclassical analysis, which roughly speaking, concentrates on the behavior of solutions to $-\hbar^2\Delta + V$ and analogous operators as $\hbar \to 0$, but these bounds are typically independent of local features of V and $\mathbf{B}$. The major achievement of Chapter 6 is similar estimates from above and below which is only possible by a careful account of the impact of the electric and magnetic potentials.

1.4 Notation and conventions

Although there are many overlapping notions, results, and definitions among the different chapters in this thesis, each chapter is self-contained, in the sense that every important or recurring notion, notation, or definition appearing in a chapter will be defined within that chapter, perhaps with the addition of this section. The only exception to this convention is that Chapter 5 relies heavily on several definitions from 4 (and indeed, Chapter 5 is read best after Chapter 4).

We will often write $a \lesssim b$ to mean that there exists a constant $C \geq 1$ such that $a \leq Cb$, where C may depend only on certain allowable parameters. Likewise, we write $a \approx b$ if there exists a constant $C \geq 1$ such that $\frac{1}{C} b \leq a \leq Cb$.

If (Ω, σ) is a measure space and $E \subset \Omega$ is measurable, we will always write $⨍_E f \, d\sigma = \frac{1}{\sigma(E)} \int_E f \, d\sigma$.

The ambient dimension in Chapters 2, 3, and 6 is denoted as n, while the ambient dimension in Chapters 4 and 5 is $n+1$.

For $m \in \mathbb{N}$, we denote by $\mathscr{L}^m$ the $m-$dimensional Lebesgue measure.

For any $m \geq 0$, we write $\mathscr{H}^m$ for the $m-$dimensional Hausdorff measure (see [Fed69]). For integer m, we normalize $\mathscr{H}^m$ so that it equals $\mathscr{L}^m$.

The uppercase letters at the end of the alphabet X, Y, Z will usually refer to points in the domain. The lowercase letters x, y, z will usually refer to points in the boundary of the domain.

For $X \in \mathbb{R}^n$ and $r > 0$, we write $B(X, r) \subset \mathbb{R}^n$ for the (open) ball of radius r centered at X.

If A is a Borel set in $\mathbb{R}^n$ and $F \in L^1(A, \mathscr{L}^n)$, we will often write $\int_A F \, \mathrm{d}\mathscr{L}^n(X) = \int_A F \, dX$.

In Chapters 2 and 3, $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.

The notion of $w-$elliptic operators, introduced in (1.1.2) and (1.1.3), is relevant in Chapters 2 and 3. In the former, we fix the weight w to be given by $w(X) = \delta(X)^{d+1-n}$ in the domain Ω (see (2.2.6)); while in the latter we consider a broad class of weights, and so w is not fixed in Chapter 3. In Chapters 4, 5 and 6, all operators are $1-$elliptic.

Chapter 2

Carleson perturbations of elliptic operators on domains with low dimensional boundaries

The research in this chapter was done in collaboration with S. Mayboroda.

2.1 Introduction

In this section, we continue the introduction to this chapter, already begun in Section 1.2.1. Relevant literature review lies in Sections 1.3.4 and 1.3.5.

As mentioned before, in this chapter we aim to further develop the theory of the David-Feneuil-Mayboroda degenerate elliptic operators (1.2.1) by showing that quantifiable well-posedness of the Dirichlet problem is an open property. Recall that we say that the Dirichlet problem for L with L^p data is solvable, or $(\mathrm{D})_p$ is solvable, if and only if (1.1.15) holds for each $f \in L^p(\partial\Omega, \sigma)$[1]. Let us restate the main two theorems to be proved in this chapter.

Theorem 2.1.1 (Carleson perturbation preserves Dirichlet problem solvability)**.** *Suppose that Γ is d-ADR with $d \in [1, n-1)$, $n \geq 3$ (see Definition 2.2.1). Let two operators L_0 and L be given as in (1.2.1) with associated bounded and uniformly elliptic real*

[1] As usual we understand $Lu = 0$ in a weak sense, see Definition 2.7.5.

(not necessarily symmetric) matrices $\mathcal{A}_0$ and $\mathcal{A}$ (see (2.7.2) for the definition of uniform ellipticity). Let ω_0, ω denote the respective elliptic measures (see Section 2.7.1). We define the disagreement of the matrices $\mathcal{A}$, $\mathcal{A}_0$ as

$$\mathfrak{a}(X) := \sup_{Y \in B(X,\delta(X)/2)} |\mathfrak{E}(Y)|, \quad \mathfrak{E}(Y) := \mathcal{A}(Y) - \mathcal{A}_0(Y), \qquad X \in \Omega. \tag{2.1.2}$$

Assume that $\delta(X)^{d-n}\mathfrak{a}^2\, dX$ is a Carleson measure; that is, assume that there exists a constant $C \geq 1$ such that for each surface ball $\Delta = B(x,r) \cap \Gamma$, the estimate

$$\iint_{T(\Delta)} \frac{\mathfrak{a}(X)^2}{\delta(X)^{n-d}}\, dX \leq C\sigma(\Delta) \tag{2.1.3}$$

holds, where $T(\Delta) = B(x,r) \cap \Omega$. Then, if $(\mathrm{D})_{p'}$ is solvable for L_0 for some $p' \in (1,\infty)$, then there exists $q' \in (1,\infty)$ such that $(\mathrm{D})_{q'}$ is solvable for L.

Theorem 2.1.4 (Small Carleson perturbation preserves $(\mathrm{D})_p$)**.** *Suppose that Γ is d-ADR with $d \in [1, n-1)$, $n \geq 3$ (see Definition 2.2.1). Let two operators L_0 and L be given as in (1.2.1) with associated uniformly elliptic real (not necessarily symmetric) matrices $\mathcal{A}_0$ and $\mathcal{A}$. Let ω_0, ω denote the respective elliptic measures (see Section 2.7.1). Moreover, suppose that there exists $p' \in (1,\infty)$ such that $(\mathrm{D})_{p'}$ is solvable for L_0. Define $\mathfrak{a}$ as in (2.1.2), and assume that there exists $\varepsilon_0 > 0$ so that*

$$\iint_{T(\Delta)} \frac{\mathfrak{a}(X)^2}{\delta(X)^{n-d}}\, dX \leq \varepsilon_0\sigma(\Delta), \qquad \textit{for all } \Delta \subset \Gamma. \tag{2.1.5}$$

Then for all ε_0 small enough, depending only on n, d, the d-ADR constant of Γ, the ellipticity of L_0 and L, and the RH_p characteristic of the Poisson kernel $\frac{d\omega_{L_0}}{d\sigma}$ (see Theorem 2.7.40 and Definition 2.7.28), we have that $(\mathrm{D})_{p'}$ is solvable for L as well, and the RH_p characteristic of $\frac{d\omega_L}{d\sigma}$ depends only on the same parameters as ε_0 does.

Analogues of our Theorems 2.1.1 and 2.1.4 in the co-dimension 1 setting are well-known. For instance, when the domain Ω has interior Corkscrew points and Harnack Chains, and $\partial\Omega$ is $(n-1)-$ADR, then Cavero, Hofmann, and Martell [CHM19] proved the analogues of Theorems 2.1.1 and 2.1.4 for symmetric matrices. Their results have been generalized to non-symmetric matrices by Akman, Hofmann, Martell, and Toro [AHMT]; see also the work of Cavero, Hofmann, Martell, and Toro [CHMT20] for an alternative proof of the co-dimensional 1 analogue of Theorem 2.1.1.

We note that when $\Gamma = \mathbb{R}^n \backslash \mathbb{R}^d$, a version of Theorem 2.1.1 with a certain operator $L_{\rm DFM}$ satisfying some structural assumptions is already part of the main result in [DFM19a], where Carleson measure perturbations from their operator $L_{\rm DFM}$ are allowed. Likewise, a version of Theorem 2.1.4 with a specific operator $L_{\rm FMZ}$ satisfying some structural assumptions is already in [FMZ] for $\Gamma = \mathbb{R}^n \backslash \mathbb{R}^d$. The novelty of our main results is that they hold for any $d \in [1, n-1)$, any $d-$Ahlfors-David regular set Γ (with d not necessarily an integer), and for any real operator L_0 verifying the well-posedness of $({\rm D})_p$, for some $p \in (1, \infty)$. We also briefly remark that Theorem 2.1.1 may not be deduced from Theorem 2.1.4 due to the dependence of the latter on the RH_p characteristic of $\frac{d\omega_{L_0}}{d\sigma}$.

Let us discuss a couple of immediate corollaries of our result. First, in the work [DMb] (also [Fen]) the authors obtain the following result. Suppose that $\Gamma \subset \mathbb{R}^n$ is a $d-$dimensional uniformly rectifiable set in $\mathbb{R}^n$ with $d \leq n-2$, $d \in \mathbb{N}$, that μ is a uniformly rectifiable measure on Γ, and that

$$L_{\mu,\alpha} = -\operatorname{div}\Big(\frac{1}{D_{\mu,\alpha}^{n-d-1}}\nabla\Big), \qquad \text{in } \mathbb{R}^n\backslash\Gamma,$$

where $D_{\mu,\alpha}$ is the regularized distance

$$D_{\mu,\alpha}(X) := \Big(\int_\Gamma |X-y|^{-d-\alpha}\, d\mu(y)\Big)^{1/\alpha}, \qquad \alpha > 0,\ X \in \mathbb{R}^n\backslash\Gamma. \tag{2.1.6}$$

Then $\omega_{\mu,\alpha} \in A_\infty(\mu)$. We remark that in this setting, μ is equivalent to $\sigma = \mathscr{H}^d|_\Gamma$, and for any $\alpha > 0$, $D_{\mu,\alpha} \approx_\alpha \delta(X)$. Using our Theorem 2.1.1, we are able to extend their class of solvable problems.

Corollary 2.1.7 (Uniform rectifiability implies solvability of many operators)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a closed $d-$dimensional uniformly rectifiable set with $d \leq n-2$ an integer. Suppose that μ and $L_{\mu,\alpha}$ are as described above, with $\alpha > 0$. Let L be an operator of the form (1.2.1) with matrix $\mathcal{A}$ for which the disagreement of $\mathcal{A}$ with $\mathcal{A}_{\mu,\alpha}$ satisfies the Carleson measure perturbation (2.1.3). Then $\omega_L \in A_\infty(\mathscr{H}^d|_\Gamma)$.*

Next, we recall the "magic" α example in [DEM]. Let Γ be a (possibly purely unrectifiable) closed unbounded $d-$Ahlfors-David regular set in $\mathbb{R}^n$ with $d \in (0, n-2)$ not necessarily an integer, and $n \geq 3$. Let $\hat\alpha := n-d-2$. Write $D_{\hat\alpha} = D_{\sigma,\hat\alpha}$ for the regularized distance (2.1.6) with $\mu = \sigma$. It turns out that in this setting, $D_{\hat\alpha}$ solves the equation $L_{\hat{\alpha}} u = L_{\sigma,\hat{\alpha}} u = 0$ in $\mathbb{R}^n \backslash \Gamma$. Ultimately, this observation can be used to deduce that the elliptic measure of $L_{\hat{\alpha}}$ is equivalent to σ (in the sense of pointwise equivalent bounds). Our Theorem 2.1.1 implies

Corollary 2.1.8 (Open ball around "magic α" example of [DEM])**.** *Let $n \geq 3$ and let $\Gamma \subset \mathbb{R}^n$ be a closed d-ADR set with $d \in [1, n-2)$. Suppose that L is an operator of the form (1.2.1) with matrix $\mathcal{A}$ for which the disagreement of $\mathcal{A}$ with $\mathcal{A}_{\sigma,\hat{\alpha}}$ satisfies the Carleson measure perturbation (2.1.3). Then $\omega_L \in A_\infty(\sigma)$.*

In other words, the counterexample of [DEM] to certain free boundary problems extends to give an open set of counterexamples.

We now discuss an example where the Carleson perturbations (2.1.3) arise naturally in the high co-dimensional setting. In [DFM19a], David, Feneuil, and Mayboroda studied the absolute continuity of the elliptic measure of the operator $L_\alpha = L_{\sigma,\alpha}$, $\alpha > 0$, with respect to the surface measure σ on a domain Ω with low-dimensional Lipschitz boundary Γ whose Lipschitz constant is small. They found a new bi-Lipschitz mapping $\rho : \mathbb{R}^n \backslash \mathbb{R}^d \to \Omega$ ([DFM19a, Section 1.4]) for which the pull-back of L_α to the domain $\Omega_0 := \mathbb{R}^n \backslash \mathbb{R}^d$ is the operator L on Ω with associated matrix

$$\mathcal{A}(x,t) := \Big(\frac{|t|}{D_\alpha(\rho(x,t))}\Big)^{n-d-1} |\det(\mathrm{Jac}(x,t))| (\mathrm{Jac}(x,t))^T (\mathrm{Jac}(x,t))^{-1}, \quad (2.1.9)$$

for $(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d}$ and where Jac is the differential of the mapping ρ. The idea is of course not new. However, the important new feature of the particular mapping in [DFM19a] is that it is almost an isometry in t (and this feature is necessary for the proof to work). In particular, the disagreement of the matrix $\mathcal{A}$ in (2.1.9) with a matrix of the form

$$\mathcal{A}_0 := \begin{pmatrix} \mathcal{A}_0^1 & 0 \\ 0 & bI_{n-d} \end{pmatrix}$$

satisfies the Carleson measure condition (2.1.3), where $\mathcal{A}_0^1$ is a $d \times d$ uniformly elliptic matrix, I_{n-d} is the $(n-d) \times (n-d)$ identity matrix, and $b \approx 1$ is a scalar function such that $|t|\nabla b$ verifies a similar Carleson measure condition. In the same paper they show that $\omega_L \in A_\infty(\sigma)$ and $\omega_{L_0} \in A_\infty(\sigma)$, and we remark that we can obtain the former conclusion from the latter using our Theorem 2.1.1.

Perturbation results are critical for the study of well-posedness, with applications to inverse problems and numerical analysis, because they allow some degree of relaxation of the assumptions on the coefficients needed for solvability. Indeed, necessary and sufficient conditions for solvability are very difficult to come by, whence a sensible strategy to establish a "fat" domain of operator invertibility is to first impose relatively strong conditions on the coefficients to make the problem tractable, and then in a second step exploit perturbation results.

Our method of proof follows the program of [HM12] (see also [CHM19]), where the main result of [FKP91] was given a new proof via an extrapolation theorem first presented by [LM95] and based on ideas of the Corona construction in [Car62], [CG75], where one aims to reduce matters to small perturbations on certain so-called sawtooth domains. Under this perspective, the overarching goal is to prove that the membership to a properly defined A_∞ class (see Definition 2.7.28) of the elliptic measure is preserved when the operator undergoes a perturbation ultimately controlled by its mass near the $d-$dimensional boundary.

One of the main tools allowing one to attack uniformly rectifiable boundaries is the analysis on so-called sawtooth domains "shielding" bad parts of the boundary and providing a systematic comparison of our solutions with nice ones on a sawtooth domain. In contrast to the co-dimension 1 case, our sawtooth domains will in general have mixed-dimensional boundaries, and therefore properties like Ahlfors-David regularity or uniform rectifiability cannot possibly transfer as-is to sawtooth domains. Nevertheless, we show that the sawtooth domains which we generate through a dyadic decomposition of the boundary [Chr90, DS93] and which have been seen in [HM14, MZ19] satisfy an axiomatic elliptic PDE theory for sets with boundaries of mixed dimensions presented in [DFM]; in particular, we construct a measure on the boundary of the sawtooth domain which behaves sufficiently like a surface measure, as well as a suitable analogue of the elliptic measure on such sets. This allows us to show a global analogue of the Dahlberg-Jerison-Kenig sawtooth projection lemma [DJK84]; in our setting, a similar result is shown in the upcoming work [DMb].

Similarly, comparison principle techniques are much more subtle to use, due to the fact that the coefficients of our operators must see the boundary of the domain, and hence classical arguments in which restriction of the domain of the operator plays a crucial role are not available to us. Accordingly, our arguments introduce some new ideas even in the classical case of co-dimension 1.

Furthermore, we do not require either of the matrices to be symmetric in any of our main results. Though an analogue of Theorem 2.1.1 for the $1-$sided chord-arc domains was already shown in [CHMT20] for the non-symmetric case, their methods are different and do not go through the extrapolation technique (besides, they do not show the non-symmetric case for their analogue of our Theorem 2.1.4). We will see in this chapter that the extrapolation technique does not need the symmetry of the matrices to work, essentially by being careful about the role of the adjoint operator.

Finally, we remark on a couple more subtle technical differences in our approach to the proof of Theorem 2.1.1 than what has been seen in the literature. First, we circumvent the use of "discrete" tent spaces or the use of a dyadic averaging operator in our proof of Theorem 2.1.1 (though we have verified that both methods can work), in favor of a simple direct approach to exploiting the smallness of a "discrete" Carleson measure in a continuous setting (see (2.9.10)). Second, as mentioned above, we use a *global* rather than local sawtooth projection lemma. This method allows us to only verify the axioms of the mixed-dimension elliptic PDE theory for *unbounded* sawtooth domains, and brings with it other simplifications in the geometric arguments; however, it introduces the complication that the globally constructed sawtooth domains do not locally coincide with the local sawtooth domains. Nevertheless, we shall see that this issue can be circumvented by realizing that the discrepancy between these sets is negligible from our point of view (see Lemma 2.7.38).

In Section 2.2, we recap geometric results for our d-ADR boundary. In Section 2.3, we collect standard results on the dyadic decomposition of the boundary and related notions. In Section 2.4, we give a careful construction of the dyadically-generated sawtooth domains, reproving many results shown in the co-dimension 1 setting. In Section 2.5, we construct a "surface" measure on the boundary of the sawtooth domain and prove that our sawtooth domains satisfy the axioms of the mixed-dimension elliptic PDE theory of [DFM]. In Section 2.6, we present the continuous Carleson measures, discrete Carleson measures, and the extrapolation theorem proved in [DMb]. In Section 2.7, we review the elliptic PDE theory for sets of high co-dimension, considered in [DFM19b]. Section 2.8 sees us proving the sawtooth projection lemma. Finally, in Sections 2.9 and 2.10, we give the proofs of our main results, Theorem 2.1.1 and Theorem 2.1.4.

2.2 Geometry of domains with low dimensional boundaries

Throughout, our ambient space is $\mathbb{R}^n$, $n \geq 3$. We now introduce the class of sets that the boundaries of our domains will reside in.

Definition 2.2.1 (Ahlfors-David regular sets)**.** Let $\Gamma \subset \mathbb{R}^n$ be a closed set and $d \in (0, n)$. We say that Γ is $d-$*Ahlfors-David-regular* (or d-ADR for short) if there exists a number $C_d \geq 1$ such that for any $x \in \Gamma$ and $r > 0$,

$$C_d^{-1} r^d \leq \mathscr{H}^d(\Gamma \cap B(x,r)) \leq C_d r^d, \tag{2.2.2}$$

where $\mathscr{H}^d$ is the $d-$dimensional Hausdorff measure. We shall often denote $\mathscr{H}^d|_\Gamma$ by σ, and refer to it as the *surface measure*. If $E \subset \mathbb{R}^n$ is a bounded, closed set, then we say that E is d-ADR if (2.2.2) holds for each $x \in E$ and $r \in (0, \operatorname{diam} E)$.

Henceforth, we take $d \in (0, n-1)$ always in this chapter. The set Γ will in this chapter always be a closed (unbounded) d-ADR set, and $\Omega := \mathbb{R}^n \backslash \Gamma$, so that Ω is an open set and $\partial\Omega = \Gamma$. Given $x \in \Gamma$ and $r > 0$, we call

$$\Delta = \Delta(x,r) = \Gamma \cap B(x,r) \tag{2.2.3}$$

a *surface ball*. It is easy to see that, by virtue of (2.2.2), $\mathscr{H}^d|_\Gamma$ is doubling on Γ (see Definition 2.2.12 below). Thus, if $d|_\Gamma$ is the restriction of the Euclidean distance on $\mathbb{R}^n$ to Γ, then $(\Gamma, d|_\Gamma)$ is a doubling metric space (that is, it admits a doubling measure), and hence the triple $(\Gamma, d|_\Gamma, \sigma)$ is a space of homogeneous type (see [Chr90]). It is obvious that if Γ is a d-ADR set, then so is any surface ball in Γ. Given a surface ball $\Delta = \Delta(x,r)$ and $c > 0$, we denote by $c\Delta$ the set $\Gamma \cap B(x, cr)$.

We now see that the mass of a surface ball cannot be too concentrated at its center.

Lemma 2.2.4 (Non-degeneracy of surface balls)**.** *Suppose that Γ is a closed d-*ADR *set. Then for any $x \in \Gamma$ and any $r > 0$,*

$$\operatorname{diam} \Delta(x,r) \geq \tfrac{1}{2^{1/d}} C_d^{-2/d} r.$$

Proof. We just need to prove that $\Delta(x,r) \backslash \Delta(x, C_d^{-2/d} r / 2^{1/d}) \neq \varnothing$, and this will be true provided that $\sigma\big(\Delta(x,r)\backslash\Delta(x,C_d^{-2/d}r/2^{1/d})\big) > 0$. Observe that

$$\begin{aligned}\sigma\big(\Delta(x,r)\backslash\Delta(x,C_d^{-2/d}r/2^{1/d})\big) &= \sigma(\Delta(x,r)) - \sigma(\Delta(x,C_d^{-2/d}r/2^{1/d}))\\ &\geq C_d^{-1}r^d - C_d\big(C_d^{-2/d}r/2^{1/d}\big)^d = \big[1-\tfrac{1}{2}\big]C_d^{-1}r^d > 0,\end{aligned}$$

as desired. □

If $X \in \Omega$, then

$$\delta(X) := \operatorname{dist}(X,\Gamma). \tag{2.2.5}$$

We note in passing that since Γ is closed, for each $X \in \Omega$ there exists $x \in \Gamma$ such that $|X - x| = \delta(X)$. It will be useful to denote

$$w(X) := \delta(X)^{-n+d+1}, \tag{2.2.6}$$

and let m be the Borel measure on Ω given by $m(E) := \int_E w(X)\,dX$, $E \subseteq \Omega$.

Next, we want to be able to use the openness of Ω in a quantitative way. The framework that we use is the following definition.

Definition 2.2.7 (Corkscrew points)**.** Fix $x \in \Gamma$ and $r > 0$. Then a point $X \in \Omega$ is called a *Corkscrew point* (with *Corkscrew constant* c) for the surface ball $\Delta(x,r) \subset \Gamma$ if there exists $c > 0$ such that

$$B(X,cr) \subset B(x,r) \cap \Omega.$$

For domains with a boundary of codimension less than or equal to 1, it is not true in general that every surface ball has a Corkscrew point. The situation for d-ADR sets with $d \in (0, n-1)$ is different.

Lemma 2.2.8 (Existence of Corkscrews; Lemma 11.6 of [DFM19b])**.** *Suppose that* $d < n-1$*. Then there exists* $c \in (0,1)$*, depending only on* n*,* d*, and* C_d*, such that every surface ball in* Γ *has a Corkscrew point with Corkscrew constant* c*.*

Furthermore, domains with high co-dimensional d-ADR boundaries enjoy quantitative connectedness as well, which we now describe precisely.

Definition 2.2.9 (Harnack Chain property)**.** Let $U \subset \mathbb{R}^n$ be an open set. We say that U has the *Harnack Chain property* if there exists a uniform constant $\tilde{C}$ such that for any $s > 0$,

$\Lambda \geq 1$, and any two points $X_1, X_2 \in U$ with $\operatorname{dist}(X_i, \partial U) \geq s$ and $|X_1 - X_2| \leq \Lambda s$, there exists a chain of balls $\{B_j\}_{j=1}^N \subset U$ verifying the following properties.

(i) $X_1 \in B_1$ and $X_2 \in B_N$.

(ii) $B_j \cap B_{j+1} \neq \varnothing$ for each $j = 1, \ldots, N-1$.

(iii) $N \leq C(\Lambda)$.

(iv) $\tilde{C}^{-1} \operatorname{diam} B_j \leq \operatorname{dist}(B_j, \partial U) \leq \tilde{C} \operatorname{diam} B_j$ for each $j = 1, \ldots, N$.

Such a chain of balls $\{B_j\}_j$ is called a *Harnack Chain*.

When $U = \Omega = \mathbb{R}^n \backslash \Gamma$ and Γ is $d-$ADR, we always have Harnack Chains; in fact, the Harnack Chains for Ω are thick tubes.

Lemma 2.2.10 (Existence of Harnack Chains; Lemma 2.1 of [DFM19b])**.** *Suppose that* $d < n-1$*. Then there exists a constant* $c_{\mathcal{H}} \in (0,1)$*, that depends only on* d, n, C_d*, such that for* $\Lambda \geq 1$ *and* $X_1, X_2 \in \Omega$ *with* $\delta(X_i) \geq s$ *and* $|X_1 - X_2| \leq \Lambda s$*, we can find two points* $Y_i \in B(X_i, s/2)$ *verifying that*

$$\operatorname{dist}([Y_1, Y_2], \Gamma) \geq (c_{\mathcal{H}} \Lambda^{-\frac{d}{n-1-d}}) s,$$

where $[Y_1, Y_2]$ *is the straight line segment in* $\mathbb{R}^n$ *with endpoints* Y_1 *and* Y_2*. That is, there is a thick tube in* Ω *that connects the balls* $B(X_i, s/2)$*.*

Let $\{B_j\}_{j=1}^N$ *be a finite collection of balls of equal radii whose centers lie in the line segment* $[Y_1, Y_2]$*, whose centers are spaced out by the radii (so that they cover* $[Y_1, Y_2]$*), and such that* $N \leq C(\Lambda)$*. We call*

$$\mathcal{H} := B(X_1, s/2) \bigcup_{j=1}^N B_j \bigcup B(X_2, s/2)$$

a well-tempered Harnack Chain connecting X_1 and X_2.

Remark 2.2.11. As was shown in [DFM19b] and mentioned above, the closed unbounded d-ADR set Γ that we consider has ample interior Corkscrew points and Harnack Chains. Therefore, our boundary Γ is *axiomatically* very similar to the boundaries of the so-called 1-*sided chord-arc domains*, which are open sets $\tilde{\Omega} \subset \mathbb{R}^n$ whose boundary $\partial\tilde{\Omega}$ is $(n-1)$-Ahlfors-David regular, and having interior Corkscrews and Harnack Chains. For this reason, as we shall see in the rest of this chapter, many of the results here have direct analogues in the setting of 1-sided chord-arc domains that was explored in the seminal paper [HM14], and often with very similar proofs, that we decide to omit in some cases, and in some other occasions, we decide to give different proofs of the expected results.

A key difference for us is that the sawtooth domains (to be defined further below) will have boundaries of mixed dimension, whence the usual global ADR notion is meaningless for them. Another issue is that we cannot rely on comparison principles for domains with different boundaries than Γ, because the coefficients of the operator explicitly depend on the distance to the boundary; instead we are restricted to work with "global" comparison principles. Still, we are able to overcome these issues. Hence we supply very careful and detailed proofs of our results leading up to the analogue of the Dahlberg-Jerison-Kenig sawtooth-projection lemma.

We will need to study non-negative doubling Borel measures on Γ.

Definition 2.2.12 (Doubling measures)**.** Fix a surface ball $\Delta(x,r) \subseteq \Gamma$, $r \in (0,+\infty]$ (by convention, $\Delta(x,+\infty) = \Gamma$). We say that a non-negative Borel measure μ on Δ is *doubling* on Δ if there exists a constant M large enough such that for each surface ball Δ' with $2\Delta' \subset \Delta$, we have that

$$\mu(2\Delta') \leq C_{\text{doubling}}\mu(\Delta').$$

The following definition gives a quantitative version of absolute continuity between measures.

Definition 2.2.13 (A_∞ measures)**.** Given a doubling non-negative Borel measure ν on Γ, and a fixed surface ball $\Delta \subseteq \Gamma$, we say that the doubling measure μ is of class $A_\infty(\nu,\Delta)$ if for each $\varepsilon > 0$, there exists a number $\xi = \xi(\varepsilon) > 0$ such that for every surface ball $\Delta' \subseteq \Delta$, and every Borel set $E \subset \Delta'$, we have that

$$\frac{\nu(E)}{\nu(\Delta')} < \xi \implies \frac{\mu(E)}{\mu(\Delta')} < \varepsilon. \tag{2.2.14}$$

After reviewing the elliptic PDE theory for the sets we are studying, we will need to study a more precise quantification of absolute continuity than just membership to A_∞. Given a doubling Borel measure μ on Γ, a *weight* $\mathfrak{w}$ on Γ is a non-negative $L^1_{\text{loc}}(\Gamma,\mu)$ function. A weight induces a non-negative Borel measure as follows: for any $\mu-$measurable set $E \subset \Gamma$ we write $\mathfrak{w}(E) := \int_E \mathfrak{w}\, d\mu$.

Definition 2.2.15 (The Reverse Hölder class RH_p)**.** Given a non-negative doubling Borel measure μ on Γ, a fixed surface ball $\Delta_0 \subset \Gamma$, a weight $\mathfrak{w} \in L^1_{\text{loc}}(\Delta_0, \mu)$, and $p \in (1, \infty)$, we say that $\mathfrak{w} \in RH_p(\mu, \Delta_0)$ if there exists a constant C_p such that for every surface ball $\Delta \subset \Delta_0$,

$$\Big(\frac{1}{\mu(\Delta)} \int_\Delta \mathfrak{w}^p \, d\mu\Big)^{\frac{1}{p}} \leq C_p \frac{1}{\mu(\Delta)} \int_\Delta \mathfrak{w} \, d\mu. \tag{2.2.16}$$

We denote by the $RH_p(\mu, \Delta_0)$ *characteristic of* $\mathfrak{w}$ the smallest number C_p such that (2.2.16) holds for all $\Delta \subset \Delta_0$. When μ is the surface measure σ, we simply write $RH_p(\sigma, \Delta_0) = RH_p(\Delta_0)$. Furthermore, if ν, μ are two doubling non-negative Borel measures on Γ, $\Delta_0 \subset \Gamma$, and $p \in (1, \infty)$, we say that $\mu \in RH_p(\nu, \Delta_0)$ if $\mu \ll \nu$ and the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ lies in $RH_p(\nu, \Delta)$.

Let us record some properties of the A_∞ class. For our setting, the following results have appeared in [Jaw86] and [ST89].

Theorem 2.2.17 (Properties of A_∞ measures; Theorem 1.4.13 of [Ken94]; [ST89])**.** *Let μ, ν be doubling non-negative Borel measures on Γ, and let Δ be a surface ball. The following statements hold.*

(i) If $\mu \in A_\infty(\nu, \Delta)$, then μ is absolutely continuous with respect to ν on Δ.

(ii) The class A_∞ is an equivalence relation.

(iii) We have that $\mu \in A_\infty(\nu, \Delta)$ if and only if there exist uniform constants $C > 0, \theta > 0$, such that for each surface ball $\Delta' \subseteq \Delta$ and each Borel set $E \subseteq \Delta'$, we have that

$$\frac{\mu(E)}{\mu(\Delta')} \lesssim \Big(\frac{\nu(E)}{\nu(\Delta')}\Big)^\theta.$$

(iv) We have that $\mu \in A_\infty(\nu, \Delta)$ if and only if there exist $\varepsilon \in (0,1), \delta \in (0,1)$ so that (2.2.14) holds for all surface balls $\Delta' \subset \Delta$ and all Borel sets E (see [GR85a]).

(v) The characterization $A_\infty(\nu) = \bigcup_{p>1} RH_p(\nu)$ holds.

(vi) We have that $\mu \in RH_p(\nu)$ if and only if the uncentered Hardy-Littlewood maximal function adapted to Γ,

$$(M_\mu f)(x) := \sup_{\Delta \ni x} \frac{1}{\mu(\Delta)} \int_\Delta |f| \, d\mu, \tag{2.2.18}$$

verifies the estimate $\|M_\mu f\|_{L^{p'}(d\nu)} \lesssim \|f\|_{L^{p'}(d\nu)}$, where p' is the Hölder conjugate of p, so that $\frac{1}{p} + \frac{1}{p'} = 1$.

To conclude this minimal set-up for d-ADR sets, we give a meaning to the non-tangential maximal functions and square functions, which are essential concepts in theory of the Dirichlet problems with rough data.

Definition 2.2.19 (Non-tangential maximal function and square function)**.** For any $x \in \Gamma$ and $\alpha > 0$, we define the *non-tangential cone* $\gamma^\alpha(x)$ with vertex x and aperture α as

$$\gamma^\alpha(x) = \Big\{ X \in \Omega \,:\, |X - x| < (1+\alpha)\delta(X) \Big\},$$

We often omit the superscript α. Define the *square function* as

$$Su(x) = \Big(\iint_{\gamma(x)} |\nabla u(X)|^2 \delta(X)^{2-n} \, dX \Big)^{\frac{1}{2}}.$$

Finally, the *non-tangential maximal function* is given by $Nu(x) = \sup\limits_{X \in \gamma(x)} |u(X)|$. Given a measurable function f on Γ, we say that $u \to f$ *non-tangentially* if for $\sigma-$almost every $x \in \Gamma$, we have that $\lim\limits_{\gamma(x) \ni X \to x} u(X) = f(x)$.

2.3 Dyadic decomposition of sets of high co-dimension

In the following lemma, we exhibit a family of partitions for Γ which are analogous to dyadic cubes. The original construction of such a dyadic grid for d-ADR sets with $d = n-1$ is found in [Dav88]; in the book [Dav91] there is a simpler proof which adapts to our setting. See also [Chr90] for a different proof in the even more general case of spaces of homogeneous types.

Lemma 2.3.1 (Dyadic cubes for $d-$Ahlfors-David regular set)**.** [Dav91]. *There exist constants $a_0 \in (0,1], A_0 \in [1,\infty), \zeta \in (0,1)$, depending only on d, n, and the d-ADR constant C_d, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets* ("dyadic cubes")

$$\mathbb{D}^k = \mathbb{D}^k(\Gamma) := \{ Q_j^k \subset \Gamma \,:\, j \in \mathscr{J}^k \},$$

where $\mathscr{J}^k$ denotes some indexing set depending on k, satisfying the following properties.

(i) For each $k \in \mathbb{Z}$, $\Gamma = \bigcup_{j \in \mathscr{J}^k} Q_j^k$.

(ii) If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \varnothing$.

(iii) *For each pair (j, k) and each $m < k$, there is a unique $i \in \mathscr{J}^m$ such that $Q_j^k \subset Q_i^m$. When $m = k-1$, we call Q_i^m the* dyadic parent *of Q_j^k, and we say that Q_j^k is a* dyadic child *of Q_i^m.*

(iv) *diam $Q_j^k < A_0 2^{-k}$.*

(v) *Each Q_j^k contains some surface ball $\Delta(x_j^k, a_0 2^{-k}) = B(x_j^k, a_0 2^{-k}) \cap \Gamma$.*

(vi) *$\mathscr{H}^d(\{x \in Q_j^k : dist(x, \Gamma \backslash Q_j^k) \leq \rho 2^{-k}\}) \leq A_0 \rho^\zeta \mathscr{H}^d(Q_j^k)$, for all (j, k) and all $\rho \in (0, a_0)$.*

Let us define some notions and state some useful properties of this construction.

• We shall denote by $\mathbb{D} = \mathbb{D}(\Gamma)$ the collection of all relevant Q_j^k; that is,

$$\mathbb{D} = \mathbb{D}(\Gamma) := \bigcup_{k \in \mathbb{Z}} \mathbb{D}^k(\Gamma). \tag{2.3.2}$$

Henceforth, we refer to the elements of $\mathbb{D}$ as *dyadic cubes*, or *cubes*. For $Q \in \mathbb{D}$, we write $\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subseteq Q\}$, and $\mathbb{D}_Q^k = \mathbb{D}^k(\Gamma) \cap \mathbb{D}_Q$.

• Note carefully that if Q_i^{k+1} is the dyadic parent of Q_j^k, then it is possible that, *as sets*, $Q_i^{k+1} = Q_j^k$. In other words, if $Q \in \mathbb{D}$, then the set $\mathbb{K}(Q) := \{k \in \mathbb{Z} : Q = Q_j^k$ for some $j\}$ may in general have cardinality greater than or equal to 1. We call $\mathbb{K}(Q)$ the *generational bandwith* of Q. By Lemma 2.2.4 and properties (iv) and (v) above, we have that if $k \in \mathbb{K}(Q)$, then

$$2^{-1/d} C_d^{-2/d} a_0 2^{-k} \leq \operatorname{diam} Q \leq A_0 2^{-k}, \tag{2.3.3}$$

which implies that $\mathbb{K}(Q)$ is finite, and in fact,

$$1 \leq \operatorname{card}(\mathbb{K}(Q)) \leq \log_2 \left[2^{\frac{d+1}{d}} C_d^{2/d} A_0 a_0^{-1}\right]. \tag{2.3.4}$$

Define the *dyadic generation* of $Q \in \mathbb{D}$ as the oldest generation that Q belongs to; that is,

$$k(Q) = \min_{k \in \mathbb{K}(Q)} k,$$

and note that, if $k \in \mathbb{K}(Q)$, then $k(Q) \leq k \leq k(Q) + \log_2 \left[2^{\frac{d+1}{d}} C_d^{2/d} A_0 a_0^{-1}\right]$. We call the number $\ell(Q) = 2^{-k(Q)}$ the *length* of Q. Given a fixed $Q \in \mathbb{D}$, we call a cube $Q' \in \mathbb{D}_Q \backslash \{Q\}$ a *proper child* of Q if $\ell(Q') < \ell(Q)$ and $\ell(Q') \geq \ell(Q'')$ for any other

$Q'' \in \mathbb{D}_Q$. Likewise, given $Q \in \mathbb{D}$, we call a cube $\hat{Q} \in \mathbb{D}$ with $\hat{Q} \supset Q$ a *proper parent* of Q if $\ell(\hat{Q}) > \ell(Q)$ and $\hat{Q} \subseteq \tilde{Q}$ for any $\tilde{Q} \in \mathbb{D}$ with $\tilde{Q} \supsetneq Q$. If $Q' \in \mathbb{D}_Q$ is a proper child of Q, then we have that

$$\ell(Q) > \ell(Q') \geq \frac{a_0}{2^{\frac{2d+1}{d}} C_d^{2/d} A_0} \ell(Q) =: c_{\mathbb{K}} \ell(Q). \tag{2.3.5}$$

If Q' is a proper child of Q, then by the partitioning property of the dyadic cubes we must have that there exists a collection $\{Q''\}$ of proper children of Q such that $\cup Q'' = Q$. In the sequel, if we say that Q' is a *child* of Q, we mean that Q' is a dyadic child of Q, leaving open the possibility that $Q' = Q$ as sets.

• (Almost inscription and subscription of surface balls). Properties (iv) and (v) also imply that for each cube $Q \in \mathbb{D}$, there is a point $x_Q \in \Gamma$ such that

$$\Delta(x_Q, a_0 \ell(Q)) \subset Q \subseteq \Delta(x_Q, A_0 \ell(Q)). \tag{2.3.6}$$

We call x_Q the *center* of Q. We note that (2.3.6) and (2.2.2) imply the following estimate on the surface measure of Q:

$$C_d^{-1} a_0^d \ell(Q)^d \leq \sigma(Q) \leq C_d A_0^d \ell(Q)^d. \tag{2.3.7}$$

• (Number of children of Q). Fix $Q \in \mathbb{D}$ and let $\{Q_j\}_{j \in J}$ be the collection of all (dyadic) children of Q. It must be the case by property (i) that $Q = \cup_{j \in J} Q_j$. Observe the elementary estimate

$$\sigma(Q) = \sigma(\cup_{j \in J} Q_j) = \sum_{j \in J} \sigma(Q_j) \geq C_d^{-1} a_0^d \frac{\ell(Q)}{2} \operatorname{card} J,$$

where we used (2.3.7) in the last inequality. Putting the previous estimate together with the upper bound in (2.3.7) gives that

$$\operatorname{card}(\{\text{children of } Q\}) \leq C_d^2 [a_0^{-1} A_0 2]^d. \tag{2.3.8}$$

• (Corkscrew points for Q). We denote by X_Q a point in Ω which is a Corkscrew point (with Corkscrew constant $\tilde{c} > 0$) for the surface ball $\Delta(x_Q, a_0 \ell(Q))$. Such a point is called a *Corkscrew point for Q* (with Corkscrew constant $\tilde{c} > 0$).

• The inequality in (vi) says that the boundary of a dyadic cube Q_j^k is uniformly thin; indeed, one may easily deduce from it that $\mathscr{H}^d(\partial Q) = 0$ for any $Q \in \mathbb{D}$.

• By $\mathcal{F} = \{Q_j\}_j$ we denote a family of pairwise disjoint dyadic cubes in $\mathbb{D}$, which we identify *as subsets of* Γ and not as elements of $\cup_k \mathbb{D}^k$. Accordingly, if $Q_j \in \mathcal{F}$, then its parent $\hat{Q} \in \mathbb{D}$ does not belong to $\mathcal{F}$, and we have that $\ell(Q_j) < \ell(\hat{Q}) \leq c_{\mathbb{K}}^{-1} \ell(Q_j)$. We refer to such a collection $\mathcal{F}$ as a *disjoint family*.

• We define the projection operator $\mathcal{P}_{\mathcal{F}} : L^1_{\text{loc}}(\Gamma, \sigma) \to L^1_{\text{loc}}(\Gamma, \sigma)$ by

$$(\mathcal{P}_{\mathcal{F}} f)(x) := f(x) \mathbf{1}_{\Gamma \setminus (\cup_j Q_j)}(x) + \sum_j \Big(\frac{1}{\sigma(Q_j)} \int_{Q_j} f \, d\sigma \Big) \mathbf{1}_{Q_j}(x), \qquad x \in \Gamma. \quad (2.3.9)$$

One has that $\mathcal{P}_{\mathcal{F}} \circ \mathcal{P}_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}$, $\mathcal{P}_{\mathcal{F}}$ is self-adjoint, and $\|\mathcal{P}_{\mathcal{F}} f\|_{L^p(\Gamma,\sigma)} \leq \|f\|_{L^p(\Gamma,\sigma)}$ for every $p \in [1, \infty]$. Observe that if μ is a non-negative finite Borel measure on Γ and $E \subseteq \Gamma$ is a Borel set, then we may naturally define the measure $\mathcal{P}_{\mathcal{F}} \mu$ as follows:

$$\mathcal{P}_{\mathcal{F}} \mu(E) := \int_\Gamma \mathcal{P}_{\mathcal{F}}(\mathbf{1}_E) \, d\mu = \mu(E \setminus \cup_j Q_j) + \sum_j \frac{\sigma(E \cap Q_j)}{\sigma(Q_j)} \mu(Q_j). \qquad (2.3.10)$$

In particular, $\mathcal{P}_{\mathcal{F}} \mu(\Gamma) = \mu(\Gamma)$. Notice that $\mathcal{P}_{\mathcal{F}} \mu$ is defined in such a way that it coincides with μ in $\Gamma \setminus (\cup_j Q_j)$ and in each Q_j we replace μ by $\mu(Q_j)/\sigma(Q_j) \, d\sigma$.

2.3.1 The theory of quantitative absolute continuity adapted to the dyadic grid

We will need to make sense of the doubling property adapted to our dyadic grids.

Definition 2.3.11 (Dyadically doubling measures)**.** We say that a Borel measure μ on $Q_0 \in \mathbb{D}$ is *dyadically doubling* in Q_0 if $0 < \mu(Q) < \infty$ for every $Q \in \mathbb{D}_{Q_0}$ and there exists a constant $C_\mu \geq 1$ such that $\mu(Q) \leq C_\mu \mu(Q') < \infty$ for every $Q \in \mathbb{D}_{Q_0}$ and for every dyadic child Q' of Q.

Lemma 2.3.12 (Doubling implies dyadically doubling)**.** *Fix $Q_0 \in \mathbb{D}$ and suppose that μ is a doubling Borel measure on the surface ball $\Delta_0 := \Delta(X_{Q_0}, 2A_0 \ell(Q_0))$. Then μ is dyadically doubling in Q_0.*

The following lemma gives us that if a measure is dyadically doubling, then so is its projection. We omit its easy proof as it is essentially the same as that in [HM14] (see Remark 2.2.11).

Lemma 2.3.13 (Lemma B.1 of [HM14]). *Fix $Q_0 \in \mathbb{D}$, let $\mathcal{F} \subset Q_0$ be a disjoint family, and let μ be a dyadically doubling measure in Q_0. Then $\mathcal{P}_{\mathcal{F}}\mu$ is dyadically doubling in Q_0.*

Now we define quantitative absolute continuity on our dyadic grid.

Definition 2.3.14 (A_∞^{dyadic} and RH_p^{dyadic}). Fix $Q_0 \in \mathbb{D}$, and let μ, ν be two dyadically doubling measures on Q_0. We say that $\mu \in A_\infty^{\text{dyadic}}(\nu, Q_0)$ if there exist constants $0 < \alpha, \beta < 1$ such that for every $Q \in \mathbb{D}_{Q_0}$ and for every Borel set $F \subset Q$, we have that

$$\frac{\nu(F)}{\nu(Q)} > \alpha \quad \Longrightarrow \quad \frac{\mu(F)}{\mu(Q)} > \beta.$$

Given $p \in (1, \infty)$ and μ, ν as above, we say that $\mu \in RH_p^{\text{dyadic}}(\nu, Q_0)$ if and only if $\mu \ll \nu$ in Q_0 and there exists a constant $C_p \geq 1$ such that for every $Q \in \mathbb{D}_{Q_0}$, we have the estimate

$$\Big(\frac{1}{\nu(Q)} \int_Q \big(\frac{d\mu}{d\nu}\big)^p \, d\nu\Big)^{\frac{1}{p}} \leq C_p \frac{1}{\nu(Q)} \int_Q \frac{d\mu}{d\nu} \, d\nu.$$

Analogously to the continuous setting of Theorem 2.2.17 (v) (and essentially by the same arguments; see the proof of Lemma B.7 of [HM14]), we have the characterization

$$A_\infty^{\text{dyadic}}(\nu, Q_0) = \textstyle\bigcup_{p>1} RH_p^{\text{dyadic}}(\nu, Q_0).$$

We write $A_\infty^{\text{dyadic}}(Q_0) = A_\infty^{\text{dyadic}}(\sigma, Q_0)$ and $RH_p^{\text{dyadic}}(Q_0) = RH_p^{\text{dyadic}}(\sigma, Q_0)$.

The next result gives that the A_∞^{dyadic} property is passed on from a measure to its projection. Its proof is essentially the same as that of Lemma 4.1 in [HM10] (see Remark 2.2.11); and so we omit the details.

Lemma 2.3.15 (Lemma 4.1 of [HM10]). *Fix $Q_0 \in \mathbb{D}$, let $\mathcal{F} \subset Q_0$ be a disjoint family, and suppose that $\mu \in A_\infty^{\text{dyadic}}(Q_0)$. Then $\mathcal{P}_{\mathcal{F}}\mu \in A_\infty^{\text{dyadic}}(Q_0)$.*

As expected, we have the symmetry of the A_∞^{dyadic} class. It is proven in [HM14] in a similar setting (see Remark 2.2.11), but their proof generalizes to our situation immediately.

Lemma 2.3.16 (Symmetry of A_∞^{dyadic}, Lemma B.7 and Remark B.10 of [HM14])**.** *Let $Q_0 \in \mathbb{D}$ and let μ, ν be two dyadically doubling measures on Q_0. Assume that there exist positive constants C_0, θ_0, such that for all $Q \in \mathbb{D}_{Q_0}$ and all Borel sets $F \subseteq Q$,*

$$\frac{\nu(F)}{\nu(Q)} \leq C_0\Big(\frac{\mu(F)}{\mu(Q)}\Big)^{\theta_0}. \tag{2.3.17}$$

Then, there exist $C_1, \theta_1 > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and all Borel sets $F \subseteq Q$,

$$\frac{\mu(F)}{\mu(Q)} \leq C_1\Big(\frac{\nu(F)}{\nu(Q)}\Big)^{\theta_1}.$$

Furthermore, $\mu \in A_\infty^{\text{dyadic}}(\nu, Q_0)$ if and only if (2.3.17) holds for some C_0, θ_0 and all $Q \in \mathbb{D}_{Q_0}$, $F \subset Q$.

To finalize this section, we present a generalization of Lemma B.7 in [HM12], which allows us to conclude that a measure is A_∞^{dyadic} if it satisfies a certain local Reverse Hölder inequality. The result has been shown in a setting very similar to ours in Lemma 3.1 of [CHM19] (see Remark 2.2.11); the proof is essentially the same and so we omit it.

Lemma 2.3.18 (Local RH implies A_∞^{dyadic}, Lemma 3.1 of [CHM19])**.** *Fix $Q_0 \in \mathbb{D}$ and $\varepsilon \in (0,1)$. Let $v \in L^1(Q_0)$ be a function such that there exists $C_0 \geq 1$ verifying that $0 < v(Q) \leq C_0 v(\varepsilon\Delta_Q)$ for every $Q \in \mathbb{D}_{Q_0}$, where $\Delta_Q := \Delta(x_Q, a_0\ell(Q))$. Assume also that there exist $C_1 \geq 1$ and $p \in (1,\infty)$ such that*

$$\Big(\frac{1}{\sigma(\varepsilon\Delta_Q)}\int_{\varepsilon\Delta_Q} v^p\, d\sigma\Big)^{\frac{1}{p}} \leq C_1 \frac{1}{\sigma(\varepsilon\Delta_Q)}\int_{\varepsilon\Delta_Q} v\, d\sigma, \qquad \textit{for each } Q \in \mathbb{D}_{Q_0}.$$

Then $v \in RH_p^{\text{dyadic}}(Q_0)$ (hence, $\nu \in A_\infty^{\text{dyadic}}(Q_0)$) with RH_p characteristic depending only on n, d, C_d, p, ε, C_0, C_1.

2.4 Sawtooth domains

Given a domain Ω, the so-called sawtooth domains $\Omega_{\mathcal{F}} \subset \Omega$ can (morally) be thought of as 1-sided NTA domains which hide certain parts $\mathcal{F}$ of $\partial\Omega$ in the exterior of $\Omega_{\mathcal{F}}$. These domains will be decisively used in step 2 of the proof of Theorem 2.1.1 in Section 2.9.4 to pass from certain interior estimates to boundary estimates via a Dahlberg-Jerison-Kenig projection lemma (Lemma 2.8.1). The use of this machinery requires that we have a robust elliptic PDE theory on our sawtooth domains. We will see that our dyadically-generated sawtooth domains will be mixed-dimensional, so that an elliptic PDE theory for them is highly non-trivial. As such, we will give a careful construction with the goal to prove in the following section, Section 2.5, that our dyadic sawtooth domains satisfy the mixed-dimension theory of [DFM]. Indeed, while in [DFM] it is shown that certain sawtooth domains over Lipschitz graphs satisfy their axioms, our dyadic sawtooth domains over arbitrary d-ADR sets (with possibly fractional dimension) were not considered, and the verification is considerably more subtle.

2.4.1 Construction of sawtooth domains

In this subsection, we construct the sawtooth domains for d-ADR sets with $d < n-1$. The abstract construction here was first considered for 1-sided NTA domains with $(n-1)$-ADR boundaries in [HM14], and developed for the setting $d < n-1$ in [MZ19]. If $d < n-1$, we have no further assumptions, since the $d-$ADR property of Γ gives the existence of Corkscrew points and Harnack Chains.

We begin the construction. Since $\Omega = \mathbb{R}^n \backslash \Gamma$ is an open set in $\mathbb{R}^n$, there exists a collection of closed dyadic *Whitney boxes*, denoted by $\mathcal{W} = \mathcal{W}(\Omega)$, so that the interiors of the boxes never overlap pairwise, the boxes form a covering of Ω, and moreover they satisfy the conditions

$$4\text{diam } I \leq \text{dist}(4I, \Gamma) \leq \text{dist}(I, \Gamma) \leq 40\text{diam } I, \qquad \text{for each } I \in \mathcal{W}, \tag{2.4.1}$$

and

$$\frac{1}{4}\text{diam } I_1 \leq \text{diam } I_2 \leq 4\text{diam } I_1$$

whenever $\partial I_1 \cap \partial I_2 \neq \varnothing$ (see, for instance, [Ste70a]).

Notation 2.4.2. Let X_I denote the *center* of I and $\ell(I)$ the *side-length* of I, so that diam $I = \sqrt{n}\ell(I)$. We also write $k(I) = k$ if $\ell(I) = 2^{-k}$. We say that two Whitney boxes *touch*, or that they are *adjacent*, if their boundaries intersect. If $X \in I$ and $I \in \mathcal{W}$, then $4 \,\text{diam}\, I \leq \delta(X) \leq 41 \,\text{diam}\, I$.

We want to associate to each $Q \in \mathbb{D}$ a "Whitney region" in Ω, which we will construct by taking a union of certain dilated Whitney boxes. Hence, we ought to understand which

Whitney boxes should be part of a Whitney region associated to the cube Q. The main two properties we desire to embed in such a region are, first, that it houses the Corkscrew points for Q, and second, that these Corkscrew points are joined together by Harnack Chains that remain within the region. We will also want to fit in parameters that allow us to control the non-tangential aperture of these regions. In preparation to define these regions, we supply the technically relevant results.

$\mathcal{W}_Q^{\mathrm{cs}}$: The initial core of the Whitney region

First, we will make sure that a Whitney region houses Corkscrew points (see Definition 2.2.7).

Lemma 2.4.3 (Whitney boxes contain Corkscrew points)**.** *Fix $Q \in \mathbb{D}$ and suppose that $X \in \Omega$ is a Corkscrew point with Corkscrew constant $c > 0$ for the surface ball $\Delta(x_Q, a_0\ell(Q)/2)$. Then there exists $I \in \mathcal{W}$ such that $X \in I$ and satisfying*

$$\frac{a_0 c}{82\sqrt{n}}\ell(Q) \le \ell(I) \le \frac{a_0}{8\sqrt{n}}\ell(Q), \qquad \operatorname{dist}(I, Q) \le a_0\ell(Q)/2. \tag{2.4.4}$$

Proof. Since $\mathcal{W}$ is a covering of Ω, it follows that there exists $I \in \mathcal{W}$ such that $X \in I$. We now prove the bounds in (2.4.4). The upper estimate for $\ell(I)$ in (2.4.4) is deduced from the following chain of inequalities:

$$4\sqrt{n}\ell(I) = 4\operatorname{diam} I \le \operatorname{dist}(I, \Gamma) \le \operatorname{dist}(I, x_Q) \le |X - x_Q| < a_0\ell(Q)/2,$$

where we have used (2.4.1), $x_Q \in \Gamma$, $X \in I$, and that $X \in B(x_Q, a_0\ell(Q)/2)$. By observing that $\operatorname{dist}(I, Q) \le \operatorname{dist}(I, x_Q)$, we arrive at the desired estimate for $\operatorname{dist}(I, Q)$. It remains now to give the lower bound for $\ell(I)$. Since $B(X, ca_0\ell(Q)/2) \subset \Omega$, we have that $\operatorname{dist}(X, \Gamma) \ge ca_0\ell(Q)/2$. Note that

$$40\sqrt{n}\ell(I) = 40\operatorname{diam} I \ge \operatorname{dist}(I, \Gamma) \ge \operatorname{dist}(X, \Gamma) - \operatorname{diam} I \ge ca_0\ell(Q)/2 - \sqrt{n}\ell(I),$$

whence the desired result follows. $\square$

Thus, for each cube $Q \in \mathbb{D}$ the collection

$$\mathcal{W}_Q^{\mathrm{cs}} := \Big\{ I \in \mathcal{W} \,:\, \frac{a_0 c}{82\sqrt{n}}\ell(Q) \le \ell(I) \le \frac{a_0}{8\sqrt{n}}\ell(Q), \quad \operatorname{dist}(I,Q) \le a_0\ell(Q)/2 \Big\} \tag{2.4.5}$$

contains all the Corkscrew points for $\Delta(x_Q, a_0\ell(Q)/2)$ with Corkscrew constant c (which evidently are Corkscrew points of Q). We note that without loss of generality, we may assume that a Corkscrew point for Q is located at the center of some $I \in \mathcal{W}_Q^{\mathrm{cs}}$ (with possibly smaller Corkscrew constant), as gives us the following result.

Lemma 2.4.6 (Corkscrew points lie at the centers of Whitney boxes)**.** *Fix $Q \in \mathbb{D}$ and X a Corkscrew point for the surface ball $\Delta(x_Q, a_0\ell(Q)/2)$ with Corkscrew constant $c > 0$. If $I \in \mathcal{W}_Q^{\mathrm{cs}}$ contains X, then X_I (see Notation 2.4.2) is a Corkscrew point for Q with Corkscrew constant $\tilde{c} = c/(1000\sqrt{n})$. Moreover, $B(X_I, \tilde{c}a_0\ell(Q)) \subset \operatorname{int}(\frac{1}{2}I)$.*

Proof. Suppose that $I \in \mathcal{W}_Q^{\mathrm{cs}}$ contains X. Then $|X - X_I| \le \operatorname{diam} I \le a_0\ell(Q)/4$, and

$$|X_I - x_Q| \le |X_I - X| + |X - x_Q| < a_0\ell(Q)/4 + a_0\ell(Q)/2 < a_0\ell(Q).$$

Hence $X_I \in B(x_Q, a_0\ell(Q))$. Now let $\tilde{c}$ be as in the statement of the lemma, and observe that for any $Y \in B(X_I, \tilde{c}a_0\ell(Q))$, we have that

$$|X_I - Y| \le \tilde{c}a_0\ell(Q) < \tfrac{a_0 c}{82\sqrt{n}}\tfrac{\ell(Q)}{8} < \ell(I)/8,$$

$$|Y - x_Q| \le \operatorname{diam} I + |X - x_Q| < a_0\ell(Q),$$

as desired. ☐

Corollary 2.4.7. *For any $Q \in \mathbb{D}$, there exists $I \in \mathcal{W}_Q^{\mathrm{cs}}$ such that X_I is a Corkscrew point for Q with Corkscrew constant $\tilde{c} = \tilde{c}(c, n)$.*

$\mathcal{W}_Q^0$: The fattened core

It may happen that $\mathcal{W}_Q^{\mathrm{cs}}$ is too meager a region to use it to pass to "continuous" sawtooth-domains, or to pass between "adjacent" Whitney regions. We introduce parameters

$\eta \in (0,1)$ and $K \geq 1$ and define

$$\mathcal{W}_Q^0 := \left\{ I \in \mathcal{W} \, : \, \frac{a_0 c}{82\sqrt{n}} \eta \ell(Q) \leq \ell(I) \leq \frac{a_0}{4\sqrt{n}} K \ell(Q), \quad \text{dist}(I, Q) \leq a_0 K \ell(Q) \right\} \tag{2.4.8}$$

so that we may enlargen $\mathcal{W}_Q^0$ according to aperture considerations. Immediately we have the following two technical results.

Lemma 2.4.9 (Transversal adjacency of Whitney regions)**.** *If $\eta \in (0, c_{\mathbb{K}})$, then for any $Q \in \mathbb{D}$, the Whitney region $\mathcal{W}_Q^0$ contains all Corkscrew points of the proper children of Q (with Corkscrew constant c).*

Proof. Upon using (2.3.5), the proof is very similar to that of (2.4.4), and thus we omit it. □

Lemma 2.4.10 (Parallel adjacency of Whitney regions)**.** *Fix $Q_1, Q_2 \in \mathbb{D}$ and suppose that $\ell(Q_1) \leq \ell(Q_2) \leq c_{\mathbb{K}}^{-1} \ell(Q_1)$, and $\text{dist}(Q_1, Q_2) \leq 500\ell(Q_2)$. If $\eta \in (0, c_{\mathbb{K}})$ and $K \geq 500 A_0 a_0^{-1}$, then $\mathcal{W}_{Q_1}^0 \cap \mathcal{W}_{Q_2}^0 \neq \varnothing$.*

Proof. Recall that $\mathcal{W}_{Q_1}^{\text{cs}} \neq \varnothing$, so fix $I \in \mathcal{W}_{Q_1}^{\text{cs}}$. It is easy to see that $\frac{a_0 c}{82\sqrt{n}} c_{\mathbb{K}} \ell(Q_2) \leq \ell(I) \leq \frac{a_0}{4\sqrt{n}} K \ell(Q_2)$, while the triangle inequality gives us that

$$\begin{aligned} \text{dist}(I, Q_2) &\leq \text{dist}(I, Q_1) + \text{diam}\, Q_1 + \text{dist}(Q_1, Q_2) \\ &\leq a_0 \ell(Q_1)/2 + A_0 \ell(Q_1) + 500\ell(Q_2) \leq 500 A_0 \ell(Q_2) \leq a_0 K \ell(Q_2). \end{aligned}$$

Thus I verifies the conditions to be an element of $\mathcal{W}_{Q_2}^0$. □

Henceforth, we assume that $\eta < c_{\mathbb{K}}$, $K \geq 500 A_0 a_0^{-1}$. We need to augment $\mathcal{W}_Q^0$ one final time: we must provide it with enough new boxes so that Harnack Chains (Definition 2.2.9) connecting its old boxes are contained within a region that stays far from Γ. When $d < n-1$, the technical result needed to accomplish this is

Lemma 2.4.11 (Harnack Chains of $\mathcal{W}_Q^0$ if $d < n-1$)**.** *Fix $Q \in \mathbb{D}$, X_Q a Corkscrew point of Q with Corkscrew constant $c > 0$, and $I \in \mathcal{W}_Q^0$ (note that I may or may not contain X_Q). Then we may construct a well-tempered Harnack Chain $\mathcal{H}_I$ connecting X_Q to X_I (see Lemma 2.2.10) consisting of a number at most $N_{\mathcal{H}}$ of balls, where*

$$N_{\mathcal{H}} = \tfrac{4}{c_{\mathcal{H}}} \left[\tfrac{41}{2c\eta} \left(1 + \tfrac{5}{4}K + \tfrac{A_0}{a_0}\right) \right]^{\frac{n-1}{n-1-d}},$$

and
$$\operatorname{dist}(\mathcal{H}_I,\Gamma)\geq \frac{1}{2}c_{\mathcal{H}}\Big[\tfrac{41}{2c}\big(1+\tfrac{5}{4}K+\tfrac{A_0}{a_0}\big)\Big]^{\frac{-d}{n-1-d}}\big(\tfrac{2}{41}a_0c\big)\eta^{\frac{n-1}{n-1-d}}\ell(Q).$$

Proof. Since $I\in\mathcal{W}^0_Q$, we have that
$$\delta(X_I)=\operatorname{dist}(X_I,\Gamma)\geq\operatorname{dist}(I,\Gamma)\geq 4\sqrt{n}\ell(I)\geq\frac{2}{41}a_0c\eta\ell(Q).$$

Furthermore, $\delta(X_Q)\geq cr_Q=a_0c\ell(Q)\geq\frac{2}{41}a_0c\ell(Q)$. Thus we take $s:=\frac{2}{41}a_0c\eta\ell(Q)$ in the setup of Lemma 2.2.10. Next, we estimate
$$\begin{aligned}|X_I-X_Q|&\leq\operatorname{diam}I+\operatorname{dist}(I,Q)+\operatorname{diam}Q+|X_Q-x_Q|\\&\leq\sqrt{n}\ell(I)+a_0K\ell(Q)+A_0\ell(Q)+a_0\ell(Q)\\&\leq\big((\tfrac{5}{4}K+1)a_0+A_0\big)\ell(Q)=\Big(\tfrac{41}{2c\eta}\big[1+\tfrac{5}{4}K+\tfrac{A_0}{a_0}\big]\Big)\Big(\tfrac{2}{41}a_0c\eta\ell(Q)\Big).\end{aligned}$$

Hence we take $\Lambda:=\frac{41}{2c\eta}\big[1+\frac{5}{4}K+\frac{A_0}{a_0}\big]$. We now invoke the conclusion of Lemma 2.2.10 to find two points $Y_I\in B(X_I,a_0c\eta\ell(Q)/41), Y_Q\in B(X_Q,a_0c\eta\ell(Q)/41)$ such that
$$\operatorname{dist}([Y_I,Y_Q],\Gamma)\geq c_{\mathcal{H}}\Big[\tfrac{41}{2c}\big(1+\tfrac{5}{4}K+\tfrac{A_0}{a_0}\big)\Big]^{\frac{-d}{n-1-d}}\big(\tfrac{2}{41}a_0c\big)\eta^{\frac{n-1}{n-1-d}}\ell(Q)=:a_1\eta^{\frac{n-1}{n-1-d}}\ell(Q).\tag{2.4.12}$$

Consider a finite covering of $[Y_I,Y_Q]$ by balls B_j with centers in $[Y_I,Y_Q]$, radii all equal to $\frac{a_1}{2}\eta^{\frac{n-1}{n-1-d}}\ell(Q)$, and centers spaced by the radii. It is clear that the union of the balls B_j is a well-tempered Harnack Chain $\mathcal{H}_I$ satisfying the second desired estimate. Let $N'_{\mathcal{H}}$ be the cardinality of the covering. Since
$$\frac{|Y_I-Y_Q|}{\frac{a_1}{2}\eta^{\frac{n-1}{n-1-d}}\ell(Q)}\leq\frac{|Y_I-X_I|+|X_I-X_Q|+|X_Q-Y_Q|}{\frac{a_1}{2}\eta^{\frac{n-1}{n-1-d}}\ell(Q)}\leq\frac{2\Lambda s}{\frac{c_{\mathcal{H}}}{2}\Lambda^{\frac{-d}{n-1-d}}s},$$
and $N'_{\mathcal{H}}a_1\eta^{\frac{n-1}{n-1-d}}\ell(Q)/2\leq|Y_I-Y_Q|+a_1\eta^{\frac{n-1}{n-1-d}}\ell(Q)/2$, the first desired estimate follows. □

Actually, Harnack Chains cannot stray too far from the boxes in $\mathcal{W}^0_Q$, as gives the following result.

Lemma 2.4.13 (Non-degeneracy of boxes in Harnack Chains of $\mathcal{W}^0_Q$)**.** *Suppose that*

$J \in \mathcal{W}$ intersects the Harnack Chain of Lemma 2.4.11. Then

$$\frac{a_1}{82\sqrt{n}}\eta^{\frac{n-1}{n-1-d}}\ell(Q) \leq \ell(J) \leq \frac{2a_0K + A_0}{4\sqrt{n}}\ell(Q), \qquad \operatorname{dist}(J,Q) \leq (2a_0K + A_0)\ell(Q), \tag{2.4.14}$$

where $a_1 = a_1(n, d, C_d, c, c_{\mathcal{H}}, K)$ is the quantity defined in (2.4.12).

Proof. Given that $J \cap \mathcal{H}_I \neq \varnothing$, then there exists $X \in J$ and $X \in \mathcal{H}_I$, so that in particular $\operatorname{dist}(X,\Gamma) \geq \frac{1}{2}a_1\eta^{\frac{n-1}{n-1-d}}\ell(Q)$. On the other hand,

$$\operatorname{dist}(X,\Gamma) \leq \operatorname{diam} J + \operatorname{dist}(J,\Gamma) \leq \sqrt{n}\ell(J) + 40\sqrt{n}\ell(J) = 41\sqrt{n}\ell(J),$$

so that the lower bound for $\ell(J)$ in (2.4.14) follows immediately. Now, note that $4\sqrt{n}\ell(J) \leq \operatorname{dist}(J,\Gamma) \leq \operatorname{dist}(X,\Gamma)$. Since $X \in \mathcal{H}_I$, there exists a ball B such that $X \in B$, where B has center $Y_B \in [Y_I, Y_Q]$ and has radius $\frac{a_1}{2}\eta^{\frac{n-1}{n-1-d}}\ell(Q)$. Consider the estimate

$$\begin{aligned}
\operatorname{dist}(Y_B,\Gamma) &\leq \max\left\{|Y_I - x_Q|, |Y_Q - x_Q|\right\} \\
&\leq \tfrac{1}{41}a_0c\eta\ell(Q) + \max\left\{|X_I - x_Q|, |X_Q - x_Q|\right\} \\
&\leq \tfrac{1}{41}a_0c\eta\ell(Q) + \operatorname{dist}(I,Q) + \operatorname{diam} Q + \operatorname{diam} I \\
&\leq \left[\tfrac{1}{41}c\eta + \tfrac{5}{4}K + \tfrac{A_0}{a_0}\right]a_0\ell(Q),
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{dist}(J,\Gamma) &\leq \operatorname{dist}(X,\Gamma) \leq \operatorname{dist}(Y_B,\Gamma) + \tfrac{a_1}{2}\eta^{\frac{n-1}{n-1-d}}\ell(Q) \\
&\leq \left[\tfrac{1}{41}c\eta + \tfrac{5}{4}K + \tfrac{A_0}{a_0}\right]a_0\ell(Q) + \tfrac{a_1}{2}\eta^{\frac{n-1}{n-1-d}}\ell(Q) \leq (2a_0K + A_0)\ell(Q).
\end{aligned}$$

This last estimate gives the rest of the bounds in (2.4.14). □

$\mathcal{W}_Q$: Bridging the gaps via Harnack Chains

We proceed with the construction of the sawtooth domains. Fix $Q \in \mathbb{D}$ and let X_Q be a Corkscrew point for Q, which we now fix, and which belongs to some Whitney box in $\mathcal{W}_Q^0$. For each $I \in \mathcal{W}_Q^0$, we let $\mathcal{H}_I$ be any well-tempered Harnack Chain connecting X_I to X_Q manufactured in Lemma 2.4.11. Then we let $\mathcal{W}_Q$ be the set of all $J \in \mathcal{W}$ which meet at least one of the Harnack Chains $\mathcal{H}_I$ with $I \in \mathcal{W}_Q^0$; that is,

$$\mathcal{W}_Q := \big\{ J \in \mathcal{W} \,:\, \text{there exists } I \in \mathcal{W}_Q^0 \text{ for which } \mathcal{H}_I \cap J \neq \varnothing \big\}. \tag{2.4.15}$$

Note that $\mathcal{W}_Q$ contains Harnack Chains (in the sense of Definition 2.2.9) between any two of its points. Let us sketch a proof of this fact. Let $J_1, J_2 \in \mathcal{W}_Q$ and $Z_1 \in J_1$, $Z_2 \in J_2$; immediately note that $\delta(Z_i) \gtrsim \ell(Q)$ and $|Z_1 - Z_2| \lesssim \ell(Q)$. By construction, there exist $I_i \in \mathcal{W}_Q^0$, $i = 1, 2$, such that $\mathcal{H}_{I_i} \cap J_i \neq \varnothing$, so let $Y_i \in \mathcal{H}_{I_i} \cap J_i$. Since $\mathcal{H}_{I_i}$ connects X_{I_i} to (the fixed point) X_Q, and since $Y_i \in \mathcal{H}_{I_i}$, it follows that Y_1 and Y_2 are linked by a Harnack Chain. Since J_i is an $n-$dimensional open cube, Y_i can be linked to Z_i via a Harnack Chain as well. Thus there is a Harnack Chain that contains $Z_1, Y_1, X_{I_1}, X_Q, X_{I_2}, Y_2, Z_2$, as desired.

We clearly have that $\mathcal{W}_Q^0 \subset \mathcal{W}_Q$, and we note that if $J \in \mathcal{W}_Q$, then J satisfies the assumptions of Lemma 2.4.13 and therefore (2.4.14) holds, giving that

$$\mathcal{W}_Q \subseteq \big\{ I \in \mathcal{W} \,:\, a_2\ell(Q) \leq \ell(I) \leq A_2\ell(Q), \qquad \operatorname{dist}(I, Q) \leq 4\sqrt{n}A_2\ell(Q) \big\}, \tag{2.4.16}$$

where a_2, A_2 are the corresponding multiplicative constants in the first inequality chain in (2.4.14). In particular, once η, K are fixed, for any $Q \in \mathbb{D}$, the cardinality of $\mathcal{W}_Q$ is uniformly bounded, which is a corollary of the following result.

Lemma 2.4.17 (Number of Whitney boxes in a Whitney region)**.** *Fix $Q \in \mathbb{D}$ and for positive numbers α, β, γ with $\alpha < \beta$, define the set*

$$\mathcal{W}(\alpha, \beta, \gamma) = \big\{ I \in \mathcal{W} \,:\, \alpha\ell(Q) \leq \ell(I) \leq \beta\ell(Q), \qquad \operatorname{dist}(I, Q) \leq \gamma\ell(Q) \big\}.$$

Then

$$\operatorname{card}(\mathcal{W}(\alpha, \beta, \gamma)) \leq \Big[\frac{\sqrt{n}\beta + \gamma + A_0}{\alpha}\Big]^n |B(0,1)| \tag{2.4.18}$$

Proof. Fix $Q \in \mathbb{D}$, let $X \in I$ be arbitrary with $I \in \mathcal{W}(\alpha, \beta, \gamma)$, and consider the estimate

$$|X - x_Q| \leq \operatorname{diam} I + \operatorname{dist}(I, Q) + \operatorname{diam} Q \leq \sqrt{n}\beta\ell(Q) + \gamma\ell(Q) + A_0\ell(Q).$$

It follows that $\cup_{I \in \mathcal{W}(\alpha,\beta,\gamma)} I \subseteq B(x_Q, (\sqrt{n}\beta + \gamma + A_0)\ell(Q))$. Hence, note that

$$\begin{aligned}\operatorname{card}(\mathcal{W}(\alpha,\beta,\gamma))[\alpha\ell(Q)]^n &= \sum_{I\in\mathcal{W}(\alpha,\beta,\gamma)} [\alpha\ell(Q)]^n \leq \sum_{I\in\mathcal{W}(\alpha,\beta,\gamma)} \ell(I)^n \\ &= \big|\bigcup_{I\in\mathcal{W}(\alpha,\beta,\gamma)} I\big| \leq |B(x_Q,(\sqrt{n}\beta+\gamma+A_0)\ell(Q))| \\ &= [(\sqrt{n}\beta+\gamma+A_0)\ell(Q)]^n|B(0,1)|.\end{aligned}$$

The desired result follows immediately. □

Corollary 2.4.19 (Cardinality of $\mathcal{W}_Q$)**.** *For each $Q \in \mathbb{D}$, we have that*

$$\operatorname{card}(\mathcal{W}_Q) \leq \Big[\frac{5\sqrt{n}A_2 + A_0}{a_2}\Big]^n |B(0,1)| =: N_0.$$

U_Q: The Whitney region

Next, we choose a small dilation parameter $\theta \in (0,1)$ so that for any $I \in \mathcal{W}$, the concentric dilation $I^* = (1+\theta)I$ still satisfies the Whitney property

$$\text{diam } I \approx \text{diam } I^* \approx \text{dist}(I^*,\Gamma) \approx \text{dist}(I,\Gamma),$$

with uniform constants not depending on the choices of η, K. More precisely, it can be easily shown that, as long as $\theta \in (0,8)$, we have for each $I \in \mathcal{W}$ the estimates

$$\begin{gathered}\operatorname{diam} I \leq \operatorname{diam} I^* \leq (1+\theta)\operatorname{diam} I, \\ (1-\tfrac{\theta}{8})\operatorname{dist}(I,\Gamma) \leq \operatorname{dist}(I^*,\Gamma) \leq \operatorname{dist}(I,\Gamma), \\ \tfrac{1}{40}\operatorname{dist}(I^*,\Gamma) \leq \operatorname{diam} I^* \leq 2\tfrac{1+\theta}{8-\theta}\operatorname{dist}(I^*,\Gamma).\end{gathered}$$

For later use, we record also that if $X \in \partial I^*$ and $\theta \in (0,1)$, then

$$2\operatorname{diam} I \leq \delta(X) \leq 82\operatorname{diam} I. \tag{2.4.20}$$

Moreover, by taking θ small enough we can guarantee that dist$(I^*, J^*) \approx$ dist(I, J):

Lemma 2.4.21 (Distances between dilated Whitney boxes)**.** *Suppose that $I, J \in \mathcal{W}$. Then for each $\theta \in (0, 1/(4\sqrt{n}))$, we have the estimates*

$$\big[1-4\sqrt{n}\theta\big]\operatorname{dist}(I,J) \leq \operatorname{dist}(I^*,J^*) \leq \operatorname{dist}(I,J). \tag{2.4.22}$$

Furthermore, if θ is as above and $I, J \in \mathcal{W}$ are distinct, then $I^* \cap \frac{1}{2}J = \varnothing$.

Proof. Without loss of generality, suppose that $\ell(I) \geq \ell(J)$. If I and J are adjacent, so that $\partial I \cap \partial J \neq \varnothing$, then $\operatorname{dist}(I, J) = 0$ and $\operatorname{dist}(I^*, J^*) = 0$, so that (2.4.22) holds trivially. Now suppose that I and J are not adjacent. The upper bound in (2.4.22) holds trivially. By the Pidgeonholing Principle, we must have that $\operatorname{dist}(I, J) \geq \frac{1}{4}\ell(I)$, for otherwise a point of J lies in a Whitney box adjacent to I, which implies that J is adjacent to I and we have a contradiction. Next, let $X^* \in I^*$ and $Y^* \in J^*$ be the points such that $\operatorname{dist}(I^*, J^*) = |X^* - Y^*|$. For each $X \in I$ and $Y \in J$, observe the basic estimate

$$|X^* - Y^*| \geq |X - Y| - |Y - Y^*| - |X^* - X| \geq \operatorname{dist}(I, J) - |Y - Y^*| - |X^* - X|.$$

Choose X so that $|X - X^*| = \operatorname{dist}(X^*, I)$ and Y so that $|Y - Y^*| = \operatorname{dist}(Y^*, J)$. Note that $\operatorname{dist}(X^*, I) \leq \frac{\theta}{2}\sqrt{n}\ell(I)$ and $\operatorname{dist}(Y^*, J) \leq \frac{\theta}{2}\sqrt{n}\ell(J) \leq \frac{\theta}{2}\sqrt{n}\ell(I)$. It follows that $\operatorname{dist}(I^*, J^*) \geq \left[1 - 4\sqrt{n}\theta\right] \operatorname{dist}(I, J)$, which ends the proof of (2.4.22). Now suppose that $I, J \in \mathcal{W}$ are distinct and without loss of generality say that $\ell(I) \geq \ell(J)$. If they are not adjacent, then $I^* \cap \frac{1}{2}J = \varnothing$ follows immediately from (2.4.22). Hence we need only consider the case that I and J are adjacent, and in this case we have that $\ell(J) \geq \frac{1}{4}\ell(I)$. Let $X^* \in I^*$ and $Y \in \frac{1}{2}J$ be points such that $\operatorname{dist}(I^*, \frac{1}{2}J) = |X^* - Y|$, and choose $X \in I$ so that $|X^* - X| = \operatorname{dist}(X^*, I) \leq \frac{\theta}{2}\sqrt{n}\ell(I)$. Reckon the elementary estimate

$$|X^* - Y| \geq |Y - X| - |X - X^*| \geq \tfrac{1}{2}\ell(J) - \tfrac{\theta}{2}\sqrt{n}\ell(I) \geq \left[\tfrac{1}{8} - \tfrac{\theta}{2}\sqrt{n}\right]\ell(I) > 0,$$

which yields the desired result. □

Corollary 2.4.23. *Suppose that $I, J \in \mathcal{W}$ and $\theta \in (0, 1/(4\sqrt{n}))$. Then $I^* \cap J^* \neq \varnothing$ if and only if $\partial I \cap \partial J \neq \varnothing$.*

Given $Q \in \mathbb{D}$, we define an associated *Whitney region* U_Q as

$$U_Q := \textstyle\bigcup_{I \in \mathcal{W}_Q} I^*. \tag{2.4.24}$$

$\Omega_{\mathcal{F}}$: The sawtooth domain

For any disjoint family $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}$, we define the *discretized sawtooth relative to* $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{\mathcal{F}} \mathbb{D}_{Q_j}, \tag{2.4.25}$$

so that $\mathbb{D}_{\mathcal{F}}$ is the collection of all $Q \in \mathbb{D}$ which are not contained in any $Q_j \in \mathcal{F}$. We may also need to consider a local version of the sawtooth. If $\mathcal{F} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ is as above, then we define the *(local) discretized sawtooth relative to* $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q_0} := \mathbb{D}_{Q_0} \setminus \bigcup_{\mathcal{F}} \mathbb{D}_{Q_j}. \tag{2.4.26}$$

Finally, we define the global and local *sawtooth domains* relative to $\mathcal{F}$ via

$$\Omega_{\mathcal{F}} := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}_{\mathcal{F}}} U_Q \Big), \qquad \Omega_{\mathcal{F},Q_0} := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} U_Q \Big), \tag{2.4.27}$$

where U_Q is given in (2.4.24). Note that the discretized sawtooth is a collection of cubes living in Γ, while the sawtooth domain is an open set in Ω. The salient feature of the sawtooth domain $\Omega_{\mathcal{F}}$ is that it "hides" the disjoint family $\mathcal{F}$ (see Proposition 2.4.35 and Lemma 2.4.43). In Section 2.4.3 below, we study the geometric properties of the sawtooth domains in detail. For convenience, we set

$$\mathcal{W}_{\mathcal{F}} := \bigcup_{Q \in \mathbb{D}_{\mathcal{F}}} \mathcal{W}_Q, \qquad \mathcal{W}_{\mathcal{F},Q_0} := \bigcup_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} \mathcal{W}_Q, \tag{2.4.28}$$

where $\mathcal{W}_Q$ is given in (2.4.15), so that in particular, we may write

$$\Omega_{\mathcal{F}} = \operatorname{int}\Big(\bigcup_{I \in \mathcal{W}_{\mathcal{F}}} I^* \Big), \qquad \Omega_{\mathcal{F},Q_0} = \operatorname{int}\Big(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q_0}} I^* \Big). \tag{2.4.29}$$

We remark that

$$\Omega_{\mathcal{F},Q_0} \subset B(x_{Q_0}, 7\sqrt{n} A_2 \ell(Q_0)) \cap \Omega \tag{2.4.30}$$

for any $Q_0 \in \mathbb{D}$ and any family $\mathcal{F}$ as above, where x_{Q_0} is given in (2.3.6). Indeed, let $X \in \Omega_{\mathcal{F},Q_0}$, so that there exists $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and $I \in \mathcal{W}_Q$ with $X \in I^*$. By (2.4.16) and the triangle inequality, we see that

$$|X - x_{Q_0}| \le \operatorname{diam} I^* + \operatorname{dist}(I^*, Q) + \operatorname{diam} Q_0 \le 2\sqrt{n}\ell(I) + 4\sqrt{n}A_2\ell(Q) + A_0\ell(Q) \le 7\sqrt{n}A_2\ell(Q_0).$$

2.4.2 Some further notation

Let $\mathcal{F} = \{Q_j\}_j$ be a disjoint family. The definitions below will be stated for the global discretized sawtooth relative to $\mathcal{F}$, but it is clear that we have direct analogues for the local discrete sawtooth (see Section 2.4.1)

• We denote by $\Delta_\star$ a surface ball on $\partial\Omega_{\mathcal{F}}$. More precisely, suppose that $x_\star \in \partial\Omega_{\mathcal{F}}$ and $r > 0$. Then $\Delta_\star(x_\star, r) := B(x_\star, r) \cap \partial\Omega_{\mathcal{F}}$.

• Let $\delta_\star : \Omega_{\mathcal{F}} \to [0, \infty)$ be the distance to $\partial\Omega_{\mathcal{F}}$; that is, $\delta_\star(X) := \operatorname{dist}(X, \partial\Omega_{\mathcal{F}})$.

• We denote $\Sigma := \partial\Omega_{\mathcal{F}} \backslash \Gamma$, and observe that

$$\Sigma = \partial\Omega_{\mathcal{F}} \backslash \Gamma = \partial\big(\textstyle\bigcup_{I \in \mathcal{W}_{\mathcal{F}}} I^*\big) \backslash \Gamma \subset \bigcup_{\substack{I \,\in\, \mathcal{W}_{\mathcal{F}} \\ I \,\cap\, \partial\Omega_{\mathcal{F}} \,\neq\, \varnothing}} \partial I^*,$$

so that Σ consists of subsets of $(n-1)-$dimensional faces of Whitney boxes $I \in \mathcal{W}_{\mathcal{F}}$. One may think of Σ as the sawtooth "barrier" that hides the disjoint family $\mathcal{F}$. The relevant geometric properties of Σ are studied in Section 2.4.3 below.

• For each $Q_0 \in \mathbb{D}$, we let the *Carleson collection associated to* Q_0 be

$$\mathcal{R}_{Q_0} := \textstyle\bigcup_{Q \in \mathbb{D}_{Q_0}} \mathcal{W}_Q, \tag{2.4.31}$$

and we define the *Carleson region* R_{Q_0} as

$$R_{Q_0} := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}_{Q_0}} U_Q\Big) = \operatorname{int}\Big(\bigcup_{I \in \mathcal{R}_{Q_0}} I^*\Big). \tag{2.4.32}$$

The set in (2.4.32) generalizes the notion of a "Carleson box" above a cube (see Lemma 2.4.38); and this is the impetus for our notation of Carleson collection/region.

• The following dyadic version of the conical approach region in Definition 2.2.19 will be useful during Step 1 of the proof of Theorem 2.1.1 in Section 2.9.2. For every

$x \in \Gamma$, we define the (global and local) *dyadic non-tangential cones* as

$$\gamma_d(x) = \bigcup_{Q \in \mathbb{D}: Q \ni x} U_Q, \qquad \gamma_d^{Q_0}(x) = \bigcup_{Q \in \mathbb{D}_{Q_0}: Q \ni x} U_Q \tag{2.4.33}$$

Given an aperture $\alpha > 0$, there exists K (in the definition (2.4.8)) sufficiently large such that the standard non-tangential cone $\gamma^\alpha(x) \subset \gamma_d(x)$ for all $x \in \Gamma$; and vice versa, for fixed values of η, K and the dilation constant θ, there exists $\alpha_1 > 0$ such that the dyadic cone $\gamma_d(x) \subset \gamma^{\alpha_1}(x)$ for all $x \in \Gamma$.

2.4.3 Geometric properties of sawtooth domains

In this subsection, we collect a number of technical results, many of which are direct analogues of results shown in [HM14]. There, the authors work with 1-sided NTA domains with $(n-1)-$ADR boundaries. We are interested in borrowing their setup; the proofs of many results here are very similar to theirs, with some small modifications.

The first lemma we wish to present says that the boundary of a union of Whitney boxes consists of hyper-rectangles that do not degenerate.

Lemma 2.4.34 (Non-degeneracy of the faces of Σ)**.** *For all $\theta \in (0, 1/(16\sqrt{n}))$ and for each $I \in \mathcal{W}_{\mathcal{F}}$ (see (2.4.28)) intersecting $\partial\Omega_{\mathcal{F}}$, the set*

$$\partial_\Sigma I^* := \partial I^* \setminus \cup_{J \in \mathcal{W}_{\mathcal{F}}, J \neq I} \operatorname{int} J^* = (\partial I^* \cap \partial\Omega_{\mathcal{F}}) \setminus \cup_{J \in \mathcal{W}_{\mathcal{F}}, J \neq I} \operatorname{int} J^*$$

is a non-empty union of $(n-1)-$dimensional rectangles R embedded in the $(n-1)$-dimensional faces of ∂I^*, such that no sidelength of any R is smaller than $\theta\ell(I)/4$, thus verifying $\mathscr{H}^{n-1}(R) \geq c_n \theta^{n-1} \ell(I)^{n-1}$, where $c_n \in (0,1)$ depends only on n.

Proof. Suppose that $I \in \mathcal{W}_{\mathcal{F}}$ intersects $\partial\Omega_{\mathcal{F}}$, whence it follows that some face of I^* intersects $\partial\Omega_{\mathcal{F}}$. In the union defining $\partial_\Sigma I^*$, we need only consider those $J \in \mathcal{W}_{\mathcal{F}}$ which are adjacent to I, by Corollary 2.4.23. That $\partial_\Sigma I^*$ is a union of $(n-1)-$dimensional rectangles is clear by the construction of the sawtooth domain, as the boundary can be written as a union of faces of cubes intersecting the complements of cubes.

Now let F^* be any face of I^* so that $F^* \cap \partial\Omega_{\mathcal{F}} \neq \varnothing$, and let F be the face of I corresponding to F^*, defined as the unique face of I such that $\operatorname{dist}(\operatorname{int} F, F^*) = \frac{1}{2}\theta\ell(I)$.

Fix a maximal rectangle $R \subset F^* \cap \partial\Omega_{\mathcal{F}}$ (maximal in the sense that increasing any of its side-lengths makes it stop being a subset of $\partial\Omega_{\mathcal{F}}$) and consider two cases.

Case a) There exists $x \in R$ and $x' \in \overline{F}$ such that $\operatorname{dist}(x, x') = \frac{1}{2}\theta\ell(I)$. In this case, for each θ small enough, we must have that $x' \in \overline{J'}$, where $J' \notin \mathcal{W}_{\mathcal{F}}$ is a Whitney box adjacent to I. Since J' is adjacent to I, we have that $\ell(J) \geq \ell(I)/4$, and recall that the Whitney boxes are dyadically aligned. It follows that $F \backslash_{J\in\mathcal{W}_{\mathcal{F}}, J\neq I} \overline{J}$ contains an $(n-1)-$dimensional cube $F' \subset \partial J'$ of length $\ell(F') \geq \ell(I)/4$. For each $y' \in F'$, there exists a unique $y \in F^*$ such that $\operatorname{dist}(y, y') = \frac{1}{2}\theta\ell(I)$; accordingly we set F'^* to be the collection of $y \in F^*$ constructed in this way, and observe that $x \in F'^*$. Then we must have that $R \supset F'^* \backslash \cup_{J\in\mathcal{W}_{\mathcal{F}}, J\neq I} \overline{J}$ contains an $(n-1)-$dimensional cube of side-length $(\frac{1}{4} - 4\theta)\ell(I)$, because for any straight line segment L in F^* parallel to a coordinate axis and passing through the center of F'^*, L intersects at most two Whitney boxes $J_1, J_2 \in \mathcal{W}_{\mathcal{F}}$ different from I and such that $J_i^* \cap \overline{F'^*} \neq \varnothing$; both of which have $4\ell(I) \geq \ell(J_i) \geq \ell(I)/4$. Hence, in case a) we have established the desired result.

Case b) There is no such $x' \in \overline{F}$ as in Case a). It follows that $R \subset F^* \backslash \frac{1}{1+\theta}F^*$, and therefore $R \cap \operatorname{int} J' \neq \varnothing$ for some Whitney box $J' \notin \mathcal{W}_{\mathcal{F}}$ touching I. If $J \in \mathcal{W}_{\mathcal{F}}$ is any Whitney box adjacent to I with $\ell(J) = 2^k\ell(I)$ and such that $J^* \cap J' \neq \varnothing$, then $k \in \{-2, -1\}$. Indeed, if $k \in \mathbb{Z}$ is larger then J^* protrudes a distance (perpendicular to the face F whose boundary is intersected by $\overline{J}$) greater than or equal to $\frac{1}{2}\theta\ell(I)$, so that $R \subset \overline{J^*}$. If $R \subset \operatorname{int} J^*$ we have a contradiction to the fact that $R \subset \partial\Omega_{\mathcal{F}}$; whereas if $R \subset \partial J^*$, then there is a face F_{J^*} which is adjacent to F^* and such that $\operatorname{int}(F_{J^*} \cup F^*)$ is a connected set; we may then reduce to Case a) by considering that $R \subset F_{J^*}$. Finally, since $k \leq -1$, then J^* protrudes a distance at most $\frac{1}{4}\theta\ell(I)$. It follows that all the sides of R have length larger than or equal to $\frac{1}{2}\theta\ell(I) - \frac{1}{4}\theta\ell(I) = \frac{1}{4}\theta\ell(I)$, giving the result. $\square$

Proposition 2.4.35 (Characterization of non-hidden regions, Proposition 6.1 in [HM14])**.** *Let $\mathcal{F} = \{Q_j\}_j$ be a disjoint family. Then*

$$\Gamma \backslash \big(\textstyle\bigcup_{\mathcal{F}} Q_j\big) \subseteq \Gamma \cap \partial\Omega_{\mathcal{F}} \subseteq \Gamma \backslash \big(\textstyle\bigcup_{\mathcal{F}} \operatorname{int} Q_j\big). \tag{2.4.36}$$

Proof. Let us show by contradiction the second containment first, thus assume that there exists $x \in \Gamma \cap \partial\Omega_{\mathcal{F}} \cap \operatorname{int} Q_j$ for some $Q_j \in \mathcal{F}$. Hence, there exists $\varepsilon > 0$ for which $B(x, \varepsilon) \cap \Gamma \subset Q_j$. In particular, $B(x, \varepsilon) \cap Q = \varnothing$ for any $Q \in \mathbb{D}_{\mathcal{F}}$ that does not contain Q_j. Since $x \in \partial\Omega_{\mathcal{F}}$, there exist $X_k \in \Omega_{\mathcal{F}}$ such that $|X_k - x| \to 0$ as

$k \to \infty$. Accordingly, for each $k \in \mathbb{N}$ there exists a Whitney box I_k and a dyadic cube $Q_k \in \mathbb{D}_{\mathcal{F}}$ such that $I_k \in \mathcal{W}_{Q_k}$ and $X_k \in I_k^*$ (see (2.4.29) and (2.4.15)). Since $x \in \Gamma$ and $X_k \to x$, then $\delta(X_k) \to 0$ and therefore $\ell(I_k^*) \to 0$, which further implies that $\ell(Q_k) \to 0$ by (2.4.16). It follows that for all k large enough, $\ell(Q_k) \ll \ell(Q_j)$, so that $B(x, \varepsilon) \cap Q_k = \varnothing$. On the other hand,

$$\operatorname{dist}(Q_k, x) \leq \operatorname{dist}(Q_k, I_k^*) + \operatorname{diam} I_k^* + |X_k - x| \lesssim \ell(Q_k) + \ell(I_k) + |X_k - x| \longrightarrow 0,$$

which implies that there is $k_0 \in \mathbb{N}$ large enough and a point $q \in Q_{k_0}$ so that $|q - x| < \varepsilon$. Hence $q \in B(x, \varepsilon) \cap Q_{k_0}$, a contradiction. The second containment is thus established.

We now prove the first containment. Suppose that $x \in \Gamma \backslash (\cup_{\mathcal{F}} Q_j)$, and note that obviously $x \notin \Omega_{\mathcal{F}}$. Hence, for any generation $k \in \mathbb{Z}$, $x \in Q_k$ for some $Q_k \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}^k$. According to each $Q_k \in \mathbb{D}_{\mathcal{F}}$, there exists $I_k \in \mathcal{W}_{Q_k}$ and $I_k \subset \Omega_{\mathcal{F}}$. Then the centers $X_{I_k} \in I_k$ satisfy $|X_{I_k} - x| \leq \operatorname{diam} Q_k + \operatorname{dist}(Q_k, I_k) + \operatorname{diam} I_k \lesssim \ell(Q_k)$. Taking $k \to \infty$ gives that $\ell(Q_k) \approx 2^{-k} \to 0$, and hence that $|X_{I_k} - x| \to 0$. It follows that $x \in \partial\Omega_{\mathcal{F}}$. ☐

Although we may not quite say that $\Gamma \cap \partial\Omega_{\mathcal{F}} \subseteq \Gamma \backslash \operatorname{int}(\cup_{\mathcal{F}} Q)$, we do have a technical improvement to the upper containment in (2.4.36).

Lemma 2.4.37 (A refinement to Proposition 2.4.35)**.** *Suppose that $\mathcal{F}$ is a disjoint family and let $x \in \Gamma \cap \partial\Omega_{\mathcal{F}}$. Then for each $k \in \mathbb{Z}$, there exists $Q^k \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}^k$ verifying that $x \in \overline{Q^k}$.*

Proof. Fix $k \in \mathbb{Z}$ and recall that $\{Q_m^k\}_{m \in \mathscr{J}^k}$ is a disjoint covering of Γ. We first show that $x \in \overline{\bigcup_{\mathbb{D}^k \cap \mathbb{D}_{\mathcal{F}}} Q_m^k}$. Indeed, suppose otherwise, so that $x \in \operatorname{int}(\bigcup_{\mathbb{D}^k \backslash \mathbb{D}_{\mathcal{F}}} Q_m^k)$. Then there exists $\varepsilon > 0$ so that $B(x, \varepsilon) \cap \Gamma \subset \bigcup_{\mathbb{D}^k \backslash \mathbb{D}_{\mathcal{F}}} Q_m^k$. Since $x \in \Gamma \cap \partial\Omega_{\mathcal{F}}$, as in the proof of Proposition 2.4.35 we can procure dyadic cubes $Q_i' \in \mathbb{D}_{\mathcal{F}}$ such that $\operatorname{dist}(Q_i', x) \to 0$ and $\ell(Q_i') \to 0$ as $i \to \infty$. Accordingly, for all i large enough, we have that $Q_i' \subset B(x, \varepsilon) \cap \Gamma$ and $\ell(Q_i') \ll 2^{-k} \approx Q_m^k$ for any $Q_m^k \in \mathbb{D}^k \backslash \mathbb{D}_{\mathcal{F}}$, which prohibits Q_i' from being an ancestor of any $Q_m^k \in \mathbb{D}^k \cap \mathbb{D}_{\mathcal{F}}$, and this implies that for all i large enough, $Q_i' \subseteq Q_j$ for some $Q_j \in \mathcal{F}$, yielding a contradiction to $Q_i' \in \mathbb{D}_{\mathcal{F}}$.

To finish the proof, we observe that for any $y \in \Gamma$, the cardinality of the set $S(y) := \{Q_m^k \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}^k \; : \; \operatorname{dist}(Q_m^k, y) < 2^{-k}\}$ is uniformly finite, implying in particular that $\overline{\bigcup_{\mathbb{D}^k \cap \mathbb{D}_{\mathcal{F}}} Q_m^k} = \bigcup_{\mathbb{D}^k \cap \mathbb{D}_{\mathcal{F}}} \overline{Q_m^k}$ and thus yielding the desired result. To see that card $S(y) < +\infty$, reckon the estimate

$$
\begin{aligned}
C_d^{-1}(a_0 2^{-k})^d \operatorname{card}(S(y)) &\le \sum_{Q_m^k \in S(y)} \sigma(\Delta(x_{Q_m^k}, a_0 2^{-k})\\
&\le \sum_{Q_m^k \in S(y)} \sigma(Q_m^k) = \sigma\big(\textstyle\bigcup_{S(y)} Q_m^k\big) \le \sigma\big(B(x, 2A_0 2^{-k}) \cap \Gamma\big) \le C_d(2A_0 2^{-k})^d,
\end{aligned}
$$

so that $\operatorname{card}(S(y)) \le 2^d C_d^2 A_0^d a_0^{-d}$. $\square$

The following lemma establishes that Carleson regions are quantitatively fat.

Lemma 2.4.38 (Quantitative fatness of Carleson regions, Lemma 3.55 of [HM14])**.** *The following statements are true.*

(i) For each $Q \in \mathbb{D}$, there exists a ball $B_s := B(x_Q, s) \subseteq B(x_Q, a_0\ell(Q))$ with $s \approx \ell(Q)$ (we may, in fact, take $s = a_0\ell(Q)/4$), such that

$$B_s \cap \Omega \subset R_Q,$$

where R_Q is defined in (2.4.32).

(ii) Moreover, for a somewhat smaller choice of $s \approx \ell(Q)$ (in fact, we may choose $s = a_0 a_2 A_2^{-1}\ell(Q)/10$), we have for every pairwise disjoint family $\mathcal{F} \subset \mathbb{D}$, and for each $Q_0 \in \mathbb{D}$ containing Q, that

$$B_s \cap \Omega_{\mathcal{F},Q_0} = B_s \cap \Omega_{\mathcal{F},Q}, \tag{2.4.39}$$

where $\Omega_{\mathcal{F},Q}$ is defined in (2.4.29).

Proof. We consider (i) first. Fix $Y \in B_s \cap \Omega$, and let $I \in \mathcal{W}$ be a Whitney box that contains Y. Choose $y \in \Gamma$ such that $|Y - y| = \operatorname{dist}(Y, \Gamma) = \delta(Y) \approx \ell(I)$, and observe that

$$|y - x_Q| \le |Y - y| + |Y - x_Q| \le \delta(Y) + s \le 2s < a_0\ell(Q),$$

provided that $s < a_0\ell(Q)/2$. Hence $y \in \Delta(x_Q, a_0\ell(Q)) \subseteq Q$. Now let $Q^d \subset \mathbb{D}_Q$ be a descendant of Q (which is unique *as a set*) that contains y and that verifies the inequalities $\frac{a_0 c c_{\mathbb{K}}}{82\sqrt{n}}\ell(Q^d) \le \ell(I) \le \frac{a_0}{8\sqrt{n}}\ell(Q^d)$. Since we must have that

$$\operatorname{dist}(I, Q^d) \le |Y - y| = \delta(Y) \le 41 \operatorname{diam} I \le \tfrac{41}{8} a_0\ell(Q^d) \le a_0 K\ell(Q^d),$$

we conclude that $I \in \mathcal{W}^0_{Q^d}$ (see (2.4.8), and therefore $Y \in U_{Q^d} \subset R_Q$, as desired.

Now we consider (ii). Since $Q \subseteq Q_0$, the containment $B_s \cap \Omega_{\mathcal{F},Q_0} \supseteq B_s \cap \Omega_{\mathcal{F},Q}$ is trivial, and thus we need only verify the opposite containment. Fix $Y \in B_s \cap \Omega_{\mathcal{F},Q_0}$, and as such there exist $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$ and $I \in \mathcal{W}_{Q'}$ verifying $Y \in I^*$. Note that

$$|Y - x_Q| \geq \operatorname{dist}(I^*, \Gamma) \geq \tfrac{1}{2} \operatorname{dist}(I, \Gamma) \geq 2 \operatorname{diam} I = 2\sqrt{n}\ell(I),$$

so that $\ell(I) < s/(2\sqrt{n})$. We claim that $Q' \subseteq Q$. To see this, let $z \in Q'$ and reckon the estimate

$$\begin{aligned} |z - x_Q| &\leq \operatorname{diam} Q' + \operatorname{dist}(I^*, Q') + \operatorname{diam} I^* + |Y - x_Q| \\ &< A_0 a_2^{-1}\ell(I) + 4\sqrt{n}A_2 a_2^{-1}\ell(I) + 2\sqrt{n}\ell(I) + s \ < 5A_2 a_2^{-1} s < a_0 \ell(Q), \end{aligned}$$

provided that $s < \frac{1}{5} a_0 a_2 A_2^{-1} \ell(Q)$. Hence $Q' \subseteq \Delta_Q \subseteq Q$, as claimed. But then, $Q' \in \mathbb{D}_{\mathcal{F},Q}$, so that $I \in \mathcal{W}_{\mathcal{F},Q}$ (see (2.4.28)) and therefore $I^* \subset \Omega_{\mathcal{F},Q}$, which implies that $Y \in \Omega_{\mathcal{F},Q}$, as desired. $\square$

The next proposition gives us that the sets in (2.4.36) are the same from the perspective of a doubling measure. Its proof is essentially the same as in [HM14], thus we omit the details (see Remark 2.2.11).

Proposition 2.4.40 (Negligibility of pathologies in (2.4.36) for doubling measures, Proposition 6.3 in [HM14])**.** *Suppose that μ is a doubling measure on Γ. Then $\partial Q := \overline{Q} \backslash \operatorname{int} Q$ has $\mu-$measure 0, for every $Q \in \mathbb{D}$. In particular, the pairwise difference of any two of the sets in (2.4.36) has $\mu-$measure 0.*

We will now elicit the existence of a point that acts as a Corkscrew point simultaneously in Ω and in the sawtooth domain $\Omega_{\mathcal{F},Q_0}$.

Proposition 2.4.41 (Existence of simultaneous Corkscrews, Proposition 6.4 of [HM14])**.** *Fix $Q_0 \in \mathbb{D}$, and let $\mathcal{F} \subset \mathbb{D}_{Q_0}$ be a disjoint family. Then for each $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, there is a radius $r_Q \approx \ell(Q)$ (in fact, we may take $r_Q = 6\sqrt{n}A_2\ell(Q)$), and a point $X_Q \in \Omega_{\mathcal{F},Q_0}$ which serves as a Corskcrew point (with Corkscrew constant $\tilde{c} := \frac{1}{12\sqrt{n}} a_2 A_2^{-1}$, see Definition 2.2.7) simultaneously for $\Omega_{\mathcal{F},Q_0}$, with respect to the surface ball $\Delta_\star(y_Q, r_Q)$ (see Section 2.4.2), for some $y_Q \in \partial\Omega_{\mathcal{F},Q_0}$, and for Ω, with respect to each surface ball $\Delta(x, r_Q)$, for every $x \in Q$.*

It will be clear by our method of proof that we also have simultaneous Corkscrews for Ω and $\Omega_{\mathcal{F}}$.

Proof. Fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, and so note that there exists $I \in \mathcal{W}_Q$ with $\operatorname{int} I^* \subset \Omega_{\mathcal{F},Q_0}$ (see (2.4.15) and (2.4.29)). We fix this Whitney cube I. Observe that

$$\operatorname{dist}(X_I, \partial\Omega_{\mathcal{F},Q_0}) \geq \operatorname{dist}(X_I, \partial I^*) = \tfrac{1}{2}(1+\theta)\ell(I) \geq \tfrac{1}{2}a_2\ell(Q),$$

while on the other hand, since $X_I \in \operatorname{int}\Omega_{\mathcal{F},Q_0}$ (see Notation 2.4.2) and $\Gamma \subset \mathbb{R}^n \backslash \Omega_{\mathcal{F},Q_0}$, we have that

$$\begin{aligned}\operatorname{dist}(X_I, \partial\Omega_{\mathcal{F},Q_0}) = \operatorname{dist}(X_I, \mathbb{R}^n \backslash \Omega_{\mathcal{F},Q_0}) \leq \operatorname{dist}(X_I, \Gamma) \leq \operatorname{dist}(X_I, Q)& \\ \leq 5\sqrt{n}A_2\ell(Q).&\end{aligned}$$

It follows that, if we let $y \in \partial\Omega_{\mathcal{F},Q_0}$ be a point that satisfies $\operatorname{dist}(X_I, \partial\Omega_{\mathcal{F},Q_0}) = |X_I - y|$, then $B(y, r_Q) \supseteq I \supseteq B(X_I, \tilde{c}r_Q)$, where $\tilde{c}$ and r_Q are as in the statement of the proposition. Moreover, for any $x \in Q$, note that

$$\begin{aligned}|x - X_I| \leq \operatorname{diam} Q + \operatorname{dist}(I, Q) + \operatorname{diam} I \leq A_0\ell(Q) + 4\sqrt{n}A_2\ell(Q) + \sqrt{n}A_2\ell(Q)& \\ \leq 6\sqrt{n}A_2\ell(Q),&\end{aligned}$$

whence it is easy to see that $B(x, r_Q) \supset B(X_I, \tilde{c}r_Q)$. Letting $X_Q = X_I$ and $y_Q = y$ finishes the proof. $\square$

Owing to (2.4.30), when $Q = Q_0$ in the above proposition, we have

Corollary 2.4.42 (A uniform Corkscrew point)**.** *The point X_{Q_0} given by Proposition 2.4.41 is a Corkscrew point (with Corkscrew constant $\tilde{c} = \frac{1}{12\sqrt{n}}a_2A_2^{-1}$) with respect to $\Delta_\star(y, r_{Q_0})$ for all $y \in \partial\Omega_{\mathcal{F},Q_0}$, and for $\Delta(x, r_{Q_0})$, for all $x \in Q_0$, with $r_{Q_0} = 7\sqrt{n}A_2\ell(Q_0)$.*

The next lemma establishes a fatness of the region "hidden" by the sawtooth boundary; hence the next result has a similar spirit to (i) of Lemma 2.4.38.

Lemma 2.4.43 (Quantitative fatness of hidden regions, Lemma 5.9 in [HM14])**.** *Let $\mathcal{F} \subset \mathbb{D}$ be a disjoint family. Then for every $Q \subseteq Q_j \in \mathcal{F}$, there is a ball $B' \subset \mathbb{R}^n \backslash \Omega_{\mathcal{F}}$, centered at Γ, with radius $r' \approx \ell(Q)$, and $\Delta' := B' \cap \Gamma \subset Q$. In fact, we may take*

$B' = B(x_Q, r')$ *and* $r' = a_0\ell(Q)/(5A_2a_2^{-1})$.

Proof. Recall that $\Delta_Q = B(x_Q, a_0\ell(Q)) \cap \Gamma \subset Q$. Let $B_M := B(x_Q, a_0\ell(Q)/M)$, where $M = 5A_2a_2^{-1}$. We claim that B_M is the ball B' with the desired properties. We need only check that $B_M \subset \mathbb{R}^n \backslash \Omega_{\mathcal{F}}$, and we proceed via proof by contradiction. Thus suppose that there exists $I \in \mathcal{W}_{\mathcal{F}}$ with $I^* \cap B_M \neq \varnothing$, so that we may find $Y \in I^* \cap B_M$ and $Q_I \in \mathbb{D}_{\mathcal{F}}$ with $I \in \mathcal{W}_{Q_I}$. Then $\delta(y) < a_0\ell(Q)/M$, and therefore

$$\operatorname{diam} I^* \leq 2 \operatorname{diam} I \leq \operatorname{dist}(I, \Gamma)/2 \leq \operatorname{dist}(I^*, \Gamma) \leq \delta(y) < a_0\ell(Q)/M,$$
$$\operatorname{dist}(I^*, Q_I) \leq \operatorname{dist}(I, Q_I) \leq 4\sqrt{n}A_2\ell(Q_I) \leq 4\sqrt{n}A_2a_2^{-1}\ell(I) \leq 2A_2a_2^{-1}a_0\ell(Q)/M.$$

It follows that

$$\operatorname{dist}(Q_I, x_Q) \leq \operatorname{dist}(I^*, Q_I) + \operatorname{diam} I^* + |y - x_Q| \leq \tfrac{4A_2a_2^{-1}}{M} a_0\ell(Q),$$

and so for any $q_I \in Q_I$, we have that

$$\begin{aligned} |q_I - x_Q| \leq \operatorname{diam} Q_I + \operatorname{dist}(Q_I, x_Q) &\leq \tfrac{A_0a_2^{-1}}{\sqrt{n}} \operatorname{diam} I + \tfrac{4A_2a_2^{-1}}{M} a_0\ell(Q) \\ &\leq \tfrac{9A_2a_2^{-1}/2}{M} a_0\ell(Q) < a_0\ell(Q), \end{aligned}$$

which implies that $Q_I \subset \Delta_Q \subset Q \subset Q_j \in \mathcal{F}$, but this is a contradiction to the assumption that $Q_I \in \mathbb{D}_{\mathcal{F}}$. The desired result ensues. $\square$

We would now like to fix an $(n-1)-$ dimensional rectangle of the boundary of the Carleson region R_{Q_j} which is morally a "lift" of Δ_{Q_j}. The precise statement is as follows.

Proposition 2.4.44 (Lift of Δ_{Q_j}, Proposition 6.7 in [HM14])**.** *Let $\mathcal{F}$ be a disjoint family. Then for each $Q_j \in \mathcal{F}$, there is an $(n-1)-$dimensional cube $P_j \subset \partial\Omega_{\mathcal{F}}$, which is contained in a face of I^* for some $I \in \mathcal{W}$, and that satisfies*

$$\ell(P_j) \approx \operatorname{dist}(P_j, Q_j) \approx \operatorname{dist}(P_j, \Gamma) \approx \ell(I) \approx \ell(Q_j), \tag{2.4.45}$$

with uniform constants.

Proof. Fix $Q_j \in \mathcal{F}$ and let $\hat{Q}$ be its proper parent, so that $\ell(Q_j) < \ell(\hat{Q}) \leq c_{\mathbb{K}}^{-1}\ell(Q_j)$ and $\hat{Q} \in \mathbb{D}_{\mathcal{F}}$. Let $I \in \mathcal{W}_{\hat{Q}}^{\mathrm{cs}}$ (see (2.4.5)), and in particular $\operatorname{int} I^* \subset \Omega_{\mathcal{F}}$. On the other hand, by Lemma 2.4.43, the ball $B = B(x_{Q_j}, a_0 a_2 \ell(Q_j)/(5A_2))$ lies in $\mathbb{R}^n \backslash \Omega_{\mathcal{F}}$, so that if X is a Corkscrew point (with Corkscrew constant c) for the surface ball $B \cap \Gamma$, then there exists $J \in \mathcal{W}_{Q_j}$ such that $X \in J$ and $J \backslash \Omega_{\mathcal{F}} \neq \varnothing$, whence we have that $\Omega_{\mathcal{F}} \cap \frac{1}{2} J = \varnothing$.

Let X_I and X_J be the centers of I and J, respectively. We will connect these points via a Harnack Chain. Reckon the estimates

$$\tfrac{2}{41} a_0 c \ell(Q_j) \leq \delta(X_I) \leq \tfrac{41}{4} a_0 c_{\mathbb{K}}^{-1} \ell(Q_j),$$
$$\tfrac{1}{100} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \delta(X_J) \leq \tfrac{1}{4} a_0 a_2 A_2^{-1} \ell(Q_j),$$

$$\begin{aligned} |X_I - X_J| &\leq \operatorname{diam} I + \operatorname{dist}(I, \hat{Q}) + \operatorname{diam} \hat{Q} + |x_{Q_j} - X| + |X - X_J| \\ &\leq \tfrac{1}{4} a_0 \ell(\hat{Q}) + a_0 \ell(\hat{Q}) + A_0 \ell(\hat{Q}) + \tfrac{1}{4} a_0 a_2 A_2^{-1} \ell(Q_j) \leq 4 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j) \\ &\leq \left[\frac{400 A_2^2}{a_2^2 c c_{\mathbb{K}}} \right] \left(\tfrac{1}{100} a_0 a_2 c A_2^{-1} \ell(Q_j) \right). \end{aligned}$$

We thus apply Lemma 2.2.10 with $s := \frac{1}{100} a_0 a_2 c A_2^{-1} \ell(Q_j)$ and $\Lambda := 400 A_2^2 / (a_2^2 c c_{\mathbb{K}})$ and fix a well-tempered Harnack Chain $\mathcal{H}$ connecting X_I and X_j with Λ, s as above. Since $X_I \in \operatorname{int} \Omega_{\mathcal{F}}$, $X_J \in \operatorname{int}(\mathbb{R}^n \backslash \Omega_{\mathcal{F}})$, and $\mathcal{H}$ is centered on a straight line segment, it follows that $\mathcal{H}$ intersects $\partial \Omega_{\mathcal{F}}$ at some $Z \in \partial \Omega_{\mathcal{F}}$. By the construction of the well-tempered Harnack Chain, Z verifies

$$\tfrac{1}{200} c_{\mathcal{H}} \Lambda^{\frac{-d}{n-1-d}} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \delta(Z) \leq 13 a_0 c_{\mathbb{K}}^{-1} \ell(Q_j). \tag{2.4.46}$$

Since $Z \in \partial \Omega_{\mathcal{F}}$, there exists $I_Z \in \mathcal{W}_{\mathcal{F}}$ such that $Z \in \partial I_Z^*$, which implies by (2.4.20) and (2.4.46) that

$$\tfrac{1}{82\sqrt{n}} \tfrac{1}{200} c_{\mathcal{H}} \Lambda^{\frac{-d}{n-1-d}} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \ell(I_Z) \leq \tfrac{13}{2\sqrt{n}} a_0 c_{\mathbb{K}}^{-1} \ell(Q_j), \tag{2.4.47}$$

$$\tfrac{1}{(50)(82)} c_{\mathcal{H}} \Lambda^{\frac{-d}{n-1-d}} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \operatorname{dist}(I_Z, \Gamma) \leq 260 a_0 c_{\mathbb{K}}^{-1} \ell(Q_j)$$

By Lemma 2.4.34, there exists an $(n-1)-$dimensional cube $P_j \subset \partial I_Z^*$ of side-length $\theta \ell(I_Z)/4$ that contains Z, and hence

$$\tfrac{1}{82\sqrt{n}} \tfrac{1}{800} \theta c_{\mathcal{H}} \Lambda^{\frac{-d}{n-1-d}} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \ell(P_j) \leq \tfrac{13}{8\sqrt{n}} \theta a_0 c_{\mathbb{K}}^{-1} \ell(Q_j) \tag{2.4.48}$$

Since $\operatorname{dist}(I_Z, \Gamma) \leq \operatorname{dist}(P_j, \Gamma) \leq \operatorname{dist}(I_Z, \Gamma) + \operatorname{diam} I_Z^*$, it follows that

$$\tfrac{1}{500} c_{\mathcal{H}} \Lambda^{\frac{-d}{n-1-d}} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \operatorname{dist}(P_j, \Gamma) \leq 300 a_0 c_{\mathbb{K}}^{-1} \ell(Q_j). \tag{2.4.49}$$

Finally, we consider $\operatorname{dist}(P_j, Q_j)$. The lower bound is immediate from (2.4.49) and the fact that $\operatorname{dist}(P_j, Q_j) \geq \operatorname{dist}(P_j, \Gamma)$. As for the upper bound, we first note that $\operatorname{dist}(X_I, Q_j) \geq \operatorname{dist}(X_J, Q_j)$ and $\operatorname{diam} I \geq \operatorname{diam} J$, so that by the construction of the well-tempered Harnack Chain $\mathcal{H}$, we have that

$$\begin{aligned} \operatorname{dist}(I_Z, Q_j) \leq \operatorname{dist}(Z, Q_j) &\leq \tfrac{1}{2} c_{\mathcal{H}} \Lambda^{-\frac{d}{n-1-d}} s + \operatorname{dist}(I, Q_j) + \operatorname{diam} I \\ &\leq \tfrac{1}{2} c_{\mathcal{H}} \Lambda^{-\frac{d}{n-1-d}} s + \operatorname{dist}(I, \hat{Q}) + \operatorname{diam} \hat{Q} + \operatorname{diam} I \leq 4 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j). \end{aligned}$$

Therefore, we conclude that

$$\tfrac{1}{5000} c_{\mathcal{H}} \Lambda^{\frac{-d}{n-1-d}} a_0 a_2 c A_2^{-1} \ell(Q_j) \leq \operatorname{dist}(P_j, Q_j) \leq 11 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j). \tag{2.4.50}$$

The estimates (2.4.47), (2.4.48), (2.4.49), and (2.4.50) readily imply (2.4.45). $\square$

Notation 2.4.51. Let $\mathcal{F}$ be a disjoint family and $Q_j \in \mathcal{F}$. Let $P_j \subset \partial\Omega_{\mathcal{F}}$ be the $(n-1)-$dimensional cube constructed in Proposition 2.4.44 and satsifying (2.4.45). We denote by $x_j^\star$ the *center* of P_j, and we write $r_j := 7\sqrt{n} A_2 \ell(Q_j)$. With these choices, we have that $P_j \subseteq \Delta_\star(x_j^\star, r_j)$ and that $\overline{R_{Q_j}} \subset B(x_j^\star, r_j)$, by an argument similar to the one giving (2.4.30). Moreover, given $Q \in \mathbb{D}_{\mathcal{F}}$ and $y_Q \in \partial\Omega_{\mathcal{F}}$ as in (an analogous global version of) Proposition 2.4.41, then we may choose $\hat{r}_Q \approx r_Q$ (in fact, we may take $\hat{r}_Q = 42\sqrt{n} c_{\mathbb{K}}^{-1} A_2 \ell(Q)$) so that the containment

$$Q \bigcup \Big(\bigcup_{Q_j \in \mathcal{F}: Q_j \subset Q} B(x_j^\star, r_j) \Big) \subset B(y_Q, \hat{r}_Q) \tag{2.4.52}$$

holds. Indeed, it is easy to see that $Q \subset B(y_Q, \hat{r}_Q)$, while if $Q_j \in \mathcal{F}$ with $Q_j \subset Q$, then for any $z \in B(x_j^\star, r_j)$ we use the bound

$$|z - y_Q| \leq \operatorname{diam} B(x_j^\star, r_j) + \operatorname{dist}(P_j, Q_j) + \operatorname{diam} Q + |x_Q - y_Q| \lesssim r_Q.$$

Henceforth, we fix $y_Q, \hat{r}_Q$ as in this paragraph.

We conclude this section with the fact that there is a "lift" of any $\Delta_Q, Q \in \mathbb{D}_{\mathcal{F}}$ which does not intersect Carleson regions of $Q_j \in \mathcal{F}$, Q_j not contained in Q.

Proposition 2.4.53 (A lift of Δ_Q, Proposition 6.12 of [HM14])**.** *Let $\mathcal{F}$ be a disjoint family. For $Q_j \in \mathcal{F}$, let $B(x_j^\star, r_j)$ be the ball described in Notation 2.4.51. Then for each $Q \in \mathbb{D}_{\mathcal{F}}$, there is a surface ball*

$$\Delta_\star^Q := \Delta_\star(x_Q^\star, t_Q) \subset \big(Q \cap \partial\Omega_{\mathcal{F}}\big) \bigcup \Big(\bigcup_{Q_j \in \mathcal{F}: Q_j \subset Q} \big(B(x_j^\star, r_j) \cap \partial\Omega_{\mathcal{F}}\big)\Big),$$

with $t_Q \approx \ell(Q), x_Q^\star \in \partial\Omega_{\mathcal{F}}$, and $\operatorname{dist}(Q, \Delta_\star^Q) \lesssim \ell(Q)$.

Proof. Fix M a large number to be chosen momentarily. We split the proof in two cases.

Case 1. There exists $Q_j \subset Q$, for which $\ell(Q_j) \geq \ell(Q)/M$. In this case, we set $\Delta_\star^Q = \Delta_\star(x_k^\star, \ell(P_j)/2)$, where P_j is the cube established in Proposition 2.4.44.

Case 2. There is no Q_j as in Case 1. In this case, if $Q_j \cap Q \neq \varnothing$, then $Q_j \subset Q$ and $\ell(Q_j) < \ell(Q)/M$.

Sub-case 1. No $Q_j \in \mathcal{F}$ meets $\Delta(x_Q, a_0\ell(Q)/(4\sqrt{M})$. Then, we simply set $\Delta_\star^Q := \Delta(x_Q, \ell(Q)/(4\sqrt{M}))$, and we reckon that $\Delta_\star^Q \subset \Delta_Q \subset Q \cap \partial\Omega_{\mathcal{F}}$ by Proposition 2.4.35.

Sub-case 2. There exists $Q_k \in \mathcal{F}$ which intersects the surface ball $\Delta(x_Q, \frac{a_0\ell(Q)}{4\sqrt{M}})$. We claim that for all M large enough, we have that $P_k \subset B(x_Q, a_0\ell(Q)/(2\sqrt{M}))$. Indeed, suppose that $y \in P_k$. Then, using (2.4.45), we deduce that

$$\begin{aligned}
|y - x_Q| &\leq \operatorname{diam} P_k + \operatorname{dist}(P_k, Q_k) + \operatorname{diam} Q_k + \operatorname{dist}(Q_k, x_Q) \\
&\leq \tfrac{13}{8}\theta a_0 c_{\mathbb{K}}^{-1}\ell(Q_k) + 11A_0 c_{\mathbb{K}}^{-1}\ell(Q_k) + A_0\ell(Q_k) + \tfrac{1}{4\sqrt{M}}a_0\ell(Q) \\
&< 13A_0 c_{\mathbb{K}}^{-1}\tfrac{1}{M}\ell(Q) + \tfrac{1}{4\sqrt{M}}a_0\ell(Q) = \tfrac{13A_0a_0^{-1}c_{\mathbb{K}}^{-1}}{\sqrt{M}}\tfrac{1}{4\sqrt{M}}a_0\ell(Q) + \tfrac{1}{4\sqrt{M}}a_0\ell(Q) \\
&\qquad < \frac{1}{2\sqrt{M}}a_0\ell(Q),
\end{aligned}$$

provided that $M > (13A_0a_0^{-1}c_{\mathbb{K}}^{-1})^2$. Thus the claim is shown, and accordingly, we have the inclusion $\Delta_\star^Q := B\big(x_k^\star, \frac{a_0\ell(Q)}{2\sqrt{M}}\big) \cap \partial\Omega_{\mathcal{F}} \subseteq B\big(x_Q, \frac{a_0\ell(Q)}{\sqrt{M}}\big) \cap \Omega_{\mathcal{F}}$. In particular, note that $\Gamma \cap \Delta_\star^Q \subseteq Q$. It remains only to show that $\Delta_\star^Q$ has the desired properties. To do this, we claim that the inclusion

$$\partial\Omega_{\mathcal{F}} \subseteq (\partial\Omega_{\mathcal{F}} \cap \Gamma) \bigcup \Big(\bigcup_{Q_j \in \mathcal{F}} \big(\partial\Omega_{\mathcal{F}} \cap \overline{R_{Q_j}}\big)\Big) \tag{2.4.54}$$

holds. Indeed, consider the following elementary set-theoretic calculations:

$$\mathbb{R}^n \backslash \Omega_{\mathcal{F}} = \bigcup_{\mathcal{W}} I^* \backslash \operatorname{int}\Big(\bigcup_{\mathcal{W}_{\mathcal{F}}} I^* \Big) \subseteq \bigcup_{\mathcal{W}\backslash\mathcal{W}_{\mathcal{F}}} I^* = \bigcup_{\mathcal{R}_{Q_j}, Q_j \in \mathcal{F}} I^* \subseteq \bigcup_{Q_j \in \mathcal{F}} \overline{R_{Q_j}},$$

$$\partial\Omega_{\mathcal{F}} \backslash \Gamma = \partial(\mathbb{R}^n \backslash \Omega_{\mathcal{F}}) \backslash \Gamma \subset \big(\overline{\textstyle\bigcup_{Q_j \in \mathcal{F}} \overline{R_{Q_j}}}\big) \backslash \Gamma = \big(\textstyle\bigcup_{Q_j \in \mathcal{F}} \overline{R_{Q_j}}\big) \backslash \Gamma,$$

where in the last equality we used that the boundary points of $\bigcup_{Q_j \in \mathcal{F}} \overline{R_{Q_j}}$ which are not contained in the union, necessarily lie in Γ. From these calculations, (2.4.54) follows.

Since $B(x_j^\star, r_j) \supset \overline{R_{Q_j}}$, (2.4.54) holds, and $\Gamma \cap \Delta_\star^Q \subset Q$, we will have the desired result as soon as we show that for $Q_j \in \mathcal{F}$, if $\overline{R_{Q_j}}$ meets $B(x_Q, a_0 \ell(Q)/\sqrt{M}) \supset \Delta_\star^Q$, then $Q_j \subseteq Q$. Thus we show the latter. Suppose that $\overline{R_{Q_j}} \cap B(x_Q, a_0\ell(Q)/\sqrt{M}) \neq \varnothing$, whence there exists $Q' \in \mathbb{D}_{Q_j}$, $I \in \mathcal{W}_{Q'}$ and $X \in \overline{I^*}$ such that $X \in \overline{I^*} \cap B(x_Q, a_0\ell(Q)/\sqrt{M})$. Thus, we note that $\delta(X) \leq |X - x_Q|$, and

$$\begin{aligned} \operatorname{dist}(Q', x_Q) &\leq \operatorname{diam} Q' + \operatorname{dist}(I, Q') + \operatorname{diam} I^* + |X - x_Q| \\ &\leq A_0\ell(Q') + 4\sqrt{n}A_2\ell(Q') + 2\sqrt{n}\ell(I) + \tfrac{a_0\ell(Q)}{\sqrt{M}} \\ &\leq \tfrac{A_0 a_2^{-1}}{2\sqrt{n}}\delta(X) + 2A_2 a_2^{-1}\delta(X) + \delta(X) + \tfrac{a_0\ell(Q)}{\sqrt{M}} \leq \tfrac{5A_2 a_2^{-1}}{\sqrt{M}} a_0\ell(Q) < a_0\ell(Q), \end{aligned}$$

provided that $M > (5A_2 a_2^{-1})^2$. It follows that $Q' \subset \Delta_Q \subset Q$, which implies that $Q_j \cap Q \neq \varnothing$, and so $Q_j \subset Q$ since $Q \in \mathbb{D}_{\mathcal{F}}$. As explained above, this calculation ends the proof. $\square$

2.5 A surface measure on the boundary of a mixed-dimensional sawtooth

The goal of this section is to construct a non-negative locally finite Borel measure $\sigma_\star$ on $\partial\Omega_{\mathcal{F}}$ (see (2.4.29)) which is doubling and well-suited to work with the elliptic PDE theory of [DFM19b]; so that it supplants the role of a "surface measure" on the boundary of the sawtooth domain. When $d < n-1$ and $\mathcal{F} \neq \varnothing$, $\partial\Omega_{\mathcal{F}}$ is necessarily of mixed dimension. In [DFM], the authors established an axiomatic elliptic PDE theory for domains with boundaries of mixed dimension. Recall that m is the non-negative Borel measure on Ω given by $m(E) = \iint_E w(X)\, dX$, where $w(X) = \delta(X)^{-n+d+1}$. We must construct $\sigma_\star$

so that the triple $(\Omega_{\mathcal{F}}, m, \sigma_\star)$ satisfies the axioms (H1)-(H6) outlined in [DFM].

Our candidate for the measure $\sigma_\star$ on $\partial\Omega_{\mathcal{F}}$ is defined as follows: for each Borel set $E \subset \partial\Omega_{\mathcal{F}}$, let

$$\sigma_\star(E) = \mathscr{H}^d|_\Gamma(E \cap \Gamma) + \int_{E\setminus\Gamma} \operatorname{dist}(X,\Gamma)^{d+1-n}\, \mathrm{d}\mathscr{H}^{n-1}|_{\partial\Omega_{\mathcal{F}}\setminus\Gamma}(X). \tag{2.5.1}$$

Recall that $\Sigma = \partial\Omega_{\mathcal{F}}\setminus\Gamma$. We see that $\sigma_\star = \sigma|_{\Gamma\cap\partial\Omega_{\mathcal{F}}} + \sigma_\star|_\Sigma$, where $\sigma|_{\Gamma\cap\partial\Omega_{\mathcal{F}}} = \mathscr{H}^d|_{\Gamma\cap\partial\Omega_{\mathcal{F}}}$ and $\sigma_\star|_\Sigma$ are mutually singular, and

$$\sigma_\star|_\Sigma(E) := \int_{E\setminus\Gamma} \operatorname{dist}(X,\Gamma)^{d+1-n}\, \mathrm{d}\mathscr{H}^{n-1}|_{\partial\Omega_{\mathcal{F}}\setminus\Gamma}(X).$$

Theorem 2.5.2 (Sawtooth domains admit an elliptic PDE theory)**.** *The triple $(\Omega_{\mathcal{F}}, m, \sigma_\star)$ satisfies the following axioms.*

(H1) There exists a constant $c_1 > 0$ such that for each $x \in \partial\Omega_{\mathcal{F}}$ and each $r > 0$, there exists a point $X \in B(x,r)$ satisfying that $B(X, c_1 r) \subset \Omega_{\mathcal{F}}$.

(H2) There exists a positive integer $C_2 = N+1$ such that for each $X_1, X_2 \in \Omega_{\mathcal{F}}$ with $\delta_\star(X_i) > r$, $i = 1,2$, and $|X - Y| \leq 7c_1^{-1}r$, there exist $N+1$ points $Z_0 := X_1$, $Z_2, \ldots, Z_N := X_2$ in $\Omega_{\mathcal{F}}$ and verifying $|Z_i - Z_{i+1}| < \delta_\star(X)/2$.

(H3) The support of $\sigma_\star$ is $\partial\Omega_{\mathcal{F}}$, and $\sigma_\star$ is doubling. That is, there exists a constant $C_3 > 1$ such that for each $x \in \partial\Omega_{\mathcal{F}}$ and each $r > 0$,

$$\sigma_\star(\Delta_\star(x,2r)) \leq C_3 \sigma_\star(\Delta_\star(x,r)),$$

where $\Delta_\star$ is defined in Section 2.4.2.

(H4) The measure m is mutually absolutely continuous with respect to the Lebesgue measure; that is, there exists a weight $\tilde{w} \in L^1_{\mathrm{loc}}(\Omega_{\mathcal{F}})$ which is positive Lebesgue-a.e. in $\Omega_{\mathcal{F}}$, and such that for each Borel set $E \subset \Omega_{\mathcal{F}}$, we may write $m(E) = \iint_E \tilde{w}(X)\, dX$. In addition, m is doubling in $\overline{\Omega_{\mathcal{F}}}$, so that there exists a constant C_4 such that for each $X \in \overline{\Omega_{\mathcal{F}}}$ and each $r > 0$, we have that

$$m(B(X,2r) \cap \Omega_{\mathcal{F}}) \leq C_4 m(B(X,r) \cap \Omega_{\mathcal{F}}).$$

(H5) For each $x \in \partial\Omega_{\mathcal{F}}$ and each $r > 0$, the function ρ given by

$$\rho(x,r) := \frac{m(B(x,r) \cap \Omega_{\mathcal{F}})}{r\sigma_\star(\Delta_\star(x,r))}$$

verifies for some constant C_5 and $\varepsilon := 1/C_5$ that

$$\frac{\rho(x,r)}{\rho(x,s)} \leq C_5 \Big(\frac{r}{s}\Big)^{1-\varepsilon}, \qquad \textit{for each } x \in \partial\Omega_{\mathcal{F}}, \quad 0 < s < r.$$

(H6) If D is compactly contained in $\Omega_{\mathcal{F}}$ and $u_i \in C^\infty(\overline{D})$ is a sequence of functions such that $\iint_D |u_i|\, dm \to 0$ and $\iint_D |\nabla u_i - v|^2\, dm \to 0$ as $i \to \infty$, where v is a vector-valued function in $L^2(D, dm)$, then $v \equiv 0$.

Roadmap to the proof of Theorem 2.5.2. The proof of the theorem is split into several parts. First, we check the quantitative properties of $\Omega_{\mathcal{F}}$, hence in Proposition 2.5.3 below we show that the Corkscrew property (H1) holds, and in Proposition 2.5.6 we see that the Harnack Chain property (H2) holds. We then explore the d-ADR -"like" properties of $\sigma_\star$ in Propositions 2.5.9 and 2.5.28, on which we base our verifications of (H3) in Proposition 2.5.38 and of (H5) in Proposition 2.5.40. Finally, we justify in Remark 2.5.41 that (H4) and (H6) are easy consequences of the previously established results in [DFM19b] and the existence of interior Corkscrews for $\partial\Omega_{\mathcal{F}}$.

As stated, let us show that $\Omega_{\mathcal{F}}$ enjoys the properties (H1) and (H2).

Proposition 2.5.3 (Existence of Corkscrew points for the sawtooth domain)**.** *The sawtooth domain $\Omega_{\mathcal{F}}$ has property (H1). More precisely, for each $x \in \partial\Omega_{\mathcal{F}}$ and each $r > 0$, there exists a point $X \in B(x,r)$ such that $B(X, c_1 r) \subset \Omega_{\mathcal{F}} \cap B(x,r)$, where $c_1 > 0$ is the uniform small constant given in (2.5.4) below.*

Proof. Fix $x \in \partial\Omega_{\mathcal{F}}$ and $r > 0$. Suppose first that $x \in \Gamma \cap \partial\Omega_{\mathcal{F}}$. Fix $k \in \mathbb{Z}$ that satisfies $\frac{r}{4A_0 c_{\mathbb{K}}^{-1}} \leq 2^{-k} < \frac{r}{2A_0 c_{\mathbb{K}}^{-1}}$, and by Lemma 2.4.37, there exists $Q \in \mathbb{D}_{\mathcal{F}}$ with $2^{-k} \leq \ell(Q) \leq c_{\mathbb{K}}^{-1} 2^{-k}$ and verifying $x \in \overline{Q}$. We observe that $B(x_Q, a_0\ell(Q)) \subset B(x,r)$: let Y be an arbitrary element of the former, and consider the estimate

$$|Y - x| \leq |Y - x_Q| + |x_Q - x| \leq a_0\ell(Q) + \operatorname{diam} Q \leq (a_0 + A_0)\ell(Q) \leq 2A_0 c_{\mathbb{K}}^{-1} 2^{-k} < r,$$

as desired. Now, according to Corollary 2.4.7, there exists $I \in \mathcal{W}_Q^{\mathrm{cs}}$ (see (2.4.5)) such that its center X_I is a Corkscrew point for Q with Corkscrew constant $\tilde{c} = \frac{c}{1000\sqrt{n}}$. Moreover, $B(X_I, \tilde{c}a_0\ell(Q)) \subset \operatorname{int}(\frac{1}{2}I)$ and therefore $B(X_I, \tilde{c}a_0\ell(Q)) \subset \Omega_{\mathcal{F}}$. Reckon that $\tilde{c}a_0\ell(Q) \geq \frac{\tilde{c}c_{\mathbb{K}}a_0}{4A_0}r =: c_{11}r$, whence the ball $B(X_I, c_{11}r) \subset \Omega_{\mathcal{F}} \cap B(x,r)$ has the desired properties.

We now consider the case that $x \in \Sigma = \partial\Omega_{\mathcal{F}}\backslash\Gamma$. In this case, $\delta(x) > 0$ and consequently there exists a Whitney box $I \subset \Omega_{\mathcal{F}}$ such that $x \in \partial I^*$, which we now fix. We split into two cases: either $r \leq 10A_2a_2^{-1}\delta(x)$, or not.

We resolve the former case first. Since I^* is an $n-$cube, $\operatorname{int} I^* \subset \Omega_{\mathcal{F}}$, and $x \in \partial\Omega_{\mathcal{F}}$, it follows that the ray R containing the line segment $[x, X_I]$ has a non-empty intersection with $B(x,r)$. If $r \leq 2\operatorname{diam} I$, then take the unique point $Y \in R$ such that $|Y - x| = r/(4\sqrt{n})$. This point satisfies $|Y - x| \leq \ell(I)/2 < |X_I - x|$ since $x \in \partial I^*$, and therefore $Y \in [x, X_I] \subset I^*$. Note also that $\operatorname{dist}(Y, \partial I^*) \geq r/(4n)$. Hence the ball $B(Y, r/(8n)) = B(Y, c_{12}r)$ has the desired properties. If, instead, $r > 2\operatorname{diam} I$, then $I^* \subset B(x,r)$ and it follows that $B(X_I, \ell(I)/4) \subset B(x,r)$. On the other hand, owing to (2.4.20) we have that $\ell(I) \geq \frac{\delta(x)}{82\sqrt{n}} \geq \frac{a_2}{820A_2\sqrt{n}}r$. Thus the ball $B(X_I, a_2r/(4000\sqrt{n}A_2)) = B(X_I, c_{13}r)$ has the desired properties.

It remains only to consider the case that $r > 10A_2a_2^{-1}\delta(x)$. In this case, let $Q \in \mathbb{D}_{\mathcal{F}}$ be a dyadic cube such that $I \in \mathcal{W}_Q$, which we now fix. Observe that $\ell(Q) \leq \frac{r}{20\sqrt{n}A_2}$, and that for any generation $k \leq k(Q)$, there is a unique $Q_k \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}^k$ which contains Q. Let k be the unique generation such that $\frac{r}{20\sqrt{n}A_2} \leq 2^{-k} < \frac{r}{10\sqrt{n}A_2}$, and choose Q_k as above. According to Corollary 2.4.7, there exists a point $X_k \in \Omega$ which is the center of some Whitney box $I_k \in \mathcal{W}_{Q_k}^{\text{cs}}$ and is also a Corkscrew point for Q_k with Corkscrew constant $\tilde{c}$. Let us see that with our choice of constants, we have that $B(X_k, \tilde{c}a_0\ell(Q_k) \subset B(x,r)$. Fix $Y \in B(X_k, \tilde{c}a_0\ell(Q_k)) \subset \frac{1}{2}I$, and consider the estimate

$$
\begin{aligned}
|Y - x| &\leq \operatorname{diam} I_k + \operatorname{dist}(I_k, Q_k) + \operatorname{diam} Q_k + \operatorname{dist}(I^*, Q) + \operatorname{diam} I^* \\
&\leq \sqrt{\ell}(I_k) + a_0\ell(Q_k) + A_0\ell(Q_k) + 4\sqrt{n}A_2\ell(Q) + 2\operatorname{diam} I \\
&\leq (5a_0/4 + A_0)\ell(Q_k) + 6\sqrt{n}A_2\ell(Q) \leq 9\sqrt{n}A_2\ell(Q_k) < 10\sqrt{n}A_2\ell(Q_k) < r,
\end{aligned}
$$

as desired. Note that $\tilde{c}a_0\ell(Q_k) \geq \tilde{c}a_0\frac{r}{20\sqrt{n}A_2}$, so that $B(X_k, \frac{\tilde{c}a_0}{20\sqrt{n}A_2}r) =: B(X_k, c_{14}r)$ is a ball with the desired properties.

Finally, take

$$c_1 = \min\left\{c_{11}, c_{12}, c_{13}, c_{14}\right\}$$
$$= \min\left\{\frac{\tilde{c}c_{\mathbb{K}}a_0}{4A_0}, \frac{1}{8n}, \frac{a_2}{4000\sqrt{n}A_2}, \frac{\tilde{c}a_0}{20\sqrt{n}A_2}\right\} \quad (2.5.4)$$

and reckon that the desired result is proved. □

Remark 2.5.5. We can generalize the result of Proposition 2.5.3 as follows: for each $X \in \overline{\Omega_{\mathcal{F}}}$ and $r > 0$, there exists $\tilde{X} \in B(X, r) \cap \Omega_{\mathcal{F}}$ and $c \in (0, 1)$ (in fact, we can take $c = c_1/4$, where c_1 is the constant in (2.5.4)) such that $B(\tilde{X}, cr) \subset B(X, r) \cap \Omega_{\mathcal{F}}$. If $X \in \partial\Omega_{\mathcal{F}}$, the claim is a direct consequence of Proposition 2.5.3. If $X \in \Omega_{\mathcal{F}}$, we have two cases: either $\delta_\star(X) \geq r/2$ or $\delta_\star(X) < r/2$. In the former case, the claim is trivial (we can take $\tilde{X} = X$ and $c = 1/4$), thus we focus on the case that $\delta_\star(X) < r/2$. In this case, there exists $x \in \partial\Omega_{\mathcal{F}}$ such that $|X - x| = \delta_\star(X) < r/2$. We let $\tilde{X} \in \Omega_{\mathcal{F}}$ be the Corkscrew point (with Corkscrew constant c_1) for the ball $B(x, r/4)$ which is constructed in Proposition 2.5.3. Hence $B(\tilde{X}, \frac{c_1 r}{4}) \subset \Omega_{\mathcal{F}}$. Now, if $Y \in B(\tilde{X}, c_1 r/4)$, then

$$|Y - X| \leq |Y - \tilde{X}| + |\tilde{X} - x| + |x - X| < \tfrac{c_1 r}{4} + \tfrac{r}{4} + \tfrac{r}{2} = r,$$

which shows that $B(\tilde{X}, \frac{c_1 r}{4}) \subset B(X, r)$. The claim ensues.

Proposition 2.5.6 (Harnack Chains in the sawtooth domain)**.** *The sawtooth domain $\Omega_{\mathcal{F}}$ has the Harnack Chain property of Definition 2.2.9, with $\tilde{C} = \tilde{C}(n, d, C_d, \theta)$. In particular, $\Omega_{\mathcal{F}}$ has property (H2).*

Proof. Fix $X_1, X_2 \in \Omega_{\mathcal{F}}$ with $\delta_\star(X_i) > r$, $i = 1, 2$, and $|X_1 - X_2| \leq \Lambda r$. We seek to join X_1 and X_2 via a Harnack Chain that stays far from $\partial\Omega_{\mathcal{F}}$. Note that for any $Z \in \Omega_{\mathcal{F}}$, $\delta(Z) \geq \delta_\star(Z)$, while in the other direction we have that if $Z \in I$ for some $I \in \mathcal{W}_{\mathcal{F}}$, then $\delta_\star(Z) \geq \frac{1}{2}\theta\ell(I) \geq \frac{1}{164\sqrt{n}}\theta\delta(Z)$. For each $i = 1, 2$, fix $Q_i \in \mathbb{D}_{\mathcal{F}}$ and $I_i \in \mathcal{W}_{Q_i}$ such that $X_i \in \operatorname{int} I_i^*$. If it can be arranged that $I_1 = I_2$, then the result follows immediately from the fact that $\operatorname{int} I^*$ is an $n-$dimensional open cube. Similarly, if $I_1^* \cap I_2^* \neq \varnothing$, then a Harnack Chain connecting X_1 to X_2 can be obtained by noting that $I_1^* \cap I_2^*$ is a union of rectangles with no side-length smaller than $\theta \min_i \ell(I_i)$, whence we may use these intersections to "transfer" from X_1 to X_2 in the manner desired.

Suppose first that the estimate $\delta_\star(X_1) < \frac{\theta}{200\Lambda}\delta(X_1) =: \frac{1}{M}\delta(X_1)$ holds. Then

$$|X_1 - X_2| \le \Lambda r \le \tfrac{\theta}{200}\delta(X_1) \le \tfrac{\theta}{2}\operatorname{diam} I_1 \ll \operatorname{diam} I_1,$$

so that I_1 and I_2 touch, which implies that $I_1^* \cap I_2^* \ne \varnothing$, and as mentioned above, this gives the desired result with $\tilde{C} = \tilde{C}(n, \theta)$.

It remains to obtain the desired conclusion under the supposition that for each $i = 1, 2$, the estimate $\delta_\star(X_i) \ge \frac{1}{M}\delta(X_i)$ holds. In this case, we have that $\delta_\star(X_i) \approx_M \delta(X_i)$, and without loss of generality suppose that $\ell(Q_1) \le \ell(Q_2)$. We may connect each X_i to the respective centers of I_i, X_{I_i}, through Harnack Chains with a uniform number of balls (depending only on M). Hence we have reduced the problem to procuring a Harnack Chain between X_{I_1} and X_{I_2}. We have that Q_1 and Q_2 have comparable length, as follows: first, we have the estimate

$$\delta(X_2) \le |X_1 - X_2| + \delta(X_1) \le \Lambda r + \delta(X_1) \le 2\Lambda\delta(X_1),$$

which gives that $\ell(I_2) \le \frac{41}{2}\Lambda\ell(I_1)$, and on the other hand, for each $i = 1, 2$,

$$n^{-1/2}A_2^{-1}\delta(X_i)/41 \le A_2^{-1}\ell(I_i) \le \ell(Q_i) \le a_2^{-1}\ell(I_i) \le n^{-1/2}a_2^{-1}\delta(X_i)/4,$$

which implies that $\delta(X_1) \le \frac{41}{4}A_2a_2^{-1}\delta(X_2)$, and $\ell(I_1) \le A_2a_2^{-1}\ell(I_2)$. As such, $\ell(Q_1) \le \ell(Q_2) \le 21\Lambda A_2a_2^{-1}\ell(Q_1)$, and, furthermore,

$$\begin{aligned}\operatorname{dist}(Q_1, Q_2) &\le \operatorname{dist}(I_1, Q_1) + \operatorname{diam} I_1 + |X_1 - X_2| + \operatorname{diam} I_2 + \operatorname{dist}(I_2, Q_2) \\ &\le 5\sqrt{n}A_2(\ell(Q_1) + \ell(Q_2)) + \Lambda r \le 51\sqrt{n}A_2\Lambda\ell(Q_2). \quad (2.5.7)\end{aligned}$$

Fix $Q_2^a \in \mathbb{D}_{\mathcal{F}}$ as the unique ancestor of Q_2 that satisfies

$$\frac{51\sqrt{n}A_2\Lambda}{500}\ell(Q_2) \le \ell(Q_2^a) \le c_{\mathbb{K}}^{-1}\frac{51\sqrt{n}A_2\Lambda}{500}\ell(Q_2),$$

and then choose for $Q_1^a \in \mathbb{D}_{\mathcal{F}}$ the ancestor of Q_1 that verifies $c_{\mathbb{K}}\ell(Q_2^a) \le \ell(Q_1^a) \le \ell(Q_2^a)$. By construction, we have that $\ell(Q_1^a) \approx \ell(Q_2^a) \approx \ell(Q_2) \approx \ell(Q_1)$ with uniform constants. By Lemma 2.4.9, we see that Q_2 and its proper parent Q_2' satisfy $\mathcal{W}_{Q_2}^0 \cap \mathcal{W}_{Q_2'}^0 \ne \varnothing$, so that by the construction of $\mathcal{W}_{Q_2}$ in (2.4.15), there exists a Harnack Chain of the desired properties connecting X_{I_2} to some point X_2' lying in $I_2' \in \mathcal{W}_{Q_2'}$. It is easy to see that therefore we may inductively "ascend" through a uniformly finite (since $\ell(Q_2^a) \approx \ell(Q_2)$) sequence of Harnack Chains from X_{I_2} to a point $X_{I_2^a}$ which is the center of a Whitney cube $I_2^a \in \mathcal{W}_{Q_2^a}$. Now, from (2.5.7), we see that

$$\operatorname{dist}(Q_1^a, Q_2^a) \leq \operatorname{dist}(Q_1, Q_2) \leq 500\ell(Q_2^a),$$

so that by Lemma 2.4.10, $\mathcal{W}^0_{Q_1^a} \cap \mathcal{W}^0_{Q_2^a} \neq \varnothing$. Hence we may pass through a Harnack Chain from $X_{I_2^a}$ to a point $X_{I_1^a}$ which is the center of some $I_1^a \in \mathcal{W}_{Q_1^a}$. As before but in reverse, we proceed to "descend" from $X_{I_1^a}$ to X_{I_1} through a uniformly finite (since $\ell(Q_1^a) \approx \ell(Q_1)$) sequence of Harnack Chains. Hence, in this case the desired result is achieved with a constant $\tilde{C} = \tilde{C}(n, d, C_d, \theta)$. $\square$

We turn to a study of the properties of the measure $\sigma_\star$.

Lemma 2.5.8 (Support of $\sigma_\star$). *The measure $\sigma_\star$ is supported on $\partial\Omega_{\mathcal{F}}$.*

Proof. It is clear that $\Gamma \cap \partial\Omega_{\mathcal{F}} \subset \operatorname{supp} \sigma_\star$, so we only need to check that Σ is in the support of $\sigma_\star$. But this is easy: for any bounded open set U intersecting Σ and compactly contained in $\mathbb{R}^n \setminus \Gamma$, the set $\Sigma \cap U$ is contained in a finite union of non-empty $(n-1)-$dimensional rectangles, so that $\mathscr{H}^{n-1}(\Sigma \cap U) \in (0, \infty)$, and $\delta(X) \in (0, +\infty)$ for any $X \in \Sigma$. The claim ensues. $\square$

Proposition 2.5.9 (Upper bound for $\sigma_\star$). *Let $x \in \partial\Omega_{\mathcal{F}}$ and $r > 0$. Then*

$$\sigma_\star(\Delta_\star(x, r)) \leq V_1 r^d, \tag{2.5.10}$$

where $V_1 = V_1(n, d, C_d, a_0, A_0, \zeta, c, c_{\mathcal{H}})$. Moreover, if $\delta(x) > 0$, then

$$\sigma_\star(\Delta_\star(x, r)) \leq V_2 \delta(x)^{d+1-n} r^{n-1}. \tag{2.5.11}$$

The uniform constant V_2 in the last inequality depends only on n, d, and V_1.

Proof. Fix $x \in \partial\Omega_{\mathcal{F}}$ and $r > 0$. Let $B := B(x, r)$ and recall that $\Sigma = \partial\Omega_{\mathcal{F}} \setminus \Gamma$, $\sigma = \mathscr{H}^d|_\Gamma$. We first prove (2.5.10) by adapting ideas of the proof for Lemma 3.61 from [HM14]. Observe that

$$
\begin{aligned}
\sigma_\star(\Delta_\star(x,r)) &= \sigma_\star(B\cap\Gamma\cap\partial\Omega_{\mathcal F}) + \sigma_\star(B\cap\Sigma)\\
&\le \sigma_\star(B\cap\Gamma) + \sigma_\star(B\cap\Sigma) = \sigma(\Delta(x,r)) + \sigma_\star|_\Sigma(B\cap\Sigma)\\
&\le C_d r^d + \sigma_\star|_\Sigma(B\cap\Sigma).
\end{aligned}
$$

Thus we need only show that $\sigma_\star|_\Sigma(B(x,r)\cap\Sigma) \lesssim r^d$. We will do this by splitting $B\cap\Sigma$ into two parts: one part where the (portions of) faces in $B\cap\Sigma$ correspond to Whitney boxes having small diameter compared to r, and the other part where the (portions of) faces in $B\cap\Sigma$ correspond to Whitney boxes having large diameter compared to r. More precisely, it is clear that any $X\in\Sigma$ lies in the face of a fattened Whitney box J^*, such that $J\in\mathcal W$, $\operatorname{int} J^*\subset\Omega_{\mathcal F}$, and $\partial J^*\cap\partial\Omega_{\mathcal F}\neq\varnothing$. Then there exists a Whitney box $I\in\mathcal W$, with $I\notin\mathcal W_Q$ for any $Q\in\mathbb D_{\mathcal F}$, so that $J^*\cap I\neq\varnothing$ (otherwise, every Whitney box adjacent to J lies in $\mathcal W_{\mathcal F}$, contradicting that $J^*\cap\partial\Omega_{\mathcal F}\neq\varnothing$). Necessarily then, there exists $Q'\in\mathbb D_{Q_j}$ with $Q_j\in\mathcal F$ and verifying that $I\in\mathcal W_{Q'}$. Denote by $\mathcal F_B$ the sub-collection of those $Q_j\in\mathcal F$ such that there exists $I\in\mathcal R_{Q_j}$ (cf. (2.4.31)) intersecting $B\cap\Sigma$. Let $\mathcal F_B=\mathcal F_1\cup\mathcal F_2$ where $Q_j\in\mathcal F_B$ belongs to $\mathcal F_1$ if $\ell(Q_j)<r$, and $\mathcal F_2=\mathcal F_B\setminus\mathcal F_1$. Then, we may write

$$
\begin{aligned}
B\cap\Sigma &= B\cap\Sigma\cap\Big(\bigcup_{Q_j\in\mathcal F_B}\ \bigcup_{I\in\mathcal R_{Q_j}} I\Big)\\
&= \Big(B\cap\Sigma\cap\Big(\bigcup_{Q_j\in\mathcal F_1}\ \bigcup_{I\in\mathcal R_{Q_j}} I\Big)\Big)\bigcup\Big(B\cap\Sigma\cap\Big(\bigcup_{Q_j\in\mathcal F_2}\ \bigcup_{I\in\mathcal R_{Q_j}} I\Big)\Big)\\
&= \Big(\bigcup_{Q_j\in\mathcal F_1}(B\cap\Sigma_j)\Big)\bigcup\Big(\bigcup_{Q_j\in\mathcal F_2}(B\cap\Sigma_j)\Big),
\end{aligned}
\tag{2.5.12}
$$

where $\Sigma_j:=\Sigma\cap(\cup_{I\in\mathcal R_{Q_j}} I)$ for each $Q_j\in\mathcal F$. Our further analysis will be based on the following estimate:

$$
\sigma_\star(B\cap\Sigma_j)\lesssim\big(\min\{r\,,\,\ell(Q_j)\}\big)^d,\qquad\text{for each } Q_j\in\mathcal F. \tag{2.5.13}
$$

Suppose momentarily that (2.5.13) holds, and we will use it to control $\sigma_\star(B\cap\Sigma)$. First, we consider the contribution of $\mathcal F_1$. If $Q_j\in\mathcal F_1$ so that $\ell(Q_j)<r$, we have that $Q_j\in B^*:=B(x,(4a_0K+3A_0)r)$. Indeed, since $Q_j\in\mathcal F_1\subset\mathcal F_B$, it follows that there exists $Q'\in\mathbb D_{Q_j}$ and $I\in\mathcal W_{Q'}$ such that $B\cap I\neq\varnothing$, and thus for any $q\in Q_j$, we have

that

$$|q-x| \leq \operatorname{diam} Q_j + \operatorname{dist}(Q_j, x) \leq A_0 \ell(Q_j) + \operatorname{dist}(I, Q') + \operatorname{diam} I + \operatorname{dist}(I, x)$$
$$\leq A_0 \ell(Q_j) + (2a_0 K + A_0)\ell(Q') + \tfrac{2a_0 K + A_0}{4\sqrt{n}} \ell(Q') + r \; < (4a_0 K + 3A_0) r,$$

where we used that $\ell(Q') \leq \ell(Q_j) < r$. Then,

$$\sigma_\star\Big(\bigcup_{Q_j \in \mathcal{F}_1} (B \cap \Sigma_j)\Big) \leq \sum_{Q_j \in \mathcal{F}_1} \sigma_\star(B \cap \Sigma_j) \leq C \sum_{Q_j \in \mathcal{F}_1} \ell(Q_j)^d \leq C C_d a_0^{-d} \sum_{Q_j \in \mathcal{F}_1} \sigma(Q_j)$$
$$\leq C C_d a_0^{-d} \sigma\big(B^* \cap \Gamma\big) \leq C C_d^2 \big(4K + 3\tfrac{A_0}{a_0}\big)^d r^d, \quad (2.5.14)$$

where C is the uniform constant implicit in (2.5.13), and in the second inequality we used (2.5.13), in the third inequality we used (2.3.7), in the fourth inequality we used that the cubes $Q_j \subset B^*$ are disjoint.

Next we turn to the contribution of $\mathcal{F}_2$, still supposing that (2.5.13) holds. We begin by proving that the cardinality of $\mathcal{F}_2$ is uniformly bounded. Suppose that Q_j and Q_k belong to $\mathcal{F}_2$, so that there exist Whitney boxes $I_j \in \mathcal{R}_{Q_j}, I_k \in \mathcal{R}_{Q_k}$ intersecting $B \cap \Sigma$, and without loss of generality we may assume that $\ell(Q_k) \leq \ell(Q_j)$. Observe that for $i = j, k$ we have that $\ell(I_i) \leq \frac{2a_0 K + A_0}{4\sqrt{n}} \ell(Q_i)$ since $I_i \in \mathcal{R}_{Q_i}$. Also note that $\operatorname{dist}(I_j, I_k) \leq \operatorname{diam} B = 2r \leq 2\ell(Q_k)$, and moreover

$$4 \operatorname{diam} I_j \leq \operatorname{dist}(I_j, \Gamma) \leq \operatorname{dist}(I_j, I_k) + \operatorname{diam} I_k + \operatorname{dist}(I_k, \Gamma)$$
$$\leq 2\ell(Q_k) + \tfrac{41}{4}(2a_0 K + A_0)\ell(Q_k) \leq A(a_0 K + A_0)\ell(Q_k),$$

where $A \geq 1$ is a large real number with no dependence on any parameter. Now, we have

$$\operatorname{dist}(Q_j, Q_k)$$
$$\leq \operatorname{diam} Q_k + \operatorname{dist}(I_k, Q_k) + \operatorname{diam} I_k + \operatorname{dist}(I_k, I_j) + \operatorname{diam} I_j + \operatorname{dist}(I_j, Q_j)$$
$$\leq A(a_0 K + A_0) a_1^{-1} \eta^{-\frac{n-1}{n-1-d}} \ell(Q_k),$$

where $a_1 = a_1(n, d, C_d, c, c_{\mathcal{H}}, K)$ is the quantity defined in (2.4.12). Thus, we have shown that for any $Q_j, Q_k \in \mathcal{F}_2$, the estimate

$$\operatorname{dist}(Q_j, Q_k) \leq A(a_0K + A_0)a_1^{-1}\eta^{-\frac{n-1}{n-1-d}} \min\big\{\ell(Q_j), \ell(Q_k)\big\} =: A_1 \min\big\{\ell(Q_j), \ell(Q_k)\big\} \tag{2.5.15}$$

holds. Let us see that (2.5.15) implies the uniform boundedness of $\operatorname{card} \mathcal{F}_2$. Since for all $Q_k \in \mathcal{F}_2$ we have that $\ell(Q_k) \geq r$ by definition, then we may choose $Q_j \in \mathcal{F}_2$ so that $\ell(Q_k) \geq \ell(Q_j)$ for all $Q_k \in \mathcal{F}_2$. Fix such a $Q_j \in \mathcal{F}_2$, and reckon that by (2.5.15) and (2.3.6), for each $Q_k \in \mathcal{F}_2$ the set $Q_k \cap \Delta(x_{Q_j}, (A_0 + A_1)\ell(Q_j))$ is not empty. Accordingly, for each $Q_k \in \mathcal{F}_2$, there exists a dyadic cube $Q'_k \in \mathbb{D}_{Q_k}$ such that $c_{\mathbb{K}}\ell(Q_j) \leq \ell(Q'_k) \leq \ell(Q_j)$ and $Q'_k \subset \Delta(x_{Q_j}, 3A_1\ell(Q_j))$. We consider the estimate

$$\begin{aligned} c_{\mathbb{K}} C_d^{-1} a_0^d \ell(Q_j)^d \operatorname{card} \mathcal{F}_2 &\leq \sum_{Q_k \in \mathcal{F}_2} \sigma\big(\Delta(x_{Q'_k}, a_0\ell(Q'_k))\big) \leq \sum_{Q_k \in \mathcal{F}_2} \sigma(Q'_k) \\ &\leq \sigma(\Delta(x_{Q_j}, 3A_1\ell(Q_j)) \leq C_d(3A_1)^d \ell(Q_j)^d, \end{aligned}$$

and hence obtain that $\operatorname{card} \mathcal{F}_2 \leq c_{\mathbb{K}}^{-1} C_d^2 \big[3a_0^{-1}A_1\big]^d$. Therefore, we may conclude, using (2.5.13), that

$$\begin{aligned} \sigma_\star\big(\textstyle\bigcup_{Q_j \in \mathcal{F}_2} (B \cap \Sigma_j)\big) &\leq \sum_{Q_j \in \mathcal{F}_2} \sigma_\star(B \cap \Sigma_j) \leq C_d^2 \big[3a_0^{-1}A_1\big]^d \sup_{Q_j \in \mathcal{F}_2} \sigma_\star(B \cap \Sigma_j) \\ &\leq C_d^2 \big[3a_0^{-1}A_1\big]^d C r^d. \end{aligned} \tag{2.5.16}$$

Putting (2.5.12), (2.5.14), and (2.5.16) together, we obtain the desired result modulo the proof of (2.5.13).

We now turn to the proof of (2.5.13). Hence take $Q_j \in \mathcal{F}$ and first suppose that $\ell(Q_j) \leq Mr$ for some $M > 0$ to be fixed later. In this case, any $I \in \mathcal{R}_{Q_j}$ satisfies $\ell(I) \lesssim \ell(Q_j)$, but this estimate is too crude as there may be too many (in fact, infinitely many!) such boxes intersecting Σ_j. Therefore the idea is to control the number of Whitney boxes I of a given generation that contribute to Σ_j. To this end, recall $A_2 = \frac{2a_0K + A_0}{4\sqrt{n}}$, and define

$$\Sigma_j^k := \Sigma \bigcap \big(\textstyle\bigcup_{\{I \in \mathcal{R}_{Q_j} : \ell(I) = 2^{-k}\}} I\big), \qquad \text{so that} \quad \Sigma_j = \textstyle\bigcup_{\{k : 2^{-k} \leq A_2\ell(Q_j)\}} \Sigma_j^k.$$

Hence,

$$\begin{aligned}\sigma_\star(B\cap\Sigma_j) &= \sum_{k:2^{-k}\le A_2\ell(Q)} \sigma_\star(B\cap\Sigma_j^k)\\ &= \sum_{k:a_3\ell(Q_j)<2^{-k}\le A_2\ell(Q)} \sigma_\star(B\cap\Sigma_j^k) + \sum_{k:2^{-k}<a_3\ell(Q_j)} \sigma_\star(B\cap\Sigma_j^k) =: T_1+T_2,\end{aligned} \tag{2.5.17}$$

where $a_3>0$ is a small constant to be fixed momentarily. We bound term T_1 first. Note that if $X\in\Sigma_j^k$, then there exists $I\in\mathcal{R}_{Q_j}$ with $\ell(I)>a_3\ell(Q_j)$ and such that $X\in I$. Hence $\delta(X)\ge 4\sqrt{n}a_3\ell(Q_j)$. For convenience in the following calculation, set $\mathcal{W}_{\Sigma_j^1} := \{I\in\mathcal{R}_{Q_j} : a_3\ell(Q_j)\le\ell(I)\le A_2\ell(Q_j)\}$, and observe that

$$\begin{aligned}&\sum_{k:a_3\ell(Q_j)<2^{-k}\le A_2\ell(Q)} \sigma_\star(B\cap\Sigma_j^k)\\ &\qquad\le \sum_{k:a_3\ell(Q_j)<2^{-k}\le A_2\ell(Q)} \int_{\Sigma_j^k} \delta(X)^{d+1-n}\,\mathrm{d}\mathscr{H}^{n-1}|_\Sigma(X)\\ &\qquad\le (4\sqrt{n}a_3)^{d+1-n}\ell(Q_j)^{d+1-n}\mathscr{H}^{n-1}\big(\textstyle\bigcup_{I\in\mathcal{W}_{\Sigma_j^1}}(I\cap\Sigma)\big)\\ &\qquad\le a_3^{d+1-n}\ell(Q_j)^{d+1-n}\sum_{I\in\mathcal{W}_{\Sigma_j^1}}\mathscr{H}^{n-1}(I\cap\Sigma)\\ &\qquad\le A_n a_3^{d+1-n}\big[\tfrac{a_0K+A_0}{a_3}\big]^n\ell(Q_j)^{d+1-n}\big(\sup_{I\in\mathcal{W}_{\Sigma_j^1}}\mathscr{H}^{n-1}(I\cap\Sigma)\big),\end{aligned} \tag{2.5.18}$$

where by $A_n\ge 1$ we denote a constant depending only on n, and in the last line we used the bound (2.4.18), since the set $\mathcal{W}_{\Sigma_j^1}$ can easily be seen to be a subset of a set of the form in Lemma 2.4.17. Next, let $I\in\mathcal{W}_{\Sigma_j^1}$, and we seek to bound $\mathscr{H}^{n-1}(I\cap\Sigma)$. Observe that

$$\begin{aligned}\mathscr{H}^{n-1}(I\cap\Sigma) &\le \mathscr{H}^{n-1}\big(\textstyle\bigcup_{J\in\mathcal{W}_\mathcal{F}:J^*\cap I\neq\varnothing}\partial J^*\big)\\ &\le A_n\big(\sup_{J\in\mathcal{W}_\mathcal{F}:J^*\cap I\neq\varnothing}\mathscr{H}^{n-1}(\partial J^*)\big)\\ &\le A_n(1+\theta)^{n-1}\big(\sup_{J\in\mathcal{W}_\mathcal{F}:J^*\cap I\neq\varnothing}\ell(J)^{n-1}\big)\le A_n\ell(I)^{n-1}\le A_nA_2^{n-1}\ell(Q_j)^{n-1}.\end{aligned} \tag{2.5.19}$$

We may combine (2.5.18) with (2.5.19) to see that

$$T_1 \leq A_n a_3^{d+1-2n} \big[a_0 K + A_0\big]^{2n-1} \ell(Q_j)^d, \tag{2.5.20}$$

which is the desired bound for T_1. We remark that the estimates in (2.5.19) and (2.5.20) also allow us to say that for any $I \in \mathcal{W}$, it holds that

$$\sigma_\star(I \cap \Sigma) \leq A_n \ell(I)^d. \tag{2.5.21}$$

Now we bound T_2. Set $\mathcal{W}_{\Sigma_j^k} := \{I \in \mathcal{R}_{Q_j} : \ell(I) = 2^{-k},\ I \cap \Sigma_j^k \neq \varnothing\}$, and observe that

$$\begin{aligned} T_2 \leq \sum_{k: 2^{-k} < a_3 \ell(Q_j)} \sum_{I \in \mathcal{R}_{Q_j} : \ell(I) = 2^{-k}} \sigma_\star(I \cap \Sigma) \\ \leq \sum_{k: 2^{-k} < a_3 \ell(Q_j)} \operatorname{card}(\mathcal{W}_{\Sigma_j^k}) \big(\sup_{I \in \mathcal{W}_{\Sigma_j^k}} \sigma_\star(I \cap \Sigma) \big) \\ \leq A_n \sum_{k: 2^{-k} < a_3 \ell(Q_j)} \operatorname{card}(\mathcal{W}_{\Sigma_j^k}) 2^{-kd}, \end{aligned} \tag{2.5.22}$$

where we used (2.5.21) in the last inequality. Hence, it will suffice to show that for some large uniform constant C and some $\zeta \in (0,1)$, the estimate

$$\operatorname{card}(\mathcal{W}_{\Sigma_j^k}) \leq C(2^k \ell(Q_j))^{d-\zeta} \tag{2.5.23}$$

holds. Let us establish (2.5.23) then. If $I \in \mathcal{W}_{\Sigma_j^k}$, then there exists $Q_I \in \mathbb{D}_{Q_j}$ such that $I \in \mathcal{W}_{Q_I}$, and moreover there exists $J \in \mathcal{W}_{\mathcal{F}}$ so that $I \cap J^* \neq \varnothing$. In particular, there exists $Q' \in \mathbb{D}_{\mathcal{F}}$ such that $J \in \mathcal{W}_{Q'}$. Observe that $\ell(Q') \leq a_2^{-1} \ell(J) \leq 4a_2^{-1} \ell(I) \leq 4a_2^{-1} a_3 \ell(Q_j)$, whence $\ell(Q') < \ell(Q_j)$ provided that $a_3 \leq a_2/8$, which we assume from now on. Since $Q' \in \mathbb{D}_{\mathcal{F}}$ and $Q_j \in \mathcal{F}$, it follows that Q' and Q_j are disjoint. Consequently,

$$\begin{aligned} \operatorname{dist}(Q_I, \Gamma \backslash Q_j) \leq \operatorname{dist}(Q_I, Q') \leq \operatorname{diam} Q_I + \operatorname{dist}(I, Q_I) + \operatorname{diam} I + \operatorname{dist}(J, Q') \\ \leq (A_0 a_2^{-1} + 6\sqrt{n} A_2 a_2^{-1}) \ell(I). \end{aligned} \tag{2.5.24}$$

Since for any $q \in Q_I$ we have that $\operatorname{dist}(x, \Gamma \backslash Q_j) \leq \operatorname{diam} Q_I + \operatorname{dist}(Q_I, \Gamma \backslash Q_j)$, then by using (2.5.24), it follows that

$$Q_I \subset \left\{x \in Q_j \ : \ \operatorname{dist}(x, \Gamma\backslash Q_j) \leq \left[10\sqrt{n}a_2^{-1}A_2 2^{-k}\ell(Q_j)^{-1}\right]\ell(Q_j)\right\} =: V.$$

Observe that V is the set considered in property (vi) of Lemma 2.3.1, with $\rho = 10\sqrt{n}a_2^{-1}A_2 2^{-k}\ell(Q_j)^{-1}$. We may apply the inequality in (vi) so long as $\rho < a_0$, which in our case will be true as long as $a_3 \leq \frac{a_0 a_2}{100\sqrt{n}A_2}$. Henceforth we fix a_3 to be given by the right-hand side of the last inequality, and note that it also satisfies $a_3 \leq a_2/8$. Then, we have that

$$\sigma\big(\textstyle\bigcup_{I\in\mathcal{W}_{\Sigma_j^k}} Q_I\big) \leq \sigma(V) \leq A_0(10\sqrt{n}a_2^{-1}A_2)^{\zeta}2^{-k\zeta}\ell(Q_j)^{-\zeta}\sigma(Q_j). \tag{2.5.25}$$

Now, we reckon the estimate

$$\begin{aligned}
\operatorname{card}(\mathcal{W}_{\Sigma_j^k})2^{-kd} &= \sum_{I\in\mathcal{W}_{\Sigma_j^k}} \ell(I)^d \leq A_2^d \sum_{I\in\mathcal{W}_{\Sigma_j^k}} \ell(Q_I)^d \leq C_d(a_0^{-1}A_2)^d \sum_{I\in\mathcal{W}_{\Sigma_j^k}} \sigma(Q_I) \\
&\leq C_d(a_0^{-1}A_2)^d \sum_{\substack{\tilde{Q}\,\in\,\mathbb{D}_{Q_j}\text{ s.t.}\\ \exists I\,\in\,\mathcal{W}_{\Sigma_j^k}\cap\mathcal{W}_{\tilde{Q}}}} \sum_{I\in\mathcal{W}_{\tilde{Q}}} \sigma(\tilde{Q}) \leq C_d(a_0^{-1}A_2)^d N_0 \sum_{\substack{\tilde{Q}\,\in\,\mathbb{D}_{Q_j}\text{ s.t.}\\ \exists I\,\in\,\mathcal{W}_{\Sigma_j^k}\cap\mathcal{W}_{\tilde{Q}}}} \sigma(\tilde{Q}) \\
&\leq C_d(a_0^{-1}A_2)^d N_0\big[C_d^2(2A_0a_0^{-1})^d\big]^{\frac{A_2}{a_2}}\sigma\Big(\bigcup_{\substack{\tilde{Q}\,\in\,\mathbb{D}_{Q_j}\text{ s.t.}\\ \exists I\,\in\,\mathcal{W}_{\Sigma_j^k}\cap\mathcal{W}_{\tilde{Q}}}} \tilde{Q}\Big) \leq A_3 2^{-k\zeta}\ell(Q_j)^{d-\zeta},
\end{aligned} \tag{2.5.26}$$

where A_3 is a uniform constant. In the fifth inequality, we used Corollary 2.4.19, in the sixth inequality we used (2.3.8) and the fact that $\ell(Q_I)/\ell(Q_{I'}) \leq A_2/a_2$ for any $I, I' \in \mathcal{W}_{\Sigma_j^k}$, and in the last inequality we used (2.5.25). It is clear that (2.5.26) gives (2.5.23). Going back to (2.5.22), we can now conclude that

$$T_2 \leq A_n A_3 \ell(Q_j)^{d-\zeta} \sum_{k:2^{-k}<a_3\ell(Q_j)} 2^{-k\zeta} \leq \tfrac{2^\zeta}{2^\zeta-1}a_3^\zeta A_n A_3 \ell(Q_j)^d.$$

Putting this last estimate together with (2.5.20) and (2.5.17) gives (2.5.13) when $\ell(Q_j) \leq Mr$.

It remains only to show that (2.5.13) holds in the case that $\ell(Q_j) > Mr$. Observe that if $x \in \Sigma$, then the ball B is centered on an $(n-1)$−dimensional face of some Whitney box J^*, with $\operatorname{int} J^* \subset \Omega_{\mathcal{F}}$. Suppose that $\ell(J) \geq A'_n r$, where $A'_n \geq 1$ is chosen so that $B \cap \Sigma$ is a subset of the boundary faces of the Whitney cubes adjacent to J (including also J). It is clear that this is a constraint solely depending on n. In this case, if $X \in \partial I^*$ for any I touching J, then $\delta(X) \geq \operatorname{dist}(I^*, \Gamma) \geq 2\sqrt{n}\ell(I) \geq \sqrt{n}\ell(J)/2 \geq r/2$, and we have

$$\begin{aligned}\sigma_\star(B \cap \Sigma_j) \leq \sigma_\star(B \cap \Sigma) \leq \sigma_\star\big(\textstyle\bigcup_{I \text{ touching } J}(\partial I^* \cap B)\big) \leq \textstyle\sum_{I \text{ touching } J} \sigma_\star(\partial I^* \cap B) \\ \leq A_n \sup_{I \text{ touching } J} \sigma_\star(\partial I^* \cap B) \leq A_n \sup_{I \text{ touching } J} \int_{\partial I^* \cap B} \delta(X)^{d+1-n} \, \mathrm{d}\mathscr{H}^{n-1}|_\Sigma(X) \\ \leq A_n 2^{n-1-d} r^{d+1-n} \sup_{I \text{ touching } J} \mathscr{H}^{n-1}(B \cap \partial I^*) \leq A_n 2^{n-1-d} r^d,\end{aligned}$$

where A_n is a universal constant depending only on n, in the fourth inequality we used that the number of Whitney boxes adjacent to J is uniformly bounded (depending only on n), and in the last line we used the facts that at least one of $\mathscr{H}^{n-1}(B \cap \partial I^*) > 0$, that for any such I there exists $x' \in \partial I^*$ satisfying $B \cap \partial I^* \subset B(x_I, 2r) \cap \partial I^*$, and that each ∂I^* is an $(n-1)-$Ahlfors-David regular set.

Now suppose that either $x \in \Gamma$, or $x \in \Sigma$ with $\ell(J) \leq A'_n r$. The bound $\delta(x) \leq 42\sqrt{n}A'_n r$ holds trivially in the former case, and in the latter it holds because of the estimate $\delta(x) \leq \operatorname{dist}(J^*, \Gamma) + \operatorname{diam} J^* \leq 42\sqrt{n}\ell(J)$. For each $I \in \mathcal{R}_{Q_j}$ intersecting B, there exists $Q_I \in \mathbb{D}_{Q_j}$ such that $I \in \mathcal{W}_{Q_I}$, and we have that

$$4\sqrt{n}\ell(I) \leq \operatorname{dist}(I, \Gamma) \leq \operatorname{dist}(I, x) + \delta(x) \leq 43\sqrt{n}A'_n r.$$

Hence $\ell(I) \leq 11\sqrt{n}A'_n r$, and for any $q_I \in Q_I$, we reckon that

$$\begin{aligned}|q_I - x| &\leq \operatorname{diam} Q_I + \operatorname{dist}(Q_I, I) + \operatorname{diam} I + \operatorname{dist}(I, x) \\ &\leq A_0\ell(Q_I) + A_2\ell(Q_I) + \sqrt{n}\ell(I) + r \\ &\leq (2A_2 a_2^{-1} + \sqrt{n})\ell(I) + r \leq 100\sqrt{n}A'_n A_2 a_2^{-1} r =: A_4 r.\end{aligned}$$

Thus $Q_I \subset B(x, A_4 r)$. Now let $\{Q^i\} \subset \mathbb{D}_{Q_j}$ be a covering of $B(x, A_4 r) \cap Q_j$ such that $Mr/2 \leq \ell(Q^i) \leq Mr$ (which is possible since $\ell(Q_j) > Mr$) and such that the Q^i are pairwise disjoint. It is easy to see then that $\operatorname{dist}(Q^{i_1}, Q^{i_2}) \leq (4A_4 + A_0) \min\{\ell(Q^{i_1}), \ell(Q^{i_2})\}$, whence we deduce as in the paragraph following (2.5.15) that

$\mathrm{card}\{Q^i\} \le N_1$ where $N_1 = N_1(d, C_d, a_0, A_0, A_4)$. Now let $M = 1000nA_n' A_2 a_2^{-1} a_0^{-1}$. With M chosen in this way, our present scenario is very similar to the one for T_2 above. More precisely, suppose that I intersects Σ_j; we have that $Q_I \subset Q^i$ for some Q^i as above, and $\ell(I) \lesssim r \ll \ell(Q^i) \approx Mr$. We may find $J \in \mathcal{W}_{Q'}$ such that $J^* \cap I \neq \varnothing$ and $Q' \in \mathbb{D}_{\mathcal{F}}$, so that $\ell(Q') \approx \ell(J) \approx \ell(I) \ll Mr$. With our choice of M, we have that $\ell(Q') < \ell(Q^i)$, so that $Q' \cap Q^i = \varnothing$. This observation gives us the estimate $\mathrm{dist}(Q_I, (Q_j)^c) \lesssim \ell(I)$, and we may once again use Lemma 2.3.1 (vi) (owing to our choice of M) to control the cardinality of the Whitney boxes I intersecting Σ_j^k by $C(2^k \ell(Q^i))^{d-\zeta}$. Thus it is easy to see that we obtain the desired result in a similar way as we did for T_2, by formally replacing $\ell(Q_j)$ with $\ell(Q^i) \approx Mr$. Thus ends the proof of (2.5.13).

We turn to the proof of (2.5.11). If $r \ge \delta(x)/4$, then the desired result follows from (2.5.10); more precisely, we have that $\sigma_\star(\Delta_\star(x,r)) \le Cr^d \le 4^{n-1-d} C \delta(x)^{d+1-n} r^{n-1}$, where C is the constant from (2.5.10). Now suppose that $r < \delta(x)/4$. In this case, $B(x,r) \cap \Gamma = \varnothing$, and for any $X \in \Delta_\star(x,r)$, we have that $\delta(X) \ge 3\delta(x)/4$, whence we deduce that

$$\begin{aligned}\sigma_\star(\Delta_\star(x,r)) &\le \left(\tfrac{4}{3}\right)^{n-1-d} \delta(x)^{d+1-n} \mathscr{H}^{n-1}(\Delta_\star(x,r)) \\ &\le A_n \left(\tfrac{4}{3}\right)^{n-1-d} \delta(x)^{d+1-n} r^{n-1},\end{aligned}$$

and in the last inequality we used the $(n-1)-$Ahlfors-David regularity of each face ∂J^* intersecting B, and that the number of fattened Whitney boxes J^* which intersect $\Delta_\star(x,r)$ is uniformly bounded (depending only on n). $\square$

Remark 2.5.27. Observe that Proposition 2.5.9 implies in particular that $\mathscr{H}^{n-1}(\Sigma \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^n$. This is easily seen by fixing a compact set $K \subset \mathbb{R}^n$ and noticing that therefore $\delta(X) \le M_K$ for any $X \in \Sigma \cap K$, which implies that

$$\mathscr{H}^{n-1}(\Sigma \cap K) \le M_K^{n+1-d} \sigma_\star(\Sigma \cap K) < \infty.$$

Moreover, since $\mathscr{H}^d(\Gamma \cap K) \in (0, +\infty)$ for any compact set intersecting Γ, it follows that $\mathscr{H}^{n-1}(\Gamma) = 0$, and therefore $\partial\Omega_{\mathcal{F}}$ satisfies $\mathscr{H}^{n-1}(\partial\Omega_{\mathcal{F}} \cap K) < +\infty$ for any compact K.

We now concentrate on a lower bound for $\sigma_\star$.

Proposition 2.5.28 (Lower bound for $\sigma_\star$). *Let $x \in \partial\Omega_{\mathcal{F}}$ and $r > 0$. Suppose that M_0 is* *given by (2.5.36) below. If $\delta(x) \geq r/M_0$, then*

$$\sigma_\star(\Delta_\star(x,r)) \geq v_1 \delta(x)^{d+1-n} r^{n-1}, \tag{2.5.29}$$

where $v_1 = v_1(n,d,M_0,\theta)$. If $\delta(x) < r/M_0$, then

$$\sigma_\star(\Delta_\star(x,r)) \geq v_2 r^d, \tag{2.5.30}$$

where $v_2 = v_2(n,d,C_d,\theta,c_{\mathbb{K}},A_0,a_0,c,c_{\mathcal{H}},M_0)$.

Proof. We consider (2.5.29) first, so that $\delta(x) > 0$ which implies that $x \in \partial J^*$ for some $J \in \mathcal{W}_{\mathcal{F}}$. Hence $r/M_0 \leq \delta(x) \leq 42\sqrt{n}\ell(J)$. Observe the estimate

$$\begin{aligned}\sigma_\star(\Delta_\star(x,r)) \geq \sigma_\star(\partial J^* \cap \Sigma \cap B(x,r)) = \int_{\partial J^* \cap \Sigma \cap B(x,r)} \delta(X)^{d+1-n} \, \mathrm{d}\mathscr{H}^{n-1}|_\Sigma(X) \\ \geq (2M_0)^{d+1-n}\delta(x)^{d+1-n}\mathscr{H}^{n-1}(\partial J^* \cap \Sigma) \geq c_n (2M_0)^{d+1-n}\delta(x)^{d+1-n}\theta^{n-1}\ell(J)^{n-1} \\ \geq c_n 2^d M_0^{d+2-2n}\delta(x)^{d+1-n}\theta^{n-1}r^{n-1}\end{aligned}$$

where in the fourth inequality we made use of Lemma 2.4.34.

We proceed to prove (2.5.30), using ideas of the proof for Lemma 3.61 in [HM14]. First, observe that by Remark 2.5.27 and the criterion for sets of finite perimeter [EG92, 5.11 Theorem 1], we have that $\Omega_{\mathcal{F}}$ is a set of locally finite perimeter. Hence, by the structure theorem for sets of finite perimeter, [EG92, 5.7 Theorem 2], it follows that $\|\partial\Omega_{\mathcal{F}}\| = \mathscr{H}^{n-1} \llcorner \partial^\star\Omega_{\mathcal{F}}$. We will use these facts below.

Suppose that $\delta(x) < r/M_0$, so that there exists $\hat{x} \in \Gamma$ with $|x - \hat{x}| \leq r/M_0$. Now fix $\hat{Q} \in \mathbb{D}$ with $\hat{x} \in \hat{Q}$ and such that $c_{\mathbb{K}} r/M_0 \leq \ell(\hat{Q}) \leq r/M_0$. If $M_0 > \max\{9, A_0^2\}$, then we may guarantee that $\hat{Q} \subset B(\hat{x}, r/\sqrt{M_0}) \subset B(x,r) =: B$. We consider two cases.

Case 1. The ball $B(\hat{x}, r/\sqrt{M_0})$ meets some $Q_j \in \mathcal{F}$ with $\ell(Q_j) \geq r/M_0$. Then we may procure a dyadic cube $Q \in \mathbb{D}_{Q_j}$ with $r/2M_0 \leq \ell(Q) \leq r/M_0$ and $Q \subset B(\hat{x}, 2r/\sqrt{M_0})$. By Lemma 2.4.43, the ball $B(x_Q, r') = B(x_Q, a_0\ell(Q)/(5A_2a_2^{-1}))$ lies in $\mathbb{R}^n \backslash \Omega_{\mathcal{F}}$, while if we further assume that $M_0 \geq 16$, then for any $Y \in B(x_Q, r')$ we have that

$$|Y - x| \leq |Y - x_Q| + |x_Q - \hat{x}| + |\hat{x} - x| < \tfrac{a_0 a_2}{5A_2}\ell(Q) + \tfrac{2r}{\sqrt{M_0}} + \tfrac{r}{M_0} \leq 4r/\sqrt{M_0} < r,$$

whence it is known that $B(x_Q, r') \subset B\backslash\Omega_{\mathcal{F}}$. On the other hand, we have shown in Proposition 2.5.3 that $\partial\Omega_{\mathcal{F}}$ has interior Corkscrew points, so that there exists $X \in \Omega_{\mathcal{F}}$ verifying that $B(X, c_1 r) \subset \Omega_{\mathcal{F}} \cap B$, and c_1 is given in (2.5.4). We can now appeal to the relative isoperimetric inequality (see [EG92, 5.6 Theorem 2]) to conclude that

$$\begin{aligned}\|\partial\Omega_{\mathcal{F}}\|(B(x,r)) &\geq a_n \min\left\{\mathscr{L}^n(B(X,c_1 r))\,,\ \mathscr{L}^n(B(x_Q,r'))\right\}^{\frac{n-1}{n}} \\ &\geq a_n \min\left\{c_1, \tfrac{a_0 a_2}{10 A_2 M_0}\right\}^{n-1} r^{n-1}, \end{aligned} \tag{2.5.31}$$

where a_n is a uniform constant depending only on n. Consequently,

$$\begin{aligned}&\sigma_\star(\Delta_\star(x,r)) \geq \sigma_\star(\Delta_\star(x,r)\cap\Sigma) \geq M_0^{n-1-d} r^{d+1-n} \mathscr{H}^{n-1}(\Delta_\star \cap \Sigma) \\ &= M_0^{n-1-d} r^{d+1-n} \mathscr{H}^{n-1}(B(x,r)\cap\partial\Omega_{\mathcal{F}}) \geq M_0^{n-1-d} r^{d+1-n} \mathscr{H}^{n-1}(B(x,r)\cap\partial^\star\Omega_{\mathcal{F}}) \\ &= M_0^{n-1-d} r^{d+1-n} \|\partial\Omega_{\mathcal{F}}\|(B(x,r)) \geq a_n (a_0 a_2 A_2^{-1})^{n-1} M_0^{-d} r^d, \end{aligned} \tag{2.5.32}$$

where we used the structure theorem for sets of finite perimeter, [EG92, 5.7 Theorem 2].

Case 2. There is no Q_j as in case 1. It follows that if $Q_j \in \mathcal{F}$ meets $B(\hat{x}, r/\sqrt{M_0})$, then $\ell(Q_j) \leq r/M_0$. Let $\hat{\mathcal{F}}$ denote the collection of those $Q_j \in \mathcal{F}$ which intersect $\hat{\Delta} = \Delta(\hat{x}, r/\sqrt{M_0})$. Then we may split $\frac{1}{2}\hat{\Delta}$ as $\frac{1}{2}\hat{\Delta} = \left(\frac{1}{2}\hat{\Delta}\backslash\left(\bigcup_{\hat{\mathcal{F}}} Q_j\right)\right) \cup \left(\frac{1}{2}\hat{\Delta} \cap \left(\bigcup_{\hat{\mathcal{F}}} Q_j\right)\right)$. We now consider two sub-cases: either the estimate

$$\sigma\left(\tfrac{1}{2}\hat{\Delta}\backslash\left(\textstyle\bigcup_{\hat{\mathcal{F}}} Q_j\right)\right) \geq \tfrac{1}{2}\sigma\left(\tfrac{1}{2}\hat{\Delta}\right) \tag{2.5.33}$$

holds, or it does not. If it does, then we deduce that

$$\begin{aligned}\sigma_\star(B(x,r)\cap\partial\Omega_{\mathcal{F}}) &\geq \sigma\left(B(x,r)\cap\Gamma\backslash\textstyle\bigcup_{\mathcal{F}} Q_j\right) \geq \sigma\left(\hat{\Delta}\backslash\textstyle\bigcup_{\hat{\mathcal{F}}} Q_j\right) \geq \tfrac{1}{2}\sigma\left(\tfrac{1}{2}\hat{\Delta}\right) \\ &\geq C_d^{-1} 2^{-d-1} M_0^{-d/2} r^d =: a_5 r^d,\end{aligned}$$

which yields the desired result. We are left to consider the case that (2.5.33) does not hold. Then, instead, we have that

$$\textstyle\sum_{\mathcal{F}'} \sigma(Q_j) \geq \sigma\left(\tfrac{1}{2}\hat{\Delta} \cap \left(\bigcup_{\hat{\mathcal{F}}} Q_j\right)\right) \geq \tfrac{1}{2}\sigma\left(\tfrac{1}{2}\hat{\Delta}\right) \geq a_5 r^d, \tag{2.5.34}$$

where $\mathcal{F}'$ consists of those $Q_j \in \hat{\mathcal{F}}$ which intersect $\frac{1}{2}\Delta$. For each $Q_j \in \mathcal{F}'$, fix any one of the $(n-1)$−dimensional cubes $P_j \subset \partial\Omega_{\mathcal{F}}$ constructed in Proposition 2.4.44, and denote its center by $x_j^\star$. We now claim that for each $Q_j \in \mathcal{F}'$, the ball $B_j^* = B(x_j^\star, 16A_0 c_{\mathbb{K}}^{-1}\ell(Q_j))$ contains both an interior and an exterior Corkscrew ball for $\Omega_{\mathcal{F}}$, with respect to the surface ball $B_{Q_j}^* \cap \partial\Omega_{\mathcal{F}}$ (with Corkscrew constants that could depend on K_0).

Indeed, by virtue of Lemma 2.4.43, the ball $B_j = B(x_{Q_j}, a_0 a_2 \ell(Q_j)/(5A_2))$ is contained in $\mathbb{R}^n \backslash \Omega_{\mathcal{F}}$, and we note that for any $Y \in B_j$,

$$\begin{aligned}|Y - x_j^\star| &< \frac{a_0 a_2}{5A_2}\ell(Q_j) + \operatorname{diam} Q_j + \operatorname{dist}(P_j, Q_j) + \operatorname{diam} P_j \\ &\leq 2A_0\ell(Q_j) + 11A_0 c_{\mathbb{K}}^{-1}\ell(Q_j) + \tfrac{13}{8}\theta a_0 c_{\mathbb{K}}^{-1}\ell(Q_j) \leq 15A_0 c_{\mathbb{K}}^{-1}\ell(Q_j),\end{aligned}$$

so that $B_j \subset B_j^* \backslash \Omega_{\mathcal{F}}$. By a similar reasoning, we also have that $Q_j \subset B_j^*$. For the interior Corkscrew ball, let $\hat{Q} \in \mathbb{D}_{\mathcal{F}}$ be the proper parent of Q_j and fix $I \in \mathcal{W}_{\hat{Q}}^{\mathrm{cs}}$. Note that $B(X_I, \ell(I)/2) \subset I \subset \operatorname{int} I^* \subset \Omega_{\mathcal{F}}$, and by our choice of radius for B_j^*, we also have that $B(X_I, \ell(I)/2) \subset B_j^*$ similarly as in the estimate above. Then the ball $B(X_I, a_0 c\ell(Q_j)/(41\sqrt{n}))$ is contained in $B_j^* \cap \Omega_{\mathcal{F}}$ by (2.4.5). Thus by using the relative isoperimetric inequality in a manner analogous to (2.5.31), we deduce that $\|\partial\Omega_{\mathcal{F}}\|(B_j^*) \geq a_n(a_0 a_2/(5A_2))^{n-1}\ell(Q_j)^{n-1}$. We also have that for any $Y \in B_j^*$,

$$\delta(Y) \leq 16A_0 c_{\mathbb{K}}^{-1}\ell(Q_j) + \operatorname{diam} P_j + \operatorname{dist}(P_j, \Gamma) \leq 400A_0 c_{\mathbb{K}}^{-1}\ell(Q_j),$$

and therefore, analogous to (2.5.32), we obtain that

$$\sigma_\star(B_j^* \cap \partial\Omega_{\mathcal{F}}) \geq a_n(a_0 a_2 A_2^{-1})^{n-1} A_0^{d+1-n} c_{\mathbb{K}}^{n-1-d}\ell(Q_j)^d. \tag{2.5.35}$$

We show that for $M_0 \geq 1250A_0^2 c_{\mathbb{K}}^{-2}$, we have $B_j^* \subset B$. Fix any $Y \in B_j^*$ and observe that

$$\begin{aligned}|Y - x| &\leq \operatorname{diam}(B_j^*) + \operatorname{diam} Q_j + \tfrac{1}{2}\operatorname{diam} B\big(\hat{x}, r/\sqrt{M_0}\big) + |\hat{x} - x| \\ &\leq 32A_0 c_{\mathbb{K}}^{-1}\ell(Q_j) + A_0\ell(Q_j) + \frac{r}{\sqrt{M_0}} + \frac{r}{M_0} \leq \frac{35A_0 c_{\mathbb{K}}^{-1}}{\sqrt{M_0}} r < r,\end{aligned}$$

as claimed. Henceforth we will take

$$M_0 := 125A_0^2 c_{\mathbb{K}}^{-2}. \tag{2.5.36}$$

Let us now show that we can muster a sub-collection $\mathcal{F}'' \subset \mathcal{F}'$ of cubes $Q_j \in \mathcal{F}'$ such that the balls in $\{B_j^*\}_{Q_j \in \mathcal{F}''}$ are pairwise disjoint and

$$\sum_{Q_j \in \mathcal{F}''} \ell(Q_j) \gtrsim r^d. \tag{2.5.37}$$

Since for any $Q_j \in \mathcal{F}' \subset \hat{\mathcal{F}}$ we have that $\ell(Q_j) < r/M_0$, it follows that there exists $k_0 \in \mathbb{Z}$ such that $\ell(Q_j) \leq 2^{-k_0}$ for all $Q_j \in \mathcal{F}'$, and $\ell(Q) = 2^{-k_0}$ for some $Q \in \mathcal{F}'$. For any $k \geq k_0$, let $\mathcal{F}'_k = \{Q_j \in \mathcal{F}' : \ell(Q_j) = 2^k\}$. Fix a sub-collection $\mathcal{F}''_{k_0}$ of $\mathcal{F}'_{k_0}$ which is B_j^**−maximal* in the sense that the balls $\{B_j^*\}_{Q_j \in \mathcal{F}''_{k_0}}$ are pairwise disjoint, but where adjoining any other cube in $\mathcal{F}' \backslash \mathcal{F}''_{k_0}$ makes some of these balls overlap. Next, define inductively for each $k > k_0$ the sub-collection $\mathcal{F}''_k$ which is the union of all $\mathcal{F}''_{\tilde{k}}$, $k_0 \leq \tilde{k} < k$, and adjoined with a sub-collection of $\mathcal{F}'_k$ such that $\mathcal{F}''_k$ is B_j^*−maximal. We then set $\mathcal{F}'' = \cup_{k \geq k_0} \mathcal{F}''_k$ and observe that it satisfies that the balls in $\{B_j^*\}_{Q_j \in \mathcal{F}''}$ are pairwise disjoint, and that each $Q_m \in \mathcal{F}' \backslash \mathcal{F}''$ is such that B_m^* intersects $B_j^* \in \mathcal{F}''$ for some $Q_j \in \mathcal{F}''$ with $\ell(Q_m) \leq \ell(Q_j)$ (otherwise, Q_m would have had to belong to $\mathcal{F}''_k$ for some k).

Recall that for any $Q_j \in \mathcal{F}'$, $Q_j \subset B_j^*$. If $Q_m \in \mathcal{F}'$ intersects Q_j with $\ell(Q_m) \leq \ell(Q_j)$, then

$$\operatorname{dist}(Q_m, Q_j) \leq \operatorname{diam} B_m^* + \operatorname{diam} B_j^* \leq 32 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j),$$

and thus $Q_m \subset B(x_{Q_j}, 34 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j)) \cap \Gamma$, which, together with our observations in the last paragraph, implies that $\bigcup_{\mathcal{F}'} Q_j \subset \bigcup_{Q_j \in \mathcal{F}''} B\big(x_{Q_j}, 34 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j)\big) \cap \Gamma$. Hence,

$$\sum_{\mathcal{F}'} \sigma(Q_j) \leq \sigma\big(\textstyle\bigcup_{\mathcal{F}''} B\big(x_{Q_j}, 34 A_0 c_{\mathbb{K}}^{-1} \ell(Q_j)\big) \cap \Gamma\big) \leq C_d (34 A_0 c_{\mathbb{K}}^{-1})^d \displaystyle\sum_{\mathcal{F}''} \ell(Q_j)^d.$$

By combining this last estimate with (2.5.34), we obtain (2.5.37) with implicit constant $a_5 C_d^{-1} (34^{-1} A_0^{-1} c_{\mathbb{K}})^d$. Finally, we combine (2.5.35) and (2.5.37) to conclude that

$$\begin{aligned} \sigma_\star(B \cap \partial\Omega_{\mathcal{F}}) \geq \sum_{Q_j \in \mathcal{F}''} \sigma_\star(B_j^* \cap \partial\Omega_{\mathcal{F}}) &\geq a_n (a_2^2 A_2^{-2})^{n-1} A_0^d c_{\mathbb{K}}^{n-1-d} \sum_{Q_j \in \mathcal{F}''} \ell(Q_j)^d \\ &\geq 35^{-d} C_d^{-2} M_0^{-d/2} a_n (a_2^2 c_{\mathbb{K}} A_2^{-2})^{n-1} r^d, \end{aligned}$$

which does complete our argument for (2.5.30). $\square$

We are ready to show

Proposition 2.5.38 ($\sigma_\star$ is doubling)**.** *The measure $\sigma_\star$ verifies (H3).*

Proof. Fix $x \in \partial\Omega_{\mathcal{F}}$ and $r > 0$. We split the proof of the proposition into three cases.

Case 1. $2r < M_0\delta(x)$. Then we also have that $r < M_0\delta(x)$, and therefore,

$$\sigma_\star\big(B(x,2r)\cap\Gamma\big) \leq 2^{n-1}V_2 r^{n-1}\delta(x)^{d+1-n} \leq 2^{n-1}V_2 v_1^{-1}\sigma_\star\big(B(x,r)\cap\Gamma\big).$$

Case 2. $r < M_0\delta(x) \leq 2r$. Here, we obtain that

$$\begin{aligned}\sigma_\star&\big(B(x,2r)\cap\Gamma\big)\\ &\leq 2^d V_1 r^d = 2^d V_1 r^{d+1-n} r^{n-1} \leq 2^{-2d+1-n} M_0^{d+1-n} V_1 \delta(x)^{d+1-n} r^{n-1}\\ &\qquad\qquad\qquad\qquad\qquad\quad \leq 2^{-2d+1-n} M_0^{d+1-n} V_1 v_1^{-1}\sigma_\star\big(B(x,r)\cap\Gamma\big).\end{aligned}$$

Case 3. $r \geq M_0\delta(x)$. We easily estimate that

$$\sigma_\star\big(B(x,2r)\cap\Gamma\big) \leq 2^d V_1 r^d \leq 2^d V_1 v_2^{-1}\sigma_\star\big(B(x,r)\cap\Gamma\big),$$

as desired. $\square$

Next, we turn to verifying the growth condition (H5). In preparation, we record the following useful estimate from [DFM19b].

Lemma 2.5.39 (Behavior of m, [DFM19b] Lemma 2.3, Remark 2.4)**.** *For any $\alpha > 0$, there exists a constant $M(\alpha) > 0$, depending only on n, d, C_d, and α, such that the following statements hold for any $X \in \mathbb{R}^n$ and any $r > 0$.*

(i) If $\delta(X) \geq \alpha r$, then $M(\alpha)^{-1} r^n \delta(X)^{d+1-n} \leq m\big(B(X,r)\big) \leq M(\alpha) r^n \delta(X)^{d+1-n}$.

(ii) If $\delta(X) \leq \alpha r$, then $M(\alpha)^{-1} r^{d+1} \leq m\big(B(X,r)\big) \leq M(\alpha) r^{d+1}$.

Proposition 2.5.40 (Growth condition)**.** *The measures $\sigma_\star$ and m satisfy (H5). More precisely, there exist constants $V_5 \geq 1$ and $\varepsilon \in (0,1)$ so that for each $x \in \partial\Omega_{\mathcal{F}}$ and all r, s with $0 < s < r$, we have the estimate*

$$\frac{m\big(B(x,r)\cap\Omega_{\mathcal{F}}\big)}{m\big(B(x,s)\cap\Omega_{\mathcal{F}}\big)} \leq V_5\Big(\frac{r}{s}\Big)\frac{\sigma_\star\big(B(x,r)\cap\Gamma\big)}{\sigma_\star\big(B(x,s)\cap\Gamma\big)}.$$

Proof. Recall that M_0 is the uniform constant given in (2.5.36). Note that $m\big(B(x,r)\cap\Omega_{\mathcal{F}}\big)\leq m\big(B(x,r)\big)$ for any $x\in\partial\Omega_{\mathcal{F}}$ and $r>0$, while in the other direction, Proposition 2.5.3 implies the existence of a Corkscrew point $X=X_{x,r}\in\Omega_{\mathcal{F}}$ such that $B(X,c_1r)\subset\Omega_{\mathcal{F}}\cap B(x,r)$. Observe that such a Corkscrew point satisfies $c_1r\leq\delta(X_{x,r})\leq r+\delta(x)$. Now fix $x\in\partial\Omega_{\mathcal{F}}$ and $r,s>0$ with $0<s<r$. We consider three cases.

Case 1. $\delta(x)\geq r/M_0\geq s/M_0$. Here, note that $\delta(X_{x,s})\leq 2M_0\delta(x)$. We have that

$$
\begin{aligned}
\frac{m\big(B(x,r)\cap\Omega_{\mathcal{F}}\big)}{m\big(B(x,s)\cap\Omega_{\mathcal{F}}\big)}&\leq\frac{m\big(B(x,r)\big)}{m\big(B(X_{x,s},c_1s)\big)}\leq\frac{M(\frac{1}{M_0})r^n\delta(x)^{d+1-n}}{M(1)^{-1}(c_1s)^n\delta(X_{x,s})^{d+1-n}}\\
&\leq\Big[(2M_0)^{n-1-d}\frac{M(1)M(\frac{1}{M_0})}{c_1^n}\Big]\Big(\frac{r}{s}\Big)\frac{r^{n-1}\delta(x)^{d+1-n}}{s^{n-1}\delta(x)^{d+1-n}}\\
&\leq\Big[(2M_0)^{n-1-d}\frac{M(1)M(\frac{1}{M_0})V_2}{c_1^n v_1}\Big]\Big(\frac{r}{s}\Big)\frac{\sigma_\star\big(B(x,r)\cap\Gamma\big)}{\sigma_\star\big(B(x,s)\cap\Gamma\big)},
\end{aligned}
$$

where we have used (2.5.11) and (2.5.29), and thus established the desired estimate.

Case 2. $\delta(x)\leq s/M_0\leq r/M_0$. In this case, we have that $\delta(X_{x,s})\leq 2s$. Reckon that

$$
\begin{aligned}
\frac{m\big(B(x,r)\cap\Omega_{\mathcal{F}}\big)}{m\big(B(x,s)\cap\Omega_{\mathcal{F}}\big)}&\leq\frac{m\big(B(x,r)\big)}{m\big(B(X_{x,s},c_1s)\big)}\leq\frac{M(\frac{1}{M_0})r^{d+1}}{M(\frac{2}{c_1})^{-1}(c_1s)^{d+1}}\\
&\leq\Big[\frac{M(\frac{2}{c_1})M(\frac{1}{M_0})}{c_1^{d+1}}\Big]\Big(\frac{r}{s}\Big)\frac{r^d}{s^d}\leq\Big[\frac{M(\frac{2}{c_1})M(\frac{1}{M_0})V_1}{c_1^{d+1}v_2}\Big]\Big(\frac{r}{s}\Big)\frac{\sigma_\star\big(B(x,r)\cap\Gamma\big)}{\sigma_\star\big(B(x,s)\cap\Gamma\big)},
\end{aligned}
$$

where this time we made use of (2.5.10) and (2.5.30).

Case 3. $s/M_0<\delta(x)<r/M_0$. Now we see that $\delta(X_{x,s})\leq 2M_0\delta(x)$, and estimate

$$
\begin{aligned}
\frac{m\big(B(x,r)\cap\Omega_{\mathcal{F}}\big)}{m\big(B(x,s)\cap\Omega_{\mathcal{F}}\big)}&\leq\frac{m\big(B(x,r)\big)}{m\big(B(X_{x,s},c_1s)\big)}\leq\frac{M(\frac{1}{M_0})r^{d+1}}{M(1)^{-1}(c_1s)^n\delta(X_{x,s})^{d+1-n}}\\
&\leq\Big[(2M_0)^{n-1-d}\frac{M(1)M(\frac{1}{M_0})}{c_1^n}\Big]\Big(\frac{r}{s}\Big)\frac{r^d}{s^{n-1}\delta(x)^{d+1-n}}\\
&\leq\Big[(2M_0)^{n-1-d}\frac{M(1)M(\frac{1}{M_0})}{c_1^n}\Big]\Big(\frac{r}{s}\Big)\frac{\sigma_\star\big(B(x,r)\cap\Gamma\big)}{\sigma_\star\big(B(x,s)\cap\Gamma\big)},
\end{aligned}
$$

using (2.5.30) and (2.5.11). The desired result is established in any case. □

Remark 2.5.41. Incidentally, Lemma 2.5.39 together with Proposition 2.5.3 give that (H4) holds. Indeed, we take $\tilde{w} \equiv w|_{\partial\Omega_{\mathcal{F}}}$, and it is immediate that $dm/dX = \tilde{w}(X)$ on $\Omega_{\mathcal{F}}$, and that $\tilde{w}(X) > 0$ for ($n-$dimensional) Lebesgue-a.e. $X \in \Omega_{\mathcal{F}}$. It remains to check that m is doubling on $\overline{\Omega_{\mathcal{F}}}$.

Let $X \in \overline{\Omega_{\mathcal{F}}}$ and $r > 0$. Let $\tilde{X} \in \Omega_{\mathcal{F}}$ be the Corskcrew point (with Corkscrew constant $c_1/4$, with c_1 from (2.5.4)) for the ball $B(X, r)$ constructed in Remark 2.5.5. Observe that

$$
\begin{aligned}
m\big(B(X,2r)\cap\Omega_{\mathcal{F}}\big) \leq m\big(B(\tilde{X},3r)\cap\Omega_{\mathcal{F}}\big) \leq m\big(B(\tilde{X},3r)\big)& \\
\leq Cm\big(B(\tilde{X},\tfrac{c_1 r}{4})\big) \leq Cm\big(B(X,r)\cap\Omega_{\mathcal{F}}\big)&
\end{aligned}
$$

where in the third inequality we used that $m(B(\tilde{X},3r)) \approx m(B(\tilde{X},\frac{c_1 r}{4}))$ by virtue of Lemma 2.5.39 (i) (with implicit constant depending only on c_1), and in the last inequality we used the fact that $B(\tilde{X},\frac{c_1 r}{4}) \subset B(X,r)\cap\Omega_{\mathcal{F}}$. Thus m is doubling on $\overline{\Omega_{\mathcal{F}}}$ and (H4) is shown.

Finally, it is a trivial application of the fundamental results in [DFM19b] that m satisfies the axiom (H6), since $m|_{\Omega_{\mathcal{F}}}$ is merely the restriction of the function m on Ω which already satisfies this property.

2.6 Carleson measures, discrete Carleson measures, and extrapolation

Recall that $\sigma = \mathscr{H}^d|_\Gamma$, Γ is a closed $d-$ADR subset of Ω, $d \in [1, n-1)$, and $\Omega = \mathbb{R}^n \setminus \Gamma$.

Definition 2.6.1 (Carleson measures)**.** We say that a non-negative Borel measure λ on Ω is a *Carleson measure* if

$$
|||\lambda|||_{\mathcal{C}} := \sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(x,r)\cap\Gamma\big)}\lambda\big(B(x,r)\cap\Omega\big) < \infty.
$$

We call $|||\lambda|||_{\mathcal{C}}$ the *Carleson norm* of λ, and write $\mathcal{C}$ for the set of Carleson measures in Ω.

A main tool in our proof is the extrapolation of Carleson measures, which we use in the dyadic setting. We borrow the definitions and results from [HM14], where this result has been considered in a co-dimension 1 setting; see also [CHM19]. In the setting of higher co-dimension, this framework has appeared in [DMb].

Recall the dyadic decomposition of Γ in Lemma 2.3.1, and the definitions of $\mathbb{D}$ and $\mathbb{D}_Q$ for some $Q \in \mathbb{D}$ in (2.3.2) and the surrounding paragraph.

Definition 2.6.2 (Discrete Carleson measures)**.** Let $\{\alpha_Q\}_{Q\in\mathbb{D}}$ be a sequence of non-negative numbers indexed by $Q \in \mathbb{D}$, and for any sub-collection $\mathbb{D}' \subset \mathbb{D}$, we define

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q\in\mathbb{D}'} \alpha_Q.$$

We say that $\mathfrak{m}$ is a *discrete Carleson measure* on $\mathbb{D}$ with respect to σ (written $\mathfrak{m} \in \mathcal{C}$) if

$$\|\mathfrak{m}\|_{\mathcal{C}} := \sup_{Q\in\mathbb{D}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty.$$

Similarly, we have a local version: For a fixed $Q_0 \in \mathbb{D}$, we say that $\mathfrak{m}$ is a discrete Carleson measure on $\mathbb{D}_{Q_0}$ with respect to σ (written $\mathfrak{m} \in \mathcal{C}(Q_0)$) if

$$\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} := \sup_{Q\in\mathbb{D}_{Q_0}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty.$$

Given a disjoint family $\mathcal{F} \subset \mathbb{D}$, we define the restriction of $\mathfrak{m}$ to the sawtooth $\mathbb{D}_{\mathcal{F}}$ (see (2.4.25)) by

$$\mathfrak{m}_{\mathcal{F}}(\mathbb{D}') := \mathfrak{m}(\mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}) = \sum_{Q\in\mathbb{D}'\setminus(\cup_{\mathcal{F}}\mathbb{D}_{Q_j})} \alpha_Q, \qquad \text{for } \mathbb{D}' \subset \mathbb{D},$$

and for any $Q \in \mathbb{D}$, denote (recalling that $\mathbb{D}_{\mathcal{F},Q'} = \mathbb{D}_Q \cap \mathbb{D}_{\mathcal{F}}$ from (2.4.26))

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} := \sup_{Q'\in\mathbb{D}_Q} \frac{\mathfrak{m}(\mathbb{D}_{\mathcal{F},Q'})}{\sigma(Q')}.$$

The following result concerns the extrapolation of Carleson measures.

Theorem 2.6.3 (Extrapolation of Carleson measures; [DMb] [HM14])**.** *Let Γ be a closed d-ADR set, and $\sigma = \mathscr{H}^d|_\Gamma$. Fix $Q_0 \in \mathbb{D}$ and a dyadically doubling Borel measure μ on Q_0. Assume that there is some sequence of non-negative numbers $\{\alpha_Q\}_{Q\in\mathbb{D}(Q_0)}$ such that the corresponding $\mathfrak{m}$ satisfies $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \le M_0 < +\infty$. Suppose that there exists*

$\xi > 0$ such that for every $Q \in \mathbb{D}_{Q_0}$, and every disjoint family $\mathcal{F} \subset \mathbb{D}_Q$ verifying

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq \xi, \tag{2.6.4}$$

we have that the projection $\mathcal{P}_{\mathcal{F}}\mu$ (see (2.3.10)) satisfies the following property:

$$\forall \varepsilon \in (0,1),\ \exists C_\varepsilon > 1 \text{ such that } \left(F \subset Q,\ \frac{\sigma(F)}{\sigma(Q)} \geq \varepsilon \implies \frac{\mathcal{P}_{\mathcal{F}}\mu(F)}{\mathcal{P}_{\mathcal{F}}\mu(Q)} \geq \frac{1}{C_\varepsilon}\right). \tag{2.6.5}$$

Then, there exist $\eta_0 \in (0,1)$ and $C_0 < \infty$ such that, for every $Q \in \mathbb{D}_{Q_0}$,

$$F \subset Q,\ \frac{\sigma(F)}{\sigma(Q)} \geq 1 - \eta_0 \implies \frac{\mu(F)}{\mu(Q)} \geq \frac{1}{C_0}.$$

In other words, $\mu \in A_\infty^{\text{dyadic}}(Q_0)$ (see Definition 2.3.14).

Let us elucidate how Theorem 2.6.3 will be used to prove Theorem 2.1.1. In the hypothesis of the latter, we have that the measure $d\lambda(X) := \delta(X)^{d-n}\mathfrak{a}^2\, dX$ is a (continuous) Carleson measure, where $\mathfrak{a}$ is defined in (2.1.2). In Lemma 2.6.6 below, we define the natural discrete version of this measure, and show that it is indeed a discrete Carleson measure. This discrete measure will be $\mathfrak{m}$ in Theorem 2.6.3, while μ will be ω^{X_0}, the elliptic measure associated to the operator L with pole X_0, where X_0 is a Corkscrew point for a large ball that contains Q_0 (the definition and properties of the elliptic measure are investigated carefully in Section 2.7.1). The bulk of the proof of Theorem 2.1.1 will lie in showing that the property (2.6.5) holds, and this will come (roughly speaking) from two key steps. The first is that, by hypothesis, the disagreement of the matrices satisfies the small Carleson measure condition (2.6.4) everywhere outside of the regions hidden by the sawtooth domain, and this will be used in an argument of integration-by-parts to transfer the (local) A_∞^{dyadic} property from $\omega_{L_0}^{X_0}$ to the elliptic measure of an intermediate operator L_1 that does not "see" the difference of the matrices $\mathcal{A}, \mathcal{A}_0$ in the regions hidden by the sawtooth domain. The second step is to pass from the A_∞^{dyadic} property for $\omega_{L_1}^{X_0}$ to the desired property for $\mathcal{P}_{\mathcal{F}}\omega_L^{X_0}$ in (2.6.5) via the Dahlberg-Jerison-Kenig projection lemma (see Section 2.8), which can be done (roughly speaking) because associated elliptic measures *on the boundary of the sawtooth domain* are the same for L_1 and L. Full details of the proof of Theorem 2.1.1 are written out in Section 2.9.

Lemma 2.6.6 (A discretization of the Carleson measure condition)**.** *Suppose that $\mathcal{A}_0$,*

$\mathcal{A}$ are two uniformly elliptic matrices, such that their disagreement $\mathfrak{a}$ defined in (2.1.2) satisfies that $d\lambda(X) = \delta(X)^{d-n}\mathfrak{a}^2\,dX$ is a Carleson measure. Then, for every $Q_0 \in \mathbb{D}$, the collection $\mathfrak{m} = \{\alpha_Q\}_{Q\in\mathbb{D}_{Q_0}}$ with

$$\alpha_Q := \sum_{I\in\mathcal{W}_Q} \frac{\sup_{Y\in I^*} |\mathfrak{E}(Y)|^2}{\ell(I)^{n-d}}|I|, \qquad Q \in \mathbb{D}, \tag{2.6.7}$$

is a discrete Carleson measure, and in fact,

$$\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \le (41\sqrt{n})^{n-d}(7\sqrt{n}A_2 a_0^{-1})^d C_d^2 \|\!|\lambda|\!\|_{\mathcal{C}}.$$

Proof. Let $Q \in \mathbb{D}_{Q_0}$, write $t_Q = 7\sqrt{n}A_2\ell(Q)$, and consider the estimates

$$\begin{aligned}
\frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} &= \frac{1}{\sigma(Q)}\sum_{Q'\in\mathbb{D}_Q}\sum_{I\in\mathcal{W}_{Q'}} \frac{\sup_{Y\in I^*}|\mathfrak{E}(Y)|^2}{\ell(I)^{n-d}}|I| \\
&\le (41\sqrt{n})^{n-d}\frac{1}{\sigma(Q)}\sum_{I\in\mathcal{R}_Q}\iint_I \frac{\sup_{Y\in I^*}|\mathfrak{E}(Y)|^2}{\delta(X)^{n-d}}\,dX \\
&\le (41\sqrt{n})^{n-d}\frac{1}{\sigma(Q)}\sum_{I\in\mathcal{R}_Q}\iint_I \frac{\mathfrak{a}^2(X)}{\delta(X)^{n-d}}\,dX \le (41\sqrt{n})^{n-d}\frac{1}{\sigma(Q)}\iint_{R_Q}\frac{\mathfrak{a}^2(X)}{\delta(X)^{n-d}}\,dX \\
&\le (41\sqrt{n})^{n-d}(7\sqrt{n}A_2a_0^{-1})^d C_d^2 \frac{1}{\sigma(\Delta(x_Q,t_Q))}\iint_{B(x_Q,t_Q)\cap\Omega}\frac{\mathfrak{a}^2(X)}{\delta(X)^{n-d}}\,dX \\
&\le (41\sqrt{n})^{n-d}(7\sqrt{n}A_2a_0^{-1})^d C_d^2 \|\!|\lambda|\!\|_{\mathcal{C}},
\end{aligned}$$

where we have used in the third line that $B(X,\delta(X)/2) \supset I^*$ for any $X \in I$, and later we used that $R_Q \subset B(x_Q, 7\sqrt{n}A_2\ell(Q))$ (by the same argument as (2.4.30); for the definitions of $\mathcal{R}_Q$ and R_Q, see (2.4.31) and (2.4.32)), and (2.2.2). □

2.7 Review of the elliptic PDE theory for sets with boundaries of high co-dimension

Let us review the necessary background and theory of the David-Feneuil-Mayboroda operators [DFM19b]. Many results in this section have well-understood analogues for domains with boundaries of co-dimension 1 [CFMS81, JK82, Ken94]. Before starting, we remark that many of the results in this section have direct analogues for our saw-tooth domains by virtue of Theorem 2.5.2 and the elliptic PDE theory for sets of mixed dimension carried out in [DFM].

Formally, we write $L = -\operatorname{div} A\nabla$, with $A : \Omega \to \mathbb{M}_n(\mathbb{R})$, where $\mathbb{M}_n(\mathbb{R})$ is the set of $n \times n$ real-valued matrices, and we require that A satisfies the following weighted boundedness and ellipticity conditions:

$$\begin{aligned} \delta(X)^{n-d-1} A(X)\xi \cdot \zeta \leq C_A |\xi||\zeta|, &\quad \text{for each } X \in \Omega \text{ and every } \xi, \zeta \in \mathbb{R}^n, \\ \delta(X)^{n-d-1} A(X)\xi \cdot \xi \geq C_A^{-1} |\xi|^2, &\quad \text{for each } X \in \Omega \text{ and every } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.7.1}$$

Recall that we denote $w(X) = \delta(X)^{-n+d+1}$ and $m(E) = \iint_E w(X)\, dX$. By $\mathcal{A}$ we denote the matrix $w^{-1}A$, so that

$$\iint_\Omega A\nabla u \nabla v = \iint_\Omega \mathcal{A}\nabla u \nabla v \, dm.$$

The matrix $\mathcal{A}$ satisfies unweighted ellipticity and boundedness conditions

$$\begin{aligned} \mathcal{A}(X)\xi \cdot \zeta \leq C |\xi||\zeta|, &\quad \text{for each } X \in \Omega \text{ and every } \xi, \zeta \in \mathbb{R}^n, \\ \mathcal{A}(X)\xi \cdot \xi \geq C^{-1} |\xi|^2, &\quad \text{for each } X \in \Omega \text{ and every } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.7.2}$$

In order to rigorously define the operator L, we need a suitable domain and corresponding range. As in [DFM19b], we consider the following weighted Sobolev space,

$$W = \dot{W}^{1,2}_w(\Omega) := \big\{ u \in L^1_{\mathrm{loc}}(\Omega) \, : \, \nabla u \in L^2(\Omega, dm) \big\},$$

and set $\|u\|_W = \big(\iint_\Omega |\nabla u|^2 \, dm \big)^{\frac{1}{2}}$, $u \in W$. Actually, it is proven in [DFM19b] that $W = \big\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n, dm) \big\}$.

If $E \subset \mathbb{R}^n$ is a Borel set, we let $C_c^\infty(E)$ denote the space of compactly supported, smooth functions on E. We call W_0 the completion of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_W$. Finally, denote by $\mathcal{M}(\Gamma)$ the set of $\sigma-$measurable functions on Γ, and then set

$$H = \dot{H}^{1/2}(\Gamma) := \Big\{ g \in \mathcal{M}(\Gamma) \, : \, \int_\Gamma \int_\Gamma \frac{|g(x) - g(y)|^2}{|x-y|^{d+1}} \, d\sigma(x) \, d\sigma(y) < \infty \Big\}.$$

The significance of H is that it plays a role for W analogous in many ways to the role that the fractional Sobolev space $H^{\frac{1}{2}}$ plays for the classical Sobolev space $W^{1,2}$.

In addition to W which is a space of functions defined globally, we introduce a local version. Let $E \subset \mathbb{R}^n$ be an open set. The set of functions $W_r(E)$ is defined as

$$W_r(E) = \left\{ f \in L^1_{\text{loc}}(E) \,:\, \varphi f \in W \text{ for all } \varphi \in C_c^\infty(E) \right\}$$

where φf is seen as a function on $\mathbb{R}^n$.

The following two results establish that we can make sense of traces on Γ of functions in this weighted Sobolev space.

Theorem 2.7.3 (Trace operator, Theorem 3.4 of [DFM19b])**.** *There exists a bounded linear operator $T : W \to H$ (a trace operator) with the following properties. The trace of $u \in W$ is such that, for $\sigma-$a.e. $x \in \Gamma$, $Tu(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \iint_{B(x,r)} u(X)\, dX$, and, analogously to the Lebesgue density property, $\lim_{r\to 0} \frac{1}{|B(x,r)|} \iint_{B(x,r)} |u(X) - Tu(x)|\, dX = 0$.*

Lemma 2.7.4 (Local traces, Lemma 8.1 of [DFM19b])**.** *Let $E \subset \mathbb{R}^n$ be an open set. For every function $u \in W_r(E)$, we can define the trace of u on $\Gamma \cap E$ by*

$$Tu(x) = \lim_{r\to 0} \fint\!\!\fint_{B(x,r)} u(X)\, dX \qquad \textit{for } \sigma - \textit{almost every } x \in \Gamma \cap E,$$

and $Tu \in L^1_{\text{loc}}(\Gamma \cap E, \sigma)$. Moreover, for every choice of $f \in W_r(E)$ and $\varphi \in C_c^\infty(E)$,

$$\big(T(\varphi u)\big)(x) = \varphi(x) Tu(x) \qquad \textit{for } \sigma - \textit{almost every } x \in \Gamma \cap E.$$

Next, we give a meaning to a local solution of the problem $Lu = 0$.

Definition 2.7.5 (Local weak solutions)**.** Let $E \subseteq \Omega$ be an open set. We say that $u \in W_r(E)$ is a *solution* of $Lu = 0$ in E if for any $\varphi \in C_c^\infty(E)$,

$$\iint_\Omega A\nabla u \nabla \varphi\, dX = \iint_\Omega \mathcal{A} \nabla u \nabla \varphi\, dm = 0.$$

We have an analogous version of the Harnack inequality.

Lemma 2.7.6 (Harnack inequality; Lemma 8.9 of [DFM19b])**.** *Let B be a ball such that $3B \subseteq \Omega$, and let $u \in W_r(3B)$ be a non-negative solution in $3B$. Then $\sup_B u \leq$*

$C \inf_B u$*, where* C *depends only on* n*,* d*,* C_d*, and* C_A*.*

Now, we exhibit results concerning the *Green function*.

Lemma 2.7.7 (Green's function, Lemma 10.1 of [DFM19b])**.** *There exists a non-negative function* $g : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ *with the following properties.*

(i) For any $Y \in \Omega$ *and any* $\alpha \in C_c^\infty(\mathbb{R}^n)$ *such that* $\alpha \equiv 1$ *in a neighborhood of* y*,*

$$(1-\alpha)g(\cdot, Y) \in W_0.$$

In particular, $g(\cdot, Y) \in W_r(\mathbb{R}^n \backslash \{Y\})$ *and* $T[g(\cdot, Y)] = 0$*.*

(ii) For every choice of $Y \in \Omega$*,* $R > 0$*, and* $q \in [1, \frac{n}{n-1})$*,*

$$g(\cdot, Y) \in W^{1,q}(B(Y,R)) := \big\{u \in L^q(B(Y,R)), \nabla u \in L^q(B(Y,R))\big\}.$$

(iii) For $Y \in \Omega$ *and* $\varphi \in C_c^\infty(\Omega)$*,*

$$\iint_\Omega A \nabla_X g(X,Y) \nabla \varphi(X)\, dX = \varphi(Y).$$

In particular, $g(\cdot, Y)$ *is a solution of* $Lu = 0$ *in* $\Omega \backslash \{Y\}$*.*

(iv) For $r > 0$*,* $Y \in \Omega$*, and* $\varepsilon > 0$*,*

$$\iint_{\Omega \backslash B(Y,r)} |\nabla_X g(X,Y)|^2\, dm(X) \ \le\ \begin{cases} Cr^{1-d}, \ \text{if } 4r \ge \delta(Y), \\ \frac{Cr^{2-n}}{w(Y)}, \ \text{if } 2r \le \delta(Y),\ n \ge 3, \\ \frac{C_\varepsilon}{w(Y)} \left(\frac{\delta(Y)}{r}\right)^\varepsilon, \ \text{if } 2r \le \delta(Y),\ n = 2, \end{cases}$$

where $C > 0$ *depends on* d, n, C_d, C_A*, and* $C_\varepsilon > 0$ *depends on* d, C_d, C_A, ε*.*

(v) For $X, Y \in \Omega$ *such that* $X \ne Y$ *and* $\varepsilon > 0$*,*

$$0 \le g(X,Y) \quad \le \quad \begin{cases} C|X-Y|^{1-d}, & \text{if } 4|X-Y| \ge \delta(Y), \\ \frac{C|X-Y|^{2-n}}{w(Y)}, & \text{if } 2|X-Y| \le \delta(Y),\ n \ge 3, \\ \frac{C_\varepsilon}{w(Y)} \left(\frac{\delta(Y)}{|X-Y|}\right)^\varepsilon, & \text{if } 2|X-Y| \le \delta(Y),\ n = 2, \end{cases}$$

where $C > 0$ *depends on* d, n, C_d, C_A*, and* $C_\varepsilon > 0$ *depends on* d, C_d, C_A, ε*.*

The next result is the representation formula given by the Green's function.

Lemma 2.7.8 (Green representation formula, Lemma 10.7 of [DFM19b])**.** *Let $g : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ be the non-negative function constructed in Lemma 2.7.7. Then, for any $f \in C_c^\infty(\Omega)$, the function u defined by $u(X) = \int_\Omega g(X,Y)f(Y)\,dY$ belongs to W_0 and is a solution of $Lu = f$ in the sense that the identity*

$$\int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi\, dm = \int_\Omega f\varphi$$

holds for every $\varphi \in W_0$.

If A is a matrix satisfying (2.7.1), then its transpose A^T also satisfies (2.7.1). We denote $L^T = -\operatorname{div} A^T\nabla$, and g^T is the Green's function of Lemma 2.7.7 for the operator L^T. Lemma 10.6 of [DFM19b] tells us that

$$g(X,Y) = g^T(Y,X), \qquad \text{for all } X, Y \in \Omega, X \neq Y. \tag{2.7.9}$$

We now use Green's functions for a representation formula concerning the difference of two solutions. A proof of it in the setting of co-dimension 1 chord-arc domains may be found in [CHM19] (more specifically, see their Lemma 3.12-Lemma 3.20), and its proof in our setting is essentially the same (see Remark 2.2.11), given that our Green's function in Lemma 2.7.7 satisfies the properties analogous to those of the pioneering construction in [GW82]. Thus we omit the details of the proof, but we do provide a heuristic that formally justifies the desired identity.

Lemma 2.7.10 (Difference of solutions, [CHM19] Lemma 3.18)**.** *Suppose that A_0, A_1 are two matrices satisfying (2.7.1). Let $L_0 = -\operatorname{div} A_0\nabla$, $L_1 = -\operatorname{div} A_1\nabla$, and let E be a Borel set in Γ. Suppose that $u_i \in W$ solves $L_i u_i = 0$ in Ω and that $Tu_0 = Tu_1 = f \in H^{\frac{1}{2}}(\Gamma) \cap C_c(\Gamma)$. Then the identity*

$$u_1(X) - u_0(X) = \iint_\Omega (A_0 - A_1)^T(Y)\nabla_Y g^T_{L_1}(Y,X)\nabla u_0(Y)\,dY, \tag{2.7.11}$$

holds for almost every $X \in \Omega$, and for all $X \in \Omega \setminus \overline{\operatorname{supp}(A_0 - A_1)}$.

Heuristic for the proof. Let $F := u_1 - u_0$, and observe that $L_1 F = L_1 u_1 - L_1 u_0 =$

$-L_1 u_0$. On the other hand, $L_1 u_0 = -\operatorname{div} A_1 \nabla u_0 = -\operatorname{div}\big((A_1 - A_0)\nabla u_0\big)$. Therefore,

$$\iint_\Omega A_1 \nabla F \nabla \varphi = \iint_\Omega (A_0 - A_1)\nabla u_0 \nabla \varphi, \qquad \varphi \in C_c^\infty(\Omega).$$

Equivalently,

$$\iint_\Omega A_1^T \nabla \varphi \nabla F = \iint_\Omega (A_0 - A_1)^T \nabla \varphi \nabla u_0, \qquad \varphi \in C_c^\infty(\Omega). \tag{2.7.12}$$

Then, *formally* plugging in $\varphi = g_{L_1}^T(\cdot, X)$ and using the fact that $L_1^T g_{L_1}^T(\cdot, X) = \delta_X$, we obtain the desired result. The main issue with our logic is that in general we are not justified in plugging in $g_{L_1}^T(\cdot, X)$ for φ, because $g_{L_1}^T(\cdot, X)$ may not belong to W_0. However, we do point out that if $X \in \Omega \backslash \overline{\operatorname{supp}(A_0 - A_1)}$ (which, incidentally, is always the situation in this chapter), then we can make sense of $\varphi = g_{L^1}^T$ in the right-hand side of (2.7.12), and this realization implies the claimed identity over any such X. $\square$

2.7.1 The elliptic measure in a domain with boundary of high co-dimension

In [DFM19b], the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma, \end{cases} \tag{2.7.13}$$

was seen to have a suitably interpreted weak solution. Moreover, it was shown that there exists a family of positive regular Borel measures ω^X on Γ indexed over $X \in \Omega$, called the *elliptic measure*, such that for any boundary function $f \in C_c^0(\Gamma)$, the solution to (2.7.13) can be written as

$$u(X) := \int_\Gamma f \, d\omega^X. \tag{2.7.14}$$

Here, $C_c(\Gamma)$ is the space of continuous functions on Γ with compact support. Let us write the precise statement below. Let $C(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$.

Lemma 2.7.15 (Lemma 9.4 of [DFM19b])**.** *There exists a bounded linear operator*

$$U : C_c(\Gamma) \to C(\mathbb{R}^n)$$

such that, for every $f \in C_c(\Gamma)$, *(i) the restriction of Uf to Γ is f;*

(ii) we have that $\sup_{\mathbb{R}^n} Uf = \sup_\Gamma f$ and $\inf_{\mathbb{R}^n} Uf = \inf_\Gamma f$;

(iii) we have that $Uf \in W_r(\Omega)$ and Uf solves $L(Uf) = 0$ in Ω;

(iv) if B is a ball centered on Γ and $f \equiv 0$ in B, then Uf lies in $W_r(B)$;

(v) if $f \in C_c(\Gamma) \cap H$, then $Uf \in W$, and Uf is the unique solution of the Dirichlet problem with data f.

Lemma 2.7.16 (Harmonic measure; Lemmas 9.5 and 9.6 of [DFM19b])**.** *There exists a unique positive regular Borel measure ω^X on Γ such that $Uf(X) = \int_\Gamma f(y)\, d\omega^X(y)$ for any $f \in C_c(\Gamma)$. Besides, for any Borel set $E \subset \Gamma$,*

$$\omega^X(E) = \sup\{\omega^X(K) \,:\, E \supset K, K \text{ compact}\} = \inf\{\omega^X(V) \,:\, E \subset V, V \textit{ open}\}.$$

Moreover, for each $X \in \Omega$, ω^X is a probability measure. That is, $\omega^X(\Gamma) = 1$.

We now record some results on the elliptic measure, proved mostly in [DFM19b]. The first lemma below tells us qualitatively how the family of elliptic measures behaves over $X \in \Omega$.

Lemma 2.7.17 (Lemma 9.7 of [DFM19b])**.** *Let $E \subseteq \Gamma$ be a Borel set and define the function u_E on Ω by $u_E(X) = \omega^X(E)$. Then*

(i) if there exists $X \in \Omega$ such that $u_E(X) = 0$, then $u_E \equiv 0$;

(ii) the function u_E lies in $W_r(\Omega)$ and is a solution in Ω;

(iii) if $B \subseteq \mathbb{R}^n$ is a ball such that $E \cap B = \varnothing$, then $u_E \in W_r(B)$ and $Tu_E = 0$ on $\Gamma \cap B$.

The next lemma allows us to control from below the elliptic measure on a surface ball by the Green function in certain settings.

Lemma 2.7.18 (Green's function and the elliptic measure, Lemma 11.9 of [DFM19b])**.** *Let $x_0 \in \Gamma$ and $r > 0$ be given, and set $X_0 \in \Omega$ to be a Corkscrew point for $\Delta(x_0, r)$ given by Lemma 2.2.8. Then for all $X \in \Omega \backslash B(X_0, \delta(X_0)/4)$,*

$$r^{d-1} g(X, X_0) \leq C\omega^X(B(x_0, r) \cap \Gamma), \tag{2.7.19}$$

where $C > 0$ depends only on d, n, C_d and C_A.

The following lemma gives non-degeneracy of the elliptic measure.

Lemma 2.7.20 (Quantitative non-vanishing, Lemma 11.10 of [DFM19b])**.** *Let $\alpha > 1$, $x_0 \in \Gamma$, and $r > 0$ be given, and let $X_0 \in \Omega$ be a Corkscrew point for $\Delta(x_0, r)$. Then*

$$\begin{aligned} \omega^X(B(x_0,r)\cap\Gamma) &\geq C_\alpha^{-1} \quad \textit{for } X \in B(x_0, r/\alpha), \\ \omega^X(B(x_0,r)\cap\Gamma) &\geq C_\alpha^{-1} \quad \textit{for } X \in B(X_0, \delta(X_0)/\alpha), \end{aligned} \tag{2.7.21}$$

where $C_\alpha > 0$ depends only upon d, n, C_d, C_A, and α.

Complementary to Lemma 2.7.18, we have

Lemma 2.7.22 (Lemma 11.11 of [DFM19b])**.** *Let $x_0 \in \Gamma$ and $r > 0$ be given, and set $X_0 \in \Omega$ to be a Corkscrew point for $\Delta(x_0, r)$. Then*

$$\omega^X(B(x_0,r)\cap\Gamma) \leq Cr^{d-1}g(X,X_0) \quad \textit{for } X \in \Omega\backslash B(x_0,2r), \tag{2.7.23}$$

where $C > 0$ depends only upon d, n, C_d, and C_A.

Next, we have a doubling property of the elliptic measure on surface balls.

Lemma 2.7.24 (Harmonic measure is doubling, Lemma 11.12 of [DFM19b])**.** *For $x_0 \in \Gamma, r > 0$, and $\alpha > 1$, we have that*

$$\omega^X(B(x_0,2r)\cap\Gamma) \leq C_{\text{doubling}}\omega^X(B(x_0,r)\cap\Gamma) \quad \textit{for } X \in \Omega\backslash B(x_0, 2\alpha r),$$

where $C_{\text{doubling}} > 0$ depends only on n, d, C_d, C_A, and α.

The doubling property of the elliptic measure and the elementary properties of the dyadic cubes (recall Lemma 2.3.1) gives us the following corollaries.

Corollary 2.7.25. *Let $Q \in \mathbb{D}$, and recall that $\Delta_Q := \Delta(x_Q, a_0\ell(Q))$ is the surface ball which Q contains (see (2.3.6)). Then,*

$$\omega^X(\Delta_Q) \leq \omega^X(Q) \leq C_{\text{doubling}}^{1+\log_2(\frac{A_0}{a_0})}\omega^X(\Delta_Q), \qquad \textit{for each } X \in \Omega\backslash B(x_Q, 2A_0\ell(Q)).$$

Corollary 2.7.26 (Harmonic measure is dyadically doubling)**.** *Fix $Q_0 \in \mathbb{D}$ and $X_0 \in \Omega\backslash B(x_{Q_0}, 3A_0\ell(Q_0))$ (see Lemma 2.3.1 and (2.3.6)). Then ω^{X_0} is a dyadically doubling measure in Q_0 (see Definition 2.3.11).*

Proof. This result follows easily from Lemma 2.3.12 and Lemma 2.7.24.

The following notion is fundamental in our analysis of the absolute continuity of the elliptic measure.

Definition 2.7.27 (Poisson kernel)**.** Fix $X \in \Omega$, and suppose that $\omega_L^X \ll \sigma$. Then we denote by $k_L^X = \frac{d\omega_L^X}{d\sigma}$ the Radon-Nikodym derivative of ω_L^X with respect to σ, and refer to it as the *Poisson kernel*.

We will concern ourselves with the quantitative absolute continuity of the elliptic measure, but first we have to adapt the definitions of A_∞ and RH for it to be meaningful for the elliptic measures (as families of probability measures) that we consider here. Compare the notions in the following definition to the notions in Definitions 2.2.13, 2.2.15, and 2.3.14.

Definition 2.7.28 (A_∞, RH_p, and dyadic analogues for elliptic measure)**.** We say that the elliptic measure $\{\omega^X\}_{X\in\Omega}$ is *of class* A_∞ with respect to the surface measure σ, or simply $\omega \in A_\infty(\sigma)$, if for every $\varepsilon > 0$, there exists $\xi = \xi(\varepsilon) > 0$ such that for any surface ball Δ, every surface ball $\Delta' \subseteq \Delta$, and every Borel set $E \subset \Delta'$, we have that

$$\frac{\sigma(E)}{\sigma(\Delta')} < \xi \implies \frac{\omega^{X_\Delta}(E)}{\omega^{X_\Delta}(\Delta')} < \varepsilon,$$

where X_Δ is a Corkscrew point for Δ as in Definition 2.2.7. Analogously, we say that $\omega \in A_\infty^{\text{dyadic}}$ if for each $Q_0 \in \mathbb{D}$ and X_{Q_0} a Corkscrew point for Q_0 (see Section 2.3), we have that $\omega^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$ with uniform constants (see Definition 2.3.14).

Given $p \in (1,\infty)$, if $\omega \ll \sigma$, then we say that $\{\frac{d\omega^X}{d\sigma}\}_{X\in\Omega}$ is *of class* RH_p, or simply $k = \frac{d\omega}{d\sigma} \in RH_p$, if there exists a constant $C_0 \geq 1$ such that for each surface ball $\Delta = \Gamma \cap B(x,r)$ with a Corkscrew point $X_\Delta \in \Omega$, we have the estimate

$$\Big(\tfrac{1}{\sigma(\Delta')}\int_{\Delta'} (k^{X_\Delta})^p \, d\sigma\Big)^{1/p} \leq C_0 \tfrac{1}{\sigma(\Delta')}\int_{\Delta'} k^{X_\Delta}\, d\sigma, \qquad \text{for each surface ball } \Delta' \subseteq \Delta. \tag{2.7.29}$$

We call C_0 the RH_p characteristic of $\frac{d\omega}{d\sigma}$. Analogously, if $\omega \ll \sigma$, we say that $k = \frac{d\omega}{d\sigma} \in RH_p^{\text{dyadic}}$ if for each $Q_0 \in \mathbb{D}$ and X_{Q_0} a Corkscrew point for Q_0, we have that $k^{X_{Q_0}} \in RH_p^{\text{dyadic}}(Q_0)$ with uniform RH_p characteristic (see Definitions 2.2.15 and 2.3.14).

Next we state a global comparison principle for the elliptic measure.

Lemma 2.7.30 (Change of poles, Lemma 11.16 of [DFM19b])**.** *Let $x_0 \in \Gamma$ and $r > 0$ be given, and let $X_0 \in \Omega$ be a Corkscrew point for $\Delta(x_0, r)$. Let $E, F \subseteq \Delta_0 := B(x_0, r) \cap \Gamma$ be two Borel subsets of Γ such that both $\omega^{X_0}(E)$ and $\omega^{X_0}(F)$ are positive. Then*

$$C^{-1} \frac{\omega^{X_0}(E)}{\omega^{X_0}(F)} \leq \frac{\omega^X(E)}{\omega^X(F)} \leq C \frac{\omega^{X_0}(E)}{\omega^{X_0}(F)}, \qquad \text{for } X \in \Omega \backslash B(x_0, 2r),$$

where $C > 0$ depends only on n, d, C_d, and C_A. In particular, with the choice $F = B(x_0, r) \cap \Gamma$,

$$C^{-1} \omega^{X_0}(E) \leq \frac{\omega^X(E)}{\omega^X(\Delta_0)} \leq C \omega^{X_0}(E) \qquad \text{for } X \in \Omega \backslash B(x_0, 2r), \tag{2.7.31}$$

where again $C > 0$ depends only on n, d, C_d, and C_A.

We will also need to use a comparison principle for locally-defined solutions.

Theorem 2.7.32 (Local comparison principle, Theorem 11.17 of [DFM19b])**.** *Let $x_0 \in \Gamma$ and $r > 0$ and let $X_0 \in \Omega$ be a Corkscrew point for $\Delta(x_0, r)$. Let $u, v \in W_r(B(x_0, 2r))$ be two non-negative, not identically zero, solutions of $Lu = Lv = 0$ in $B(x_0, 2r)$, such that $Tu = Tv = 0$ on $\Gamma \cap B(x_0, 2r)$. Then*

$$C^{-1} \frac{u(X_0)}{v(X_0)} \leq \frac{u(X)}{v(X)} \leq C \frac{u(X_0)}{v(X_0)} \quad \text{for } X \in \Omega \cap B(x_0, r),$$

where $C > 0$ depends only on n, d, C_d and C_A.

Let us show that we also have a change of poles for the Poisson kernel.

Lemma 2.7.33 (Change of Poles for Poisson kernel)**.** *Let $\Delta \subset \Delta_0 \subset \Gamma$ be surface balls in Γ, and set X_0, X to be Corskcrew points of Δ_0, Δ respectively. If $\omega \ll \sigma$ then*

$$k^X(y) \approx \frac{k^{X_0}(y)}{\omega^{X_0}(\Delta)}, \qquad \text{for } \sigma - a.e.\ y \in \Delta.$$

Proof. Write $\Delta_0 = B(x_0, r_0) \cap \Gamma$ and $\Delta = B(x, r) \cap \Gamma$. Let X_0' be a Corkscrew point for $4\Delta_0$. Then $X_0' \notin B(x_0, 2r_0)$, and hence $X_0' \notin B(x, 2r)$. By the Harnack chains we have that $\omega^{X_0'}(E) \approx \omega^{X_0}(E)$, for any Borel $E \subset \Delta$. We apply (2.7.31) to see that for any Borel $E \subset \Delta$,

$$\omega^X(E) \approx \frac{\omega^{X_0'}(E)}{\omega^{X_0'}(\Delta)} \approx \frac{\omega^{X_0}(E)}{\omega^{X_0}(\Delta)}.$$

The desired result now follows by the differentiation theorem and letting $E \searrow y \in Q$. $\square$

The next lemma collects the results which allow us to compare elliptic measures (and Green functions, incidentally) for operators which agree locally near a surface ball. We remark in passing that the proof shown is based on the local comparison principle stated above, but it is also possible to obtain the results in the following lemma without appealing to the local comparison principle, by means of the identity (2.7.11), the properties of the Green function, and the Caccioppoli inequality.

Lemma 2.7.34 (Comparison of elliptic measures near the boundary)**.** *Fix $x \in \Gamma$, $r > 0$, let X_0 be a Corkscrew point (with Corkscrew constant $c < 1$) for the surface ball $\Delta_0 := \Delta(x, r)$, and suppose that A_0 and A_1 are two matrices satisfying (2.7.1) and $A_0 \equiv A_1$ in $B(x, 4c^{-1}r) \cap \Omega$. Let $L_0 = -\operatorname{div} A_0\nabla$ and $L_1 = -\operatorname{div} A_1\nabla$. The following statements hold.*

(i) For each surface ball $\Delta' \subset \Delta_0$, we have that

$$\tfrac{1}{C}\omega_1^{X_0}(\Delta') \leq \omega_0^{X_0}(\Delta') \leq C\omega_1^{X_0}(\Delta'). \tag{2.7.35}$$

(ii) The measures $\omega_1^{X_0}$ and $\omega_0^{X_0}$ are mutually absolutely continuous on Δ_0.

(iii) If $\omega_0^{X_0}|_{\Delta_0} \ll \sigma|_{\Delta_0}$, then $\omega_1^{X_0}|_{\Delta_0} \ll \sigma|_{\Delta_0}$, and $k_0^{X_0}(y) \approx k_1^{X_0}(y)$, for σ−a.e. $y \in \Delta_0$.

Proof. (i). Let $\tilde{X}_0 \in \Omega$ be a Corkscrew point for $\Delta(x, 4c^{-1}r)$, so that $\tilde{X}_0 \in \Omega \backslash B(x, 4r)$. Note that since $L_0 \equiv L_1$ in $B(x, 4r) \cap \Omega$, then $A_0^T \equiv A_1^T$ in $B(x, 4r) \cap \Omega$. As such, we may apply Theorem 2.7.32 to the Green functions $g_0^T(\cdot, \tilde{X}_0)$ and $g_1^T(\cdot, \tilde{X}_0)$, to deduce that

$$\frac{g_0^T(X_0, \tilde{X}_0)}{g_1^T(X_0, \tilde{X}_0)} \approx \frac{g_0^T(Y, \tilde{X}_0)}{g_1^T(Y, \tilde{X}_0)}, \qquad \text{for every } Y \in B(x, r) \cap \Omega. \tag{2.7.36}$$

We now use (2.7.9), (2.7.19), and (2.7.23) to obtain that $g_i^T(X_0, \tilde{X}_0) = g_i(\tilde{X}_0, X_0) \approx \frac{\omega_i^{\tilde{X}_0}(\Delta_0)}{r^{d-1}}$, $i = 0, 1$. By the Harnack inequality and Harnack chains we have that $\omega_i^{\tilde{X}_0}(\Delta_0) \approx \omega_i^{X_0}(\Delta_0) \approx 1$, $i = 0, 1$, and thus using these last results in (2.7.36), we see that

$$g_0^T(Y, \tilde{X}_0) \approx g_1^T(Y, \tilde{X}_0), \qquad \text{for every } Y \in B(x,r) \cap \Omega. \tag{2.7.37}$$

Now fix a surface ball $\Delta' = \Delta(y, r') \subset \Delta_0$ and let Y' be a Corkscrew point for Δ'. Then $Y' \in B(x,r) \cap \Omega$. Since we may write

$$g_i^T(Y', \tilde{X}_0) = g_i(\tilde{X}_0, Y') \approx \frac{\omega_i^{\tilde{X}_0}(\Delta')}{r'^{d-1}}, \qquad i = 0, 1,$$

then, using (2.7.37), we observe that $\omega_0^{\tilde{X}_0}(\Delta') \approx \omega_1^{\tilde{X}_0}(\Delta')$, and (2.7.35) immediately follows.

(ii). Let us see that (2.7.35) implies the mutual absolute continuity of $\omega_1^{X_\Delta}$ and $\omega_0^{X_\Delta}$, it suffices to use the (outer and inner) regularity of the measures, the Besicovitch covering theorem [Bes45] applied to a bounded open set, and (2.7.35). More precisely, let $E \subseteq \Delta_0$ be a Borel set such that $\omega_0^{X_0}(E) > 0$. Then by the inner regularity of $\omega_0^{X_0}$, there exists a compact set $K \subseteq E$ such that $\omega_0^{X_0}(K) \approx \omega_0^{X_0}(E)$. To prove that $\omega_1^{X_0}(E) > 0$, it suffices to show that $\omega_1^{X_0}(K) > 0$. Let $V \subseteq \Delta_0$ be an open set in the subspace topology of Δ_0 such that $V \supset K$. Then we can write $V = \Gamma \cap \left(\bigcup_{x \in V} B(x, r_x)\right)$, for suitable finite $r_x > 0$. Let $\mathcal{B} := \{B(x, r_x)\}$. The latter is a Besicovitch covering of the bounded set V, and thus applying the Besicovitch covering theorem, we can write $V = \Gamma \cap \Big(\cup_{B \in \mathcal{B}'} B\Big)$ where $\mathcal{B}'$ is a subcollection of $\mathcal{B}$ such that the balls intersect an at most uniformly finite (depending only on n) number of times. Then we may use (2.7.35) to estimate $\omega_0^{X_0}(K)$ from above by $\omega_1^{X_0}(V)$ times a constant independent of V. Since this is true for any V, it follows by the outer regularity of $\omega_1^{X_0}$ that $\omega_1^{X_0}(K) > 0$, which completes the proof of (ii).

(iii). Since $\omega_0^{X_0}|_{\Delta_0} \ll \sigma|_{\Delta_0}$ and we have seen that (ii) holds, then $\omega_1^{X_0}|_{\Delta_0} \ll \sigma|_{\Delta_0}$ follows. Now fix $y \in \Delta_0$, and for each $k \in \mathbb{N}$, let $\Delta_k = \Delta(y, r_k) \subset \Delta_0$ with $r_k \searrow 0$ as $k \to \infty$. According to (i), we may then write

$$\frac{\omega_0^{X_0}(\Delta_k)}{\sigma(\Delta_k)} \approx \frac{\omega_1^{X_0}(\Delta_k)}{\sigma(\Delta_k)}, \qquad \text{for each } k \in \mathbb{N} \text{ and every } y \in_0 .$$

Finally, we send $k \to \infty$ in the above estimate, and due to the Lebesgue Differentiation Theorem we arrive at the desired result. $\square$

The next technical result will be used in step 3 of the proof of Theorem 2.1.1 (Section 2.9.5). Morally, the lemma establishes a local equivalence of Poisson kernels on a cube Q_0 for operators whose matrices are equal in a "Carleson box" R_{Q_0} (see (2.4.32). Since our proof of Theorem 2.1.1 will use both global and local sawtooth domains $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F},Q_0}$ (see (2.4.27), which do not coincide in general (because $\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0}$ could be non-empty), we will resolve this residual issue within Lemma 2.7.38 by obtaining its conclusion for matrices that agree on the smaller set $S := R_{Q_0}\backslash(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0})$. For the purpose of the proof of Lemma 2.7.38, S behaves the same as R_{Q_0}, because the portion removed, $R_{Q_0}\cap\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0}$, consists of thin pieces anchored at the boundary of $R_{Q_0}\backslash Q_0$, and thus S retains ample access to the interior (in the subspace topology of Γ) of Q_0.

Lemma 2.7.38 (Comparison of Poisson Kernels in a cube, Lemma 3.24 in [CHM19])**.** *Fix $Q_0 \in \mathbb{D}$, let $X_0 \in \Omega$ be a Corkscrew point (with Corkscrew constant c) for the surface ball $\Delta(x_{Q_0}, 10c^{-1}\sqrt{n}A_2\ell(Q_0))$, let $\mathcal{F} \subset Q_0$ be a disjoint family, and suppose that A_0 and A_1 are two matrices satisfying (2.7.1) and $A_0 \equiv A_1$ in $R_{Q_0}\backslash(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0})$ (see (2.4.27) and (2.4.32)). Let $L_0 = -\operatorname{div} A_0\nabla$ and $L_1 = -\operatorname{div} A_1\nabla$. If the corresponding elliptic measures $\omega_0^{X_0}, \omega_1^{X_0}$ are absolutely continuous with respect to σ, then for each $t \in (0,1)$ we have that*

$$\frac{1}{C_t}k_1^{X_0}(y) \leq k_0^{X_0}(y) \leq C_t k_1^{X_0}(y), \qquad \textit{for } \sigma-\textit{almost every } y \in Q_0\backslash\Sigma_{Q_0,t},$$

and $\Sigma_{Q_0,t} := \big\{x \in Q_0 : \operatorname{dist}(x, \Gamma\backslash Q_0) < t\ell(Q_0)\big\}$.

Sketch of proof. The proof is essentially the same as in Lemma 3.24 of [CHM19] (see Remark 2.2.11), except that our claim is slightly sharper by requiring that $A_0 \equiv A_1$ only in $R_{Q_0}\backslash(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0})$ as opposed to in all of R_{Q_0}. It turns out that this is enough: in [CHM19], the authors cover $Q_0\backslash\Sigma_{Q_0,t}$ with a uniformly finite (cardinality depending on t) collection of surface balls $\{\Delta_k = \Delta(x_k, r_k)\}_k$, and chosen such that x_k and $r_k \approx_t \ell(Q_0)$ verify the containment $B_k\cap\Omega = B(x_k, Cr_k)\cap\Omega \subseteq R_{Q_0}$, for some uniform large constant $C \approx 1$. Actually, their method of proof gives that if $X \in B_k \cap \Omega$ and $X \in I \in \mathcal{W}$, then $I \in \mathcal{W}_Q$ with $Q \in \mathbb{D}_{Q_0}$. Hence, if $X \in \Omega_{\mathcal{F}} \cap B_k$, then $X \in \Omega_{\mathcal{F},Q_0}$. It follows that $(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0}) \cap B_k = \varnothing$, whence we have that $B_k \subset R_{Q_0}\backslash(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0})$. The rest of the proof is elementary; one employs Lemma 2.7.34 and the fact that $L_0 = L_1$ in B_k to get the desired result on each Δ_k, and via Harnack Chains and the Harnack Inequality, one can teleport (with constants depending on t) from a Corkscrew point for Δ_k to the fixed point X_0. $\square$

Remark 2.7.39. The main reason why we require the slightly sharper version of this result as opposed to in [HM12], [CHM19], is because we will decide to use our analogue of the Dahlberg-Jerison-Kenig sawtooth lemma, Lemma 2.8.1, on the unbounded sawtooth domain $\Omega_{\mathcal{F}}$ as opposed to the bounded sawtooth $\Omega_{\mathcal{F},Q_0}$ whose mixed-dimension elliptic PDE theory we have not fully developed (although it would not be hard to make it work given our theory in this chapter and in [DFM]; it would just be tedious rather than difficult).

We next see how to relate the solvability of the Dirichlet problem with the quantitative absolute continuity of the elliptic measure. In the setting of co-dimension 1, the following theorem has well-known analogues [Ken94].

Theorem 2.7.40 (Relationship between A_∞ and the Dirichlet problem)**.** *Assume that Γ is a closed d-*ADR *set with $d \in [1, n-1)$ not necessarily an integer. Suppose that the matrix A satisfies (2.7.1), let $L = -\operatorname{div} A\nabla$, let ω be the elliptic measure associated to L, and let $p, p' \in (1,\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then, the following statements are equivalent:*

(a) For each $f \in C_c(\Gamma)$, the solution to the Dirichlet problem u satisfies

$$\|Nu\|_{L^{p'}(\Gamma)} \leq C\|f\|_{L^{p'}(\Gamma)}, \tag{2.7.41}$$

where Nu is the non-tangential maximal function and C is a uniform constant.

(b) We have that $\omega \ll \sigma$ and $\frac{d\omega}{d\sigma} \in RH_p$ (see Definition 2.7.28).

(c) We have that $\omega \ll \sigma$, and there is a uniform constant C_0 such that for every surface ball $\Delta = \Gamma \cap B(x, r)$, there exists $X_\Delta \in \Omega$, which is a Corkscrew point for Δ, verifying the following scale-invariant L^p estimate:

$$\int_\Delta (k^{X_\Delta})^p \, d\sigma \leq C_0 \sigma(\Delta)^{1-p}. \tag{2.7.42}$$

Proof. (a) $\Longrightarrow$ (b). Fix a surface ball $\Delta = \Gamma \cap B(x_0, r)$ and $X_\Delta \in \Omega$ a Corkscrew point for the surface ball Δ. Let $X_0 \in \Omega \backslash B(x_0, 2r)$ be a Corkscrew point for the surface ball 4Δ, and immediately by Harnack Chains and the Harnack Inequality we see that $\omega^{X_0} \approx \omega^{X_\Delta}$, whence we need only prove the desired result with pole X_0 as opposed to X_Δ. We will show that $\omega^{X_0} \ll \sigma$ on Δ and that $\frac{d\omega^{X_0}}{d\sigma} \in RH_p(\sigma, \Delta)$ via the characterization in Theorem 2.2.17 (vi). Owing to Lemma 2.7.17 (i), we have that for any $X \in \Omega$, $\omega^X \ll \omega^{X_0}$. Accordingly, for each $X \in \Omega$ let $\mathscr{K}(X, \cdot) = \frac{d\omega^X}{d\omega^{X_0}}$ be the Radon-

Nikodym derivative (see [Fol99]). Since $\{\omega^X\}$ is a family of probability measures, we trivially have that $\mathscr{K}(X,\cdot) \in L^1(\Gamma, \omega^{X_0})$, for each $X \in \Omega$.

Now fix a non-negative $f \in C_c(\Delta)$, and let $u(Y) := \int_\Gamma f\, d\omega^Y$, for each$Y \in \Omega$. We claim that there exists a uniform constant $C > 0$ such that

$$\begin{aligned}(\mathcal{M}_{\omega^{X_0}} f)(x) := \sup_{\Delta(x,s)\subseteq\Delta} \frac{1}{\omega^{X_0}(\Delta(x,s))} \int_{\Delta(x,s)} f\, d\omega^{X_0} \\ \leq C(Nu)(x), \qquad \text{for each } x \in \Delta. \quad (2.7.43)\end{aligned}$$

Assume the claim for a moment. Then, since u is the solution to the Dirichlet problem with data f and (2.7.41) holds, we have the estimate

$$\|\mathcal{M}_{\omega^{X_0}} f\|_{L^{p'}(\Delta,\sigma)} \lesssim \|Nu\|_{L^{p'}(\Delta,\sigma)} \leq \|Nu\|_{L^{p'}(\Gamma,\sigma)} \lesssim \|f\|_{L^{p'}(\Gamma,\sigma)} = \|f\|_{L^{p'}(\Delta,\sigma)},$$

valid for each non-negative $f \in C_c(\Delta)$. If $f \in L^{p'}(\Delta,\sigma)$ is non-negative, we may approximate it by $C_c(\Delta)$ non-negative functions in a standard way, so that the estimate $\|\mathcal{M}_{\omega^{X_0}} f\|_{L^{p'}(\Delta,\sigma)} \lesssim \|f\|_{L^{p'}(\Delta,\sigma)}$ is valid for all non-negative $f \in L^{p'}(\Delta,\sigma)$. Consequently, according to Theorem 2.2.17 (vi), we deduce that $\frac{d\omega^{X_0}}{d\sigma} \in RH_p(\Delta)$ with RH_p characteristic independent of Δ, as desired.

Thus we proceed to prove (2.7.43). Fix $x \in \Delta$, $\alpha > 0$, and $X \in \gamma^\alpha(x)$ such that $s := |X - x|$ satisfies $\Delta(x,s) \subseteq \Delta$. By definition of $\gamma^\alpha(x)$, we have that $\delta(X) \leq s \leq (1+\alpha)\delta(X)$. Observe that by the non-negativity of f and the properties of the Radon-Nikodym derivative,

$$u(X) \geq \int_{\Delta(x,s)} f\, d\omega^X = \int_{\Delta(x,s)} f \mathscr{K}(X,\cdot)\, d\omega^{X_0}. \qquad (2.7.44)$$

Since ω^{X_0} is a doubling measure on Δ (see Lemma 2.7.24), we may use the Differentiation Theorem for doubling measures [Fol99] to obtain that for ω^{X_0} – a.e. $y \in \Delta(x,s)$,

$$\mathscr{K}(X,y) = \lim_{\Delta' \searrow y} \frac{1}{\omega^{X_0}(\Delta')} \int_{\Delta'} \mathscr{K}(X,\cdot)\, d\omega^{X_0} = \lim_{\Delta' \searrow y} \frac{\omega^X(\Delta')}{\omega^{X_0}(\Delta')},$$

(here, Δ' is a surface ball centered at y and contained in $\Delta(x,s)$). Note that necessarily we have $X_0 \in \Omega \backslash B(x,2s)$. Denote by $A_{\Delta(x,s)}$ a Corkscrew point for $\Delta(x,s)$, and so from the Comparison Principle (2.7.31) we may conclude that

$$\frac{\omega^{A_{\Delta(x,s)}}(\Delta')}{\omega^{X_0}(\Delta')} \approx \frac{1}{\omega^{X_0}(\Delta(x,s))}, \qquad \text{for all } \Delta' \searrow y \text{ and all } \Delta(x,s) \subseteq \Delta.$$

On the other hand, for any $y \in \Delta(x,s)$, $\frac{1}{1+\alpha}s \le |X-y| \le 2s$, which implies by the Harnack chains that $\omega^X(\Delta') \approx \omega^{A_{\Delta(x,s)}}(\Delta')$. Putting all these observations together and back into (2.7.44), we deduce that $(Nu)(x) \gtrsim \frac{1}{\omega^{X_0}(\Delta(x,s))}\int_{\Delta(x,s)} f\, d\omega^{X_0}$ for each $x \in \Delta$ and each $s > 0$ such that $\Delta(x,s) \subseteq \Delta$. Since $\Delta(x,s) \subseteq \Delta$ is arbitrary, the claim (2.7.43) follows.

(b) $\Longrightarrow$ (a). This is a consequence of Theorem 4.1 in [MZ19] (formally, they have symmetric A, but this assumption can be dropped).

(b) $\Longrightarrow$ (c). By assumption we already have that $\omega \ll \sigma$. Now fix $\Delta \subset \Gamma$ and X_Δ the Corkscrew point for Δ given by property (b) above. We apply (2.7.29) with $\Delta' = \Delta$ to get that $\int_\Delta (k^{X_\Delta})^p\, d\sigma \le C_0^p \sigma(\Delta)^{1-p} \omega^{X_\Delta}(\Delta)^p \le C\sigma(\Delta)^{1-p}$, where we used that ω^{X_Δ} is a probability measure. Hence (2.7.42) is established.

(c) $\Longrightarrow$ (b). By assumption we already have that $\omega \ll \sigma$. Now fix a surface ball $\Delta \subset \Gamma$ and another surface ball $\Delta' \subseteq \Delta$. According to (2.7.42) we have the estimate $\left(\frac{1}{\sigma(\Delta')}\int_{\Delta'} (k^{X_{\Delta'}})^p\, d\sigma\right)^{1/p} \le C_0^{\frac{1}{p}} \frac{1}{\sigma(\Delta')}$. Next, we use Lemma 2.7.33 applied with surface balls Δ, Δ' to obtain that there exists a uniform (in Δ, Δ') constant $\tilde{c}$ such that $k^{X_{\Delta'}} \ge \tilde{c}\frac{k^{X_\Delta}}{\omega^{X_\Delta}(\Delta')}$, $\sigma-$a.e. on Δ'. Putting these observations together, we deduce that

$$\left(\frac{1}{\sigma(\Delta')}\int_{\Delta'} (k^{X_\Delta})^p\, d\sigma\right)^{1/p} \le \frac{C_0^{\frac{1}{p}}}{\tilde{c}} \frac{1}{\sigma(\Delta')} \omega^{X_\Delta}(\Delta') = \frac{C_0^{\frac{1}{p}}}{\tilde{c}} \frac{1}{\sigma(\Delta')} \int_{\Delta'} k^{X_\Delta}\, d\sigma,$$

as desired. □

When (a) of the above theorem occurs we say that $(\mathrm{D})_{p'}$ is solvable for L or that L is solvable in $L^{p'}$. In such case, for every $f \in L^{p'}(\mathbb{R}^d)$ there exists a unique u such that $Lu = 0$ in $\mathbb{R}^n$, (2.7.41) holds and u converges non-tangentially to f for $\sigma-$a.e. $x \in \Gamma$.

Remark 2.7.45 (Equivalence of RH_p and RH_p^{dyadic}). Suppose throughout this remark that $\omega \ll \sigma$. In Theorem 2.7.40, we saw that the condition (2.7.42) is equivalent to $\frac{d\omega}{d\sigma} \in RH_p$. Consider the following dyadic analogue of condition (2.7.42): For each $Q \in \mathbb{D}$ and for

X_Q a Corkscrew point for Q, the estimate

$$\int_Q (k^{X_Q})^p \, d\sigma \lesssim \sigma(Q)^{1-p} \tag{2.7.46}$$

holds. Using Corollary 2.7.25 and the Harnack inequality to flexibly move the poles, it is not difficult to see that the condition (2.7.46) is equivalent to (2.7.42). Of course, this new condition (2.7.46) is also equivalent to the condition that $\frac{d\omega}{d\sigma} \in RH_p^{\text{dyadic}}$. It follows that $\frac{d\omega}{d\sigma} \in RH_p$ is equivalent to $\frac{d\omega}{d\sigma} \in RH_p^{\text{dyadic}}$.

To end this section, we record the L^p-control of the square function by the non-tangential maximal function under the assumption that the elliptic measure lies in A_∞.

Theorem 2.7.47 ($S < N$; Theorem 3.1 of [MZ19])**.** *Suppose that $d \in [1, n-1)$ is not necessarily an integer, Γ is d-ADR , and that A is a (not necessarily symmetric, see Remark 2.7.48 below) matrix satisfying (2.7.1). Assume that for some $p' \in (1, \infty)$, $(D)_{p'}$ is solvable for L. Write u for the solution to the Dirichlet problem $(\mathrm{D})_{p'}$ with data $f \in L^{p'}(\Gamma)$. Then, for all apertures $\alpha > 0$ we have the estimate*

$$\|Su\|_{L^{p'}(\Gamma)} \lesssim \|f\|_{L^{p'}(\Gamma)},$$

where Su is given in Definition 2.2.19, and the implicit constants depend on n, d, C_d, C_A, α, and the RH_p characteristic of $k = d\omega/d\sigma$ (see Definition 2.7.28).

Remark 2.7.48. We remark that the previous theorem is stated in [MZ19] for symmetric matrices and $d \in \mathbb{N}$ only, but in fact their method of proof generalizes to non-symmetric matrices, mainly using (2.7.9), and to all $d \in \mathbb{R}$, $d \in [1, n-1)$. For concreteness, the fact that $d \in \mathbb{N}$ was never explicitly used in the proof (recall that the construction of the dyadic cubes Lemma 2.3.1 works for all real $d \in (0, n-1)$), and the symmetry of the matrix A is used only in Step 3 of the proof of their Proposition 1.16, and explicitly arising only in their calculation (3.78), where $A = A^T$ is used to maneuver the integration by parts. However, we note that their function $G = G(X)$ in (3.78) really is $G(X_Q, X)$ (see their estimate (3.73)), while $G(X_Q, X) = G^T(X, X_Q)$ by (2.7.9), and the latter is the "correct" Green's function for which the needed cancellation $\operatorname{div} A^T G^T(\cdot, X_Q) = 0$ will hold in their (3.78) (of course, in the symmetric setting, there is no difference between these Green's functions). Thus, in the non-symmetric setting, the last two lines of their calculation (3.78) read

$$\dots = \frac{1}{2}\iint_{\mathbb{R}^n} u^2(X)A^T(X)\nabla_X G(X_Q,X)\nabla\psi_N(X)\,dX$$
$$- \iint_{\mathbb{R}^n} u(X)G(X_Q,X)A(X)\nabla u(X)\nabla\psi_N(X)\,dX =: \frac{1}{2}I - II.$$

One then uses the representation $G(X_Q,X) = G^T(X,X_Q)$ again while bounding $|I|$ and $|II|$ (see their (3.79)-(3.81)) to exploit the fact that $G^T(\cdot,X_Q)$ solves $-\operatorname{div} A^T G^(\cdot,X_Q) = 0$ in the fat sawtooth domain, so that one may use the Caccioppoli inequality and the Harnack inequality as required. At the last step when bounding $|I|$ and $|II|$, we switch from $G^T(X_I,X_Q)$ to $G(X_Q,X_I)$ and invoke Lemma 2.7.18 to obtain the required control *with* ω_L (and no dependence on ω_{L^T}). As explained here, one recovers their full Proposition 1.16 in the non-symmetric case; the rest of the proof of their Theorem 3.1 sees no obstacle from the non-symmetric point of view.

Remark 2.7.49. We note that Theorem 2.7.47 has content for all $d \in [1, n-2)$ with d not necessarily an integer, and for all closed d-ADR unbounded Γ. Indeed, for any such Γ, we consider the special operator L_{DEM} of [DEM] Theorem 6.7, and recall that $\omega_{\text{DEM}} \ll \sigma$. By Theorem 2.7.40, it follows that there exists $p > 1$ such that $(\text{D})_p$ is solvable for L_{DEM}. We thus see that the hypotheses of Theorem 2.7.47 are verified for this operator, and hence the control of the square function by the non-tangential maximal function holds.

2.8 The projection lemma for the dyadically-generated sawtooth

Having shown that the triple $(\Omega_{\mathcal{F}}, m, \sigma_\star)$ satisfies the axioms (H1)-(H6) in Section 2.5, we appeal to the elliptic PDE theory set forth in [DFM] to conclude that there is a elliptic measure $\omega_\star$ on $\partial\Omega_{\mathcal{F}}$ associated to the operator $L = -\operatorname{div} A\nabla$ whose matrix A satisfies (2.7.1). This elliptic measure $\omega_\star$ on the sawtooth boundary $\Omega_{\mathcal{F}}$ enjoys many similar properties to the elliptic measure on Γ which were reviewed in Subsection 2.7.1. In particular, we note that $\omega_\star$ has the doubling property; that is, we have Lemma 2.7.24 with Ω, Γ, and ω replaced by $\Omega_{\mathcal{F}}$, $\partial\Omega_{\mathcal{F}}$, and $\omega_\star$, respectively (see Lemma 15.43 of [DFM]).

The following lemma is an analogue of the Dahlberg-Jerison-Kenig sawtooth lemma

[DJK84]; it has already been shown in [DMb] in a similar setting, but we include its proof here too for completeness.

Lemma 2.8.1 (Dyadic sawtooth lemma, global version)**.** *Suppose that Γ is a d-ADR set with $d \in (0, n-1)$. Fix $Q_0 \in \mathbb{D}$, let $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes, and let $\mathcal{P}_{\mathcal{F}}$ be the corresponding projection operator, as in (2.3.9). Let X_0 be the Corkscrew point for Q_0 with respect to both Ω and $\Omega_{\mathcal{F}}$, whose existence is shown in Proposition 2.4.41 and Corollary 2.4.42. Let A be a matrix of essentially bounded, real coefficients satisfying the weighted ellipticity condition (2.7.1). Let $\omega = \omega^{X_0}$ and $\omega_\star = \omega_\star^{X_0}$ denote the respective elliptic measures for the domains Ω and $\Omega_{\mathcal{F}}$, with the fixed pole X_0 as above. Let $\mu = \mu^{X_0}$ be the measure defined on Q_0 as*

$$\mu(F) = \omega_\star\big(F\backslash(\cup_j Q_j)\big) + \sum_{Q_j \in \mathcal{F}} \frac{\omega(F \cap Q_j)}{\omega(Q_j)} \omega_\star(P_j), \qquad F \subset Q_0, \tag{2.8.2}$$

where P_j is the $n-$dimensional cube constructed in Proposition 2.4.44. Then $\mathcal{P}_{\mathcal{F}}\mu$ depends only on $\omega_\star$ and not on ω. More precisely,

$$\mathcal{P}_{\mathcal{F}}\mu(F) = \omega_\star\big(F\backslash(\cup_j Q_j)\big) + \sum_{Q_j \in \mathcal{F}} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega_\star(P_j), \qquad F \subset Q_0. \tag{2.8.3}$$

Moreover, there exists $\theta > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and Borel $F \subset Q$, we have that

$$\Big(\frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)}\Big)^\theta \lesssim \frac{\mathcal{P}_{\mathcal{F}}\mu(F)}{\mathcal{P}_{\mathcal{F}}\mu(Q)} \lesssim \frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)}. \tag{2.8.4}$$

Proof. We first verify that (2.8.3) holds. Let $F \subset Q_0$ be a Borel set, and observe that

$$\begin{aligned}\mathcal{P}_{\mathcal{F}}\mu(F) &= \mu\big(F\backslash(\cup_j Q_j)\big) + \sum_{Q_j \in \mathcal{F}} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \mu(Q_j) \\ &= \omega_\star\big(F\backslash(\cup_j Q_j)\big) + \sum_{Q_j \in \mathcal{F}} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \sum_{Q_k \in \mathcal{F}} \frac{\omega(Q_j \cap Q_k)}{\omega(Q_k)} \omega_\star(P_k),\end{aligned}$$

which implies (2.8.3) since $\mathcal{F}$ is a pairwise disjoint family.

We now show the right-hand side inequality in (2.8.4), so we fix $Q \in \mathbb{D}_{Q_0}$ and $F \subset Q$.

First, suppose that there exists $Q_j \in \mathcal{F}$ such that $Q \subset Q_j$. In this case, we have that

$$\frac{\mathcal{P}_{\mathcal{F}}\mu(F)}{\mathcal{P}_{\mathcal{F}}\mu(Q)} = \frac{\frac{\sigma(F\cap Q_j)}{\sigma(Q_j)}\omega_\star(P_j)}{\frac{\sigma(Q\cap Q_j)}{\sigma(Q_j)}\omega_\star(P_j)} = \frac{\frac{\sigma(F\cap Q_j)}{\sigma(Q_j)}\omega(Q_j)}{\frac{\sigma(Q\cap Q_j)}{\sigma(Q_j)}\omega(Q_j)} = \frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)}.$$

Therefore, it remains to consider the case that Q is not contained in any Q_j, $Q_j \in \mathcal{F}$; in this case, we have that $Q \in \mathbb{D}_{\mathcal{F},Q_0} \subset \mathbb{D}_{\mathcal{F}}$, and that, if $Q_j \cap Q \neq \varnothing$, then $Q_j \subsetneq Q$. Let $x_j^\star$ be the center of P_j and let $r_j \approx \ell(Q_j)$ be as in Notation 2.4.51, so that $P_j \subseteq \Delta_\star(x_j^\star, r_j)$. Since $\ell(P_j) \approx \ell(Q_j) \approx r_j$, we have that

$$\omega_\star^{X_0}(P_j) \geq \omega_\star^{X_0}\big(\Delta_\star(x_j^\star, \ell(P_j)/2)\big) \gtrsim \omega_\star^{X_0}\big(\Delta_\star(x_j^\star, r_j)\big), \tag{2.8.5}$$

where we used the doubling property of $\omega_\star^{X_0}$ [DFM, Lemma 15.43] in the last estimate[2]. Using (2.8.3), we have that

$$\begin{aligned}
\mathcal{P}_{\mathcal{F}}\mu(Q) &= \omega_\star\big(Q\backslash(\cup_j Q_j)\big) + \sum_{Q_j\in\mathcal{F}, Q_j\subsetneq Q} \frac{\sigma(Q\cap Q_j)}{\sigma(Q_j)}\omega_\star(P_j) \\
&= \omega_\star\big(Q\backslash(\cup_j Q_j)\big) + \sum_{Q_j\in\mathcal{F}, Q_j\subsetneq Q} \omega_\star(P_j) \gtrsim \omega_\star\big(Q\backslash(\cup_j Q_j)\big) + \sum_{Q_j\in\mathcal{F}, Q_j\subsetneq Q} \omega_\star\big(\Delta_\star(x_j^\star, r_j)\big) \\
&\geq \omega_\star\Big(\big(Q\cap\partial\Omega_{\mathcal{F}}\big) \bigcup \big(\textstyle\bigcup_{Q_j\in\mathcal{F}, Q_j\subsetneq Q} \Delta_\star(x_j^\star, r_j)\big)\Big) \geq \omega_\star\big(\Delta_\star^Q\big),
\end{aligned} \tag{2.8.6}$$

where $\Delta_\star^Q$ is the surface ball in Proposition 2.4.53, in the third step we used (2.8.5), and in the last line we used Proposition 2.4.53, and the doubling property of $\omega_\star$ and Propositions 2.4.35 and 2.4.40 to see that $\omega_\star(Q\backslash(\cup_j Q_j)) = \omega_\star(Q \cap \partial\Omega_{\mathcal{F}})$.

Now let X_Q be the Corkscrew point (simultaneously for Ω and $\Omega_{\mathcal{F}}$) for the cube Q. By the change of poles (Lemma 15.61 of [DFM]) and the doubling property of $\omega_\star$, for

[2]Note that if $\ell(Q_j) \approx \ell(Q_0)$, it may happen that $X_0 \in B(x_j^\star, 2r_j)$, so that we cannot conclude directly from [DFM, Lemma 15.43] the doubling property of $\omega_\star^{X_0}$ on $\Delta_\star(x_j^\star, r_j)$. However, we can use the comparison principle for elliptic measures [DFM, Lemma 15.61] and the non-degeneracy of elliptic measure [DFM, Lemma 15.1] to see that $\omega_\star^{X_0} \approx \omega_\star^{X'}$, where $X' \in \Omega_{\mathcal{F}}$ is a Corkscrew point for the surface ball $\Delta_\star(x_j^\star, A\ell(Q_0))$ with $A \approx c^{-1}$ large enough so that $X' \notin B(x_j^\star, 2r_j)$. Then [DFM, Lemma 15.43] applies for $\omega_\star^{X'}$, and hence $\omega_\star^{X_0}$ is also doubling on the desired surface ball.

any Borel set $H_\star \subset \Delta_\star(y_Q, \hat{r}_Q)$ (see Notation 2.4.51), we have that

$$\omega_\star^{X_Q}(H_\star) \approx \frac{\omega_\star^{X_0}(H_\star)}{\omega_\star^{X_0}(\Delta_\star(y_Q, \hat{r}_Q))} \approx \frac{\omega_\star^{X_0}(H_\star)}{\omega_\star^{X_0}(\Delta_\star^Q)},$$

where in the last step we have used the fact that both $\Delta_\star(y_Q, \hat{r}_Q)$ and $\Delta_\star^Q$ have radius comparable to $\ell(Q)$, and that $\operatorname{dist}(\Delta_\star^Q, Q) \lesssim \ell(Q)$, so that we can compare both of these surface balls to a bigger (with radius still equivalent to $\ell(Q)$) surface ball containing both of them, and hence achieve the stated equivalence between these surface balls by using the doubling property of $\omega_\star$ (and, if needed, Harnack Chains and Harnack Inequality).

Using this last result, (2.4.52), and (2.8.6), we see that

$$\begin{aligned} \frac{\mathcal{P}_\mathcal{F}\mu(F)}{\mathcal{P}_\mathcal{F}\mu(Q)} &\lesssim \frac{\omega_\star^{X_0}(F \backslash (\cup_j Q_j))}{\omega_\star^{X_0}(\Delta_\star^Q)} + \sum_{Q_j \in \mathcal{F}, Q_j \subseteq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \frac{\omega_\star^{X_0}(P_j)}{\omega_\star^{X_0}(\Delta_\star^Q)} \\ &\approx \omega_\star^{X_Q}(F \backslash (\cup_j Q_j)) + \sum_{Q_j \in \mathcal{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega_\star^{X_Q}(P_j). \end{aligned} \tag{2.8.7}$$

Next, we claim that the estimates

$$\omega_\star^{X_Q}(F \backslash \cup_j Q_j) \lesssim \omega^{X_Q}(F \backslash \cup_j Q_j), \qquad \omega_\star^{X_Q}(P_j) \lesssim \omega^{X_Q}(Q_j) \tag{2.8.8}$$

hold. Indeed, the first one follows immediately by the maximum principle (Lemma 12.8 in [DFM]) since $\Omega_\mathcal{F} \subset \Omega$. For the second estimate, let $u(X) := \omega^X(Q_j)$ and $u_\star(X) := \omega_\star^X(P_j)$, and we first note that $\omega^X(Q_j) \approx 1 = u_\star(X)$ for each $X \in P_j$, by (2.7.21) and the fact that $\operatorname{dist}(P_j, Q_j) \approx \operatorname{diam} P_j \approx \operatorname{diam} Q_j \approx \ell(Q_j)$. Since also we have that $u_\star(X) = 0 \leq u(X)$ for each $X \in \partial\Omega_\mathcal{F} \backslash P_j$, we may thus apply again the maximum principle to conclude that $u_\star(X_Q) \lesssim u(X_Q)$, as desired.

Finally, from (2.8.7) and (2.8.8) we deduce that

$$\begin{aligned} \frac{\mathcal{P}_\mathcal{F}\mu(F)}{\mathcal{P}_\mathcal{F}\mu(Q)} &\lesssim \omega^{X_Q}(F \backslash (\cup_j Q_j)) + \sum_{Q_j \in \mathcal{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega^{X_Q}(Q_j) \\ &\lesssim \frac{\omega^{X_0}(F \backslash (\cup_j Q_j))}{\omega^{X_0}(Q)} + \sum_{Q_j \in \mathcal{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \frac{\omega^{X_0}(Q_j)}{\omega^{X_0}(Q)} = \frac{\mathcal{P}_\mathcal{F}\omega(F)}{\omega(Q)} = \frac{\mathcal{P}_\mathcal{F}\omega(F)}{\mathcal{P}_\mathcal{F}\omega(Q)}, \end{aligned}$$

where in the second line we used the change of poles for ω (2.7.31), and in the last step we used that $\mathcal{P}_{\mathcal{F}}\omega(Q) = \omega(Q)$ (which follows from the definition of $\mathcal{P}_{\mathcal{F}}$ (2.3.10) and the dyadic nature of Q and $\mathcal{F} = \{Q_j\}$). This ends the proof of the right-hand side of (2.8.4). The left-hand side is obtained because $\mathcal{P}_{\mathcal{F}}\omega$ is dyadically doubling by Lemma 2.3.13, $\mathcal{P}_{\mathcal{F}}\mu$ is dyadically doubling by Lemma 9.51 of [DMb] (whose argument is very similar to that of Lemma B.2 from [HM14]; see our Remark 2.2.11), and A_∞^{dyadic} is an equivalence relationship among dyadically doubling measures (see Lemma 2.3.16). $\square$

2.9 Proof of Theorem 2.1.1

In this section, we give the proof of Theorem 2.1.1; we mainly follow the outline in [HM12]; see also [CHM19]. In a paragraph after Theorem 2.6.3, we briefly describe the outline of how the extrapolation of Carleson measures allows us to prove Theorem 2.1.1.

Let A_0, A be two matrices that satisfy (2.7.1), write $\mathcal{A}_0 = w^{-1}A_0$, $\mathcal{A} = w^{-1}A$, and suppose that $d\lambda(X) = \frac{\mathfrak{a}(X)^2}{\delta(X)^{n-d}}dX$ is a (continuous) Carleson measure, where $\mathfrak{a}$ is defined in (2.1.2). As in Lemma 2.6.6, the natural discretization of the Carleson measure λ is the collection $\mathfrak{m} = \{\alpha_Q\}_{Q\in\mathbb{D}}$ with

$$\alpha_Q = \sum_{I\in\mathcal{W}_Q} \frac{\sup_{Y\in I^*}|\mathfrak{E}(Y)|^2}{\ell(I)^{n-d}}|I|, \qquad Q\in\mathbb{D},$$

where $\mathfrak{E}(Y) = \mathcal{A}(Y) - \mathcal{A}_0(Y)$. Let $L_0 = -\operatorname{div} A_0\nabla$, $L = -\operatorname{div} A\nabla$, and let ω_0, ω be the elliptic measures of L_0, L respectively.

Our program is to apply Theorem 2.6.3 to eventually obtain that $\omega_L \in A_\infty^{\text{dyadic}}$ (see Definition 2.7.28). Of course, this will imply that $\omega_L \ll \sigma$ and that $\frac{d\omega_L}{d\sigma} \in RH_q^{\text{dyadic}}$ for some $q > 0$, and by Remark 2.7.45, the latter is equivalent to $\frac{d\omega_L}{d\sigma} \in RH_q$, which in turn means that $\omega_L \in A_\infty$.

Thus we ought to verify the hypotheses of Theorem 2.6.3. Fix $Q_0 \in \mathbb{D}$, and observe that $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \lesssim \|\|\lambda\|\|_{\mathcal{C}}$ by Lemma 2.6.6. Given $\xi > 0$ small enough and to be chosen later, we fix a disjoint family $\mathcal{F} = \{Q_j\}_j \in \mathbb{D}_{Q_0}$ that verifies the estimate (see Definition 2.6.2)

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)} = \sup_{Q\in\mathbb{D}_{Q_0}} \frac{\mathfrak{m}(\mathbb{D}_{\mathcal{F},Q})}{\sigma(Q)} \le \xi. \tag{2.9.1}$$

Recall that $R_{Q_0} \subset B(x_{Q_0}, 7\sqrt{n}A_0\ell(Q_0))$ (see (2.4.32) and (2.4.30)). Let $X_0 \in \Omega$ be a

Corkscrew point (with Corkscrew constant c) for $\Delta(x_{Q_0}, 10c^{-1}\sqrt{n}A_2\ell(Q_0))$. Then

$$|X_0 - x_{Q_0}| \geq \delta(X_0) \geq c10c^{-1}\sqrt{n}A_2\ell(Q_0) = 10\sqrt{n}A_2\ell(Q_0),$$

which implies that $X_0 \in \Omega\backslash B(x_{Q_0}, 10\sqrt{n}A_2\ell(Q_0)) \subset \Omega\backslash R^{**}_{Q_0} \subset \Omega\backslash R_{Q_0}$, where

$$R^{**}_{Q_0} := \text{int}\,\Big(\bigcup_{I\in\mathcal{R}_{Q_0}} I^{**}\Big), \qquad I^{**} = (1+2\theta)I. \tag{2.9.2}$$

Moreover, according to Corollary 2.7.26, we have that ω^{X_0} is dyadically doubling in Q_0, while we also have that $\delta(X_0) \approx \text{dist}(X_0, Q_0) \approx \ell(Q_0)$.

We want to show that $\mathcal{P}_{\mathcal{F}}\omega_L^{X_0}$ satisfies (2.6.5), with uniform constants and with $\omega_L^{X_0}$ in place of μ. Since L_0 is solvable in some $L^{p'}$, then by Theorem 2.7.40, $\omega_{L_0}^{X_0} \ll \sigma$ and $k_0^{X_0} \in RH_p(\Delta(x_{Q_0}, A_0\ell(Q_0))$ (with reverse Hölder characteristic independent of Q_0).

2.9.1 Step 0: A qualitative reduction

We first make a reduction that allows us to conjure *qualitative* absolute continuity properties of the elliptic measure ω_L.

Definition 2.9.3 (Tubes encasing the boundary)**.** Fix $\tau > 0$. By a $\tau-$*tube around* Γ, we mean the open set $\Gamma_\tau := \big\{X \in \Omega \,:\, \text{dist}(X, \Gamma) < \tau\big\}$.

We define A_τ as $A_\tau = A_0$ in the $\tau-$tube Γ_τ, and $A_\tau = A$ in $\mathbb{R}^n\backslash\Gamma_\tau$. In the next steps we work with $L_\tau = -\,\text{div}\, A_\tau\nabla$ in place of L. We note that the ellipticity of A_τ is controlled by those of A and A_0. The same is true of the condition on the disagreement $\mathfrak{a}$.

Let us now exploit Lemma 2.7.34 to deduce absolute continuity properties of ω_{L_τ}, with dependence on τ.

Corollary 2.9.4 (Comparability of elliptic measures in tubes)**.** *Retain the notation above. Then $\omega_\tau^{X_0} \ll \sigma$, and if τ is small enough depending on n, d, C_d only, then $k_\tau^{X_0} \in RH_p(\Delta(x_{Q_0}, A_0\ell(Q_0)))$, with the RH_p characteristic depending on τ and $\ell(Q_0)$ (see Definition 2.2.15).*

We emphasize that this is a qualitative result (the dependence on τ and $\ell(Q_0)$ is non-optimal) - see also related comments after the proof.

Proof. Fix a surface ball $\Delta = \Delta(x_0, r)$ with $r \in (0, \frac{c\tau}{4})$, and let X_Δ be a Corkscrew point for Δ. By Lemma 2.7.34, we have that $\omega_0^{X_\Delta}$ is mutually absolutely continuous with $\omega_\tau^{X_\Delta}$ on Δ. Recall that for each $i = 0, \tau$, $\omega_i^{X_\Delta}$ is mutually absolutely continuous with $\omega_i^{X_0}$. It follows that $\omega_0^{X_\Delta} \ll \sigma$ on Γ, and therefore that $\omega_\tau^{X_0} \ll \omega_\tau^{X_\Delta} \ll \omega_0^{X_\Delta} \ll \sigma$ on Δ. Since $\Delta \subset \Gamma$ was arbitrary, we have that $\omega_\tau^{X_0} \ll \sigma$ on Γ.

Next, fix a surface ball $\Delta_\tau := \Delta(x, c\tau/4)$ with $x \in \Delta(x_{Q_0}, 10c^{-1}\sqrt{n}A_2\ell(Q_0))$, and let X_{Δ_τ} be a Corkscrew point for Δ_τ. Then Lemma 2.7.34 (iii) gives that $k_0^{X_{\Delta_\tau}}(y) \approx k_\tau^{X_{\Delta_\tau}}(y)$ for $\sigma-$a.e. $y \in \Delta_\tau$. Then the Harnack Chains and Harnack inequality guarantee the estimates

$$k_\tau^{X_0}(y) \approx_\tau k_\tau^{X_{\Delta_\tau}}(y) \approx k_0^{X_{\Delta_\tau}}(y) \approx_\tau k_0^{X_0}(y), \qquad \text{for } \sigma - \text{a.e. } y \in \Delta_\tau.$$

Since we may cover $\Delta(x_{Q_0}, A_0\ell(Q_0))$ by $\{\Delta_\tau\}$ as above, it follows that $k_\tau^{X_0}(y) \approx_\tau k_0^{X_0}(y)$ for $\sigma-$a.e. $y \in \Delta(x_{Q_0}, A_0\ell(Q_0)) \supset Q_0$. The desired result ensues. $\square$

It follows that we may assume that all the elliptic measures $\omega_\tau = \omega_{L_\tau}$ are absolutely continuous with respect to σ, and $k_\tau^{X_0} = k_{L_\tau}^{X_0} \in RH_p(Q_0)$ with the RH_p characteristic depending on τ and $\ell(Q_0)$. The dependence on τ and $\ell(Q_0)$ will not be an issue because these facts are used only qualitatively.

Therefore, in Step 1 below, we will have *a priori* that $\omega_\tau^{X_0} \ll \sigma$ and that $k_\tau^{X_0} \in L^p(Q_0, \sigma)$. We eventually establish a reverse Hölder inequality for k_τ with RH exponent and characteristic independent of τ and $\ell(Q_0)$. We will finally pass to the limit using Lemma 2.9.22 below to conclude that $\omega_L \ll \sigma$ and $\frac{d\omega_L}{d\sigma} \in RH_q$. This will in turn imply as desired that L is solvable in $L^{q'}$ by Theorem 2.7.40.

2.9.2 Step 1: Exploit smallness of $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)}$

Introduce the operator L_1 defined as $L_1 = L_\tau$ in $\Omega_{\mathcal{F},Q_0}$, and $L_1 = L_0$ in $\Omega \setminus \Omega_{\mathcal{F},Q_0}$, where $\Omega_{\mathcal{F},Q_0}$ is defined in (2.4.29). We write ω_1 for the elliptic measure associated to the operator L_1, and g_1 for the Green function associated to L_1. We have that $k_0^{X_0} \in RH_p(\Delta(x_{Q_0}, A_0\ell(Q_0)))$, and in particular by Theorem 2.7.40, we have that

$$\int_{Q_0} (k_0^{X_0})^p \, d\sigma \le \int_{\Delta(x_{Q_0}, A_0\ell(Q_0))} (k_0^{X_0})^p \, d\sigma$$
$$\lesssim C_0 \sigma(\Delta(x_{Q_0}, A_0\ell(Q_0)))^{1-p} \approx \sigma(Q_0)^{1-p};$$

thus, in summary,

$$\int_{Q_0} (k_0^{X_0})^p \, d\sigma \lesssim \sigma(Q_0)^{1-p}. \tag{2.9.5}$$

Our immediate goal in Step 1 is to show that (2.9.5) remains true when $k_0^{X_0}$ is replaced by $k_1^{X_0}$, the Poisson kernel for the operator L_1 defined above.

Let $f \geq 0$ be a continuous function supported on Q_0, such that $\|f\|_{L^{p'}(Q_0,\sigma)} = 1$, and let u_0 and u_1 be the corresponding solutions to the Dirichlet problems for L_0 and L_1 with boundary data f. Set $\mathcal{E}_1(Y) = A_1(Y) - A_0(Y) = \mathcal{E}(Y)\mathbf{1}_{\Omega_{\mathcal{F},Q_0}}(Y)$, where $\mathcal{E}(Y) = A_\tau(Y) - A_0(Y)$. Then, we may write

$$\begin{aligned}
F_1(X_0) := |u_1(X_0) - u_0(X_0)| &= \Big| \iint_{\mathbb{R}^n} \nabla_Y g_1^T(Y, X_0)\mathcal{E}_1^T(Y)\nabla u_0(Y)\, dY \Big| \\
&\leq \iint_{\Omega_{\mathcal{F},Q_0}} |\nabla_Y g_1(X_0, Y)||\mathcal{E}(Y)||\nabla u_0(Y)|\, dY \\
&\leq \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \sum_{I\in\mathcal{W}_Q} \iint_{I^*} |\nabla_Y g_1(X_0, Y)||\mathcal{E}(Y)||\nabla u_0(Y)|\, dY \\
\leq \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \sum_{I\in\mathcal{W}_Q} \big(\sup_{Y\in I^*} |\mathcal{E}(Y)|\big) &\Big(\iint_{I^*} |\nabla_Y g_1(X_0, Y)|^2\, dY\Big)^{\frac{1}{2}} \Big(\iint_{I^*} |\nabla u_0(Y)|^2\, dY\Big)^{\frac{1}{2}},
\end{aligned} \tag{2.9.6}$$

where we have used (2.7.11) in the first line, and later Hölder's inequality. By definition of X_0, we have that $v(Y) = g_1(X_0, Y) = g_1^T(Y, X_0)$ is a non-negative solution of $L_1^T v = 0$ in $R_{Q_0}^{**}$ (since $X_0 \notin R_{Q_0}^{**}$), where $R_{Q_0}^{**}$ is defined in (2.9.2). Hence, we can apply Caccioppoli's inequality (see [DFM19b, Lemma 8.6]) to obtain that

$$\begin{aligned}
\iint_{I^*} |\nabla_Y g_1(X_0, Y)|^2\, dY &\lesssim \frac{1}{\ell(I)^{-n+d+1}} \iint_{I^*} |\nabla_Y g_1(X_0, Y)|^2 w(Y)\, dY \\
&\lesssim \ell(I)^{-2} \frac{1}{\ell(I)^{-n+d+1}} \iint_{I^{**}} |g_1(X_0, Y)|^2\, dm \approx \iint_{I^{**}} \frac{|g_1(X_0, Y)|^2}{\delta(Y)^2}\, dY,
\end{aligned} \tag{2.9.7}$$

for any $I \in \mathcal{W}_Q$, $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. Fix such an I and Q. We have by the Harnack inequality that $g_1(X_0, Y) \approx g_1(X_0, X_Q)$ for all $Y \in I^{**}$ and where X_Q is a Corkscrew point for Q. Then by Lemma 2.7.18, Lemma 2.7.22, and Corollary 2.7.25, for every $Y \in I^{**}$ we have that

$$\frac{g_1(X_0,Y)}{\delta(Y)} \approx \frac{g_1(X_0,X_Q)}{\delta(X_Q)} \approx \frac{\omega_1^{X_0}\big(\Delta(x_Q,a_0\ell(Q))\big)}{\ell(Q)\ell(Q)^{d-1}} \approx \frac{\omega_1^{X_0}(Q)}{\sigma(Q)}. \tag{2.9.8}$$

Putting together (2.9.7) and (2.9.8), we see that

$$\iint_{I^*} |\nabla_Y g_1(X_0,Y)|^2\,dY \lesssim \Big(\frac{\omega_1^{X_0}(Q)}{\sigma(Q)}\Big)^2 |I|. \tag{2.9.9}$$

Plugging (2.9.9) into (2.9.6), using the fact that

$$\sup_{Y\in I^*} |\mathcal{E}(Y)| \approx \ell(I)^{1-(n-d)} \sup_{Y\in I^*} |\mathfrak{E}(Y)|,$$

and using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}
F_1(X_0) &\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \sum_{I\in\mathcal{W}_Q} \ell(I)^{1-\frac{n-d}{2}} \tfrac{\omega_1^{X_0}(Q)}{\sigma(Q)} \Big(\tfrac{\sup_{Y\in I^*}|\mathfrak{E}(Y)|^2}{\ell(I)^{n-d}}|I|\Big)^{\frac12} \Big(\iint_{I^*} |\nabla u_0|^2\Big)^{\frac12} \\
&\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \Big(\frac{\alpha_Q}{\sigma(Q)}\Big)^{\frac12} \sigma(Q)\frac{\omega_1^{X_0}(Q)}{\sigma(Q)} \Big(\iint_{U_Q} |\nabla u_0(X)|^2\delta(X)^{2-n}\,dX\Big)^{\frac12} \\
&\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \Big(\tfrac{\mathfrak{m}(\mathbb{D}_{\mathcal{F},Q})}{\sigma(Q)}\Big)^{\frac12} \int_Q \big(M(k_1^{X_0}\mathbf{1}_{Q_0})\big)(x) \Big(\iint_{\gamma_d^Q(x)} |\nabla u_0(X)|^2\delta(X)^{2-n}\,dX\Big)^{\frac12} dx \\
&\lesssim \|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)}^{\frac12} \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \int_Q \big(M(k_1^{X_0}\mathbf{1}_{Q_0})\big)(x) \Big(\iint_{\gamma^{\alpha_1}(x)} |\nabla u_0(X)|^2\delta(X)^{2-n}\,dX\Big)^{\frac12} dx \\
&\lesssim \|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)}^{\frac12} \int_\Gamma \big(M(k_1^{X_0}\mathbf{1}_{Q_0})\big)(x) \big(S_{\alpha_1}u_0\big)(x)\,dx
\end{aligned} \tag{2.9.10}$$

where in the second line we used that $\ell(I) \approx \ell(Q)$ for each $I \in \mathcal{W}_Q$, the definition of α_Q, and the bounded overlap of the dylated Whitney cubes I^*; in the third line we used for each $x \in Q$ the estimate

$$\begin{aligned}
\frac{\omega_1^{X_0}(Q)}{\sigma(Q)} = \frac{1}{\sigma(Q)}\int_Q k_1^{X_0}\,d\sigma &\lesssim \frac{1}{\Delta(x,A_0\ell(Q))}\int_{\Delta(x,A_0\ell(Q))} k_1^{X_0}\mathbf{1}_{Q_0}\,d\sigma \\
&\le \big(M(k_1^{X_0}\mathbf{1}_{Q_0})\big)(x), \qquad \text{for each } x\in Q,
\end{aligned}$$

where $M = M_\sigma$ is the Hardy-Littlewood maximal function (2.2.18), in the fourth line we chose α_1 so that $\gamma_d(x) \subset \gamma^{\alpha_1}(x)$ for all $x \in \Gamma$ (see (2.4.33)), and in the fifth line we used the definition of the square function (Definition 2.2.19). Using (2.9.1), (2.9.10) and the Hölder's inequality, we furnish the estimate

$$F_1(X_0) \lesssim \xi^{\frac{1}{2}} \|M(k_1^{X_0}\mathbf{1}_{Q_0})\|_{L^p(\Gamma,\sigma)} \|S_{\alpha_1} u_0\|_{L^{p'}(\Gamma,\sigma)} \lesssim \xi^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)},$$

where we have used Theorem 2.7.47 and that the trace of u_0 is f. As such, we have that

$$|u_1(X_0) - u_0(X_0)| = F_1(X_0) \lesssim \xi^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)}. \tag{2.9.11}$$

We will now see that (2.9.11) implies the desired result. By Hölder's inequality and the definitions of u_0, u_1, f, (2.9.11) gives that

$$\begin{aligned}\int_{Q_0} f k_1^{X_0}\, d\sigma \lesssim \xi^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} + \int_{Q_0} f k_0^{X_0}\, d\sigma \\ \leq \xi^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} + \|f\|_{L^{p'}(\Gamma,\sigma)} \|k_0^{X_0}\|_{L^p(Q_0,\sigma)} \\ \leq \xi^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} + \|k_0^{X_0}\|_{L^p(Q_0,\sigma)}.\end{aligned}$$

Since the continuous functions are dense in $L^{p'}(\Gamma,\sigma)$, by taking supremum over all possible f as described earlier, we obtain that

$$\|k_1^{X_0}\|_{L^p(Q_0,\sigma)} \lesssim \xi^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} + \|k_0^{X_0}\|_{L^p(Q_0,\sigma)}. \tag{2.9.12}$$

It follows that as long as $\xi^{\frac{1}{2}}$ is small enough (depending only on the permissible constants), we may hide the first term on the right-hand side of the above inequality to the left-hand side; hence we get that $\|k_1^{X_0}\|_{L^p(Q_0,\sigma)} \lesssim \|k_0^{X_0}\|_{L^p(Q_0,\sigma)}$. And hence, since $k_0^{X_0}$ satisfies (2.9.5), we obtain that $k_1^{X_0}$ satisfies (2.9.5) as well, with the implicit constant independent of τ and Q_0.

2.9.3 Self-improvement of Step 1.

We currently have (2.9.5) for Q_0; but let us see that we can extend (2.9.5) to obtain a reverse Hölder estimate on every dyadic subcube of Q_0.

Fix $Q \in \mathbb{D}_{Q_0}$. Let X^Q be a Corkscrew point for $\Delta(x_Q, 10c^{-1}\sqrt{n}A_2\ell(Q))$, so that $X^Q \in \Omega \backslash R_Q^{**}$ (see the remarks about X_0 following (2.9.1)). Define a new operator

$L_1^Q = L_\tau$ in $\Omega_{\mathcal{F},Q}$ and $L_1^Q = L_0$ otherwise in $\Omega\backslash\Omega_{\mathcal{F},Q}$, and let $k_{L_1^Q}^{X^Q}$ denote the Poisson kernel for L_1^Q with pole at X^Q. Given our proof above, it is easy to see that

$$\int_Q (k_{L_1^Q}^{X^Q})^p \, d\sigma \leq C_1 \sigma(Q)^{1-p}, \tag{2.9.13}$$

for some C_1 independent of Q (and τ). Indeed, if $Q \subset Q_k$ for some $Q_k \in \mathcal{F}$ then we obtain that $\mathbb{D}_{\mathcal{F},Q} = \varnothing$ so that $\Omega_{\mathcal{F},Q} = \varnothing$ and so $L_1^Q \equiv L_0$ in $\Omega\backslash\Gamma$. In that case, (2.9.13) holds by hypothesis. Otherwise, trivially we have that $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq \|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)} \leq \xi$, and consequently, if Q is not contained in any $Q_k \in \mathcal{F}$, then we may repeat the previous argument with respect to Q, and we obtain (2.9.13) as before. This proves the claim.

Now, by the non-degeneracy of the elliptic measure we have that $\int_Q k_{L_1^Q}^{X^Q} \, d\sigma \gtrsim 1$, and combining this estimate with (2.9.13) we obtain that

$$\Big(\frac{1}{\sigma(Q)} \int_Q (k_{L_1^Q}^{X^Q})^p \, d\sigma\Big)^{\frac{1}{p}} \lesssim \frac{1}{\sigma(Q)} \int_Q k_{L_1^Q}^{X^Q} \, d\sigma. \tag{2.9.14}$$

Next, we want to pass from $k_{L_1^Q}^{X^Q}$ to $k_{L_1}^{X^Q}$. Notice that $L_1 \equiv L_1^Q$ in $(\Omega\backslash\Omega_{\mathcal{F},Q_0}) \cup \Omega_{\mathcal{F},Q}$, and observe that for $B_s = B(x_Q, s)$ the ball in Lemma 2.4.38 (ii) for which $\ell(Q) \lesssim s \leq \ell(Q)$ and (2.4.39) is satisfied, we have that

$$\begin{aligned} B_s \cap \Omega = B_s \cap \big[(\Omega\backslash\Omega_{\mathcal{F},Q_0}) \cup \Omega_{\mathcal{F},Q_0}\big] &= \big(B_s \cap (\Omega\backslash\Omega_{\mathcal{F},Q_0})\big) \cup \big(B_s \cap \Omega_{\mathcal{F},Q_0}\big) \\ &= \big(B_s \cap (\Omega\backslash\Omega_{\mathcal{F},Q_0})\big) \cup \big(B_s \cap \Omega_{\mathcal{F},Q}\big) = B_s \cap \big(\Omega_{\mathcal{F},Q} \cup \Omega\backslash\Omega_{\mathcal{F},Q_0}\big), \end{aligned}$$

and hence $L_1 \equiv L_1^Q$ in $B_s \cap \Omega$. Therefore, using Lemma 2.7.34 we see that there exists $\tilde{\varepsilon}$, $1 \lesssim \tilde{\varepsilon} \leq 1$ so that $k_1^{X^Q}(y) = k_{L_1}^{X^Q}(y) \approx k_{L_1^Q}^{X^Q}(y)$, for $\sigma-$a.e. $y \in \tilde{\varepsilon}\Delta_Q = \Delta(x_Q, \tilde{\varepsilon}\ell(Q))$. We use this result, (2.9.14), and the doubling property Lemma 2.7.24 to deduce the estimate

$$\begin{aligned} \Big(\frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)} \int_{\tilde{\varepsilon}\Delta_Q} (k_1^{X^Q})^p \, d\sigma\Big)^{\frac{1}{p}} &\leq \Big(\frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)} \int_Q (k_{L_1^Q}^{X^Q})^p \, d\sigma\Big)^{\frac{1}{p}} \\ &\lesssim \frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)} \int_Q k_{L_1^Q}^{X^Q} \, d\sigma \leq \frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q} \int_{\Delta(x_Q, A_0\ell(Q))} k_{L_1^Q}^{X^Q} \, d\sigma \\ &\lesssim \frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)} \int_{\tilde{\varepsilon}\Delta_Q} k_{L_1^Q}^{X^Q} \, d\sigma \approx \frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)} \int_{\tilde{\varepsilon}\Delta_Q} k_1^{X^Q} \, d\sigma. \end{aligned}$$

We can now change poles from X^Q to X_0 via Lemma 2.7.33 to obtain

$$\Big(\frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)}\int_{\tilde{\varepsilon}\Delta_Q}(k_1^{X_0})^p\,d\sigma\Big)^{\frac{1}{p}} \lesssim \frac{1}{\sigma(\tilde{\varepsilon}\Delta_Q)}\int_{\tilde{\varepsilon}\Delta_Q} k_1^{X_0}\,d\sigma, \qquad \text{for each } Q\in\mathbb{D}_{Q_0}.$$

Finally, we use Lemma 2.3.18 to furnish

Conclusion 2.9.15 (Step 1). We have that $\omega_1^{X_0}\in A_\infty^{\text{dyadic}}(Q_0)$ uniformly in τ and Q_0 (see Definition 2.7.28). Hence we deduce that $\mathcal{P}_{\mathcal{F}}\omega_1^{X_0}\in A_\infty^{\text{dyadic}}(Q_0)$ (uniformly in τ and Q_0) by Lemma 2.3.15.

2.9.4 Step 2: Hide the "bad" Carleson regions.

We define the operator L_2 such that the disagreement with L_1 lives roughly inside the Carleson regions corresponding to the family $\mathcal{F}$. More precisely, set

$$L_2=\begin{cases} L_\tau, & \text{in} \quad R_{Q_0}\backslash\Omega_{\mathcal{F}},\\ L_1, & \text{in} \quad \Omega\backslash(R_{Q_0}\backslash\Omega_{\mathcal{F}}).\end{cases}$$

Note carefully that $R_{Q_0}\backslash\Omega_{\mathcal{F}}\subseteq R_{Q_0}\backslash\Omega_{\mathcal{F},Q_0}$, but the opposite containment does not hold in general. We write $\omega_1=\omega_{L_1}^{X_0}$ and $\omega_2=\omega_{L_2}^{X_0}$ for the corresponding elliptic measures (on Γ) for L_1 and L_2 with fixed pole at X_0. We also let $\omega_{\star,1}=\omega_{\star,1}^{X_0}$ and $\omega_{\star,2}=\omega_{\star,2}^{X_0}$ denote the elliptic measures of L_1 and L_2 on $\partial\Omega_{\mathcal{F}}$, the boundary of the dyadically-generated sawtooth domain $\Omega_{\mathcal{F}}$ (see the beginning of Section 2.8). Note that $\Omega_{\mathcal{F}}\subset\Omega\backslash(R_{Q_0}\backslash\Omega_{\mathcal{F}})$, whence $L_1\equiv L_2$ in $\Omega_{\mathcal{F}}$ and consequently we have that $\omega_{\star,1}\equiv\omega_{\star,2}$.

Next, we apply Lemma 2.8.1 to both L_1 and L_2 to deduce that for all $Q\in\mathbb{D}_{Q_0}$ and $F\subset Q$, the estimate

$$\Big(\frac{\mathcal{P}_{\mathcal{F}}\omega_i(F)}{\mathcal{P}_{\mathcal{F}}\omega_i(Q)}\Big)^{\theta_i}\lesssim\frac{\mathcal{P}_{\mathcal{F}}\mu_i(F)}{\mathcal{P}_{\mathcal{F}}\mu_i(Q)}\lesssim\frac{\mathcal{P}_{\mathcal{F}}\omega_i(F)}{\mathcal{P}_{\mathcal{F}}\omega_i(Q)}$$

holds for $i=1,2$, where $\mathcal{P}_{\mathcal{F}}$ is given in (2.3.10) and μ_i is defined in (2.8.2). Observe that $\mathcal{P}_{\mathcal{F}}\mu_1\equiv\mathcal{P}_{\mathcal{F}}\mu_2$ since $\omega_{\star,1}\equiv\omega_{\star,2}$. Since $A_\infty^{\text{dyadic}}(Q_0)$ defines an equivalence relationship among dyadically doubling measures (which the projection measures $\mathcal{P}_{\mathcal{F}}\mu_i$ are; see Lemma 9.51 of [DMb] or Lemma B.2 of [HM14]), and since we showed in Step 1 that $\mathcal{P}_{\mathcal{F}}\omega_1\in A_\infty^{\text{dyadic}}(Q_0)$, we obtain in this step that $\mathcal{P}_{\mathcal{F}}\omega_2\in A_\infty^{\text{dyadic}}(Q_0)$. For definiteness, we have

Conclusion 2.9.16 (Step 2). There exist $\theta, \theta' > 0$ (independent of τ and Q_0) such that for all $Q \in \mathbb{D}_{Q_0}$ and all Borel sets $F \subseteq Q$, we have the estimate

$$\frac{1}{C}\Big(\frac{\sigma(F)}{\sigma(Q)}\Big)^{\theta} \leq \frac{\mathcal{P}_{\mathcal{F}}\omega_2^{X_0}(F)}{\mathcal{P}_{\mathcal{F}}\omega_2^{X_0}(Q)} \leq C\Big(\frac{\sigma(F)}{\sigma(Q)}\Big)^{\theta'}$$

with C uniform in τ and Q_0.

2.9.5 Step 3: Extend outside the Carleson region of Q_0

Observe that

$$\Omega = \Omega_{\mathcal{F},Q_0} \cup (R_{Q_0}\backslash\Omega_{\mathcal{F}}) \cup (R_{Q_0} \cap \Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0}) \cup (\Omega\backslash R_{Q_0}).$$

We have successfully changed the operator from L_0 to L_τ on $\Omega_{\mathcal{F},Q_0} \cup (R_{Q_0}\backslash\Omega_{\mathcal{F}})$ in the last two steps; it remains to change it in the set $\hat{\Omega} := (R_{Q_0} \cap \Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0}) \cup (\Omega\backslash R_{Q_0})$. Thus we define

$$L_3 = \begin{cases} L_\tau, & \text{in} \quad \hat{\Omega}, \\ L_2, & \text{in} \quad \Omega\backslash\hat{\Omega}. \end{cases}$$

Hence, note that L_3 is exactly L_τ in Ω, and $L_3 \equiv L_2$ in $\Omega\backslash\hat{\Omega} = R_{Q_0}\backslash(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0})$. The latter set has been discussed briefly in the remarks before Lemma 2.7.38. We will show that (2.6.5) holds, so fix $\varepsilon \in (0,1)$ and take $E \subset Q_0$ with $\frac{\sigma(E)}{\sigma(Q_0)} \geq \varepsilon$. In the case that $\mathcal{F} = \{Q_0\}$, we have the trivial estimate

$$\frac{\mathcal{P}_{\mathcal{F}}\omega_3^{X_0}(E)}{\mathcal{P}_{\mathcal{F}}\omega_3^{X_0}(Q_0)} = \frac{\frac{\sigma(E)}{\sigma(Q_0)}\omega_3^{X_0}(Q_0)}{\frac{\sigma(Q_0)}{\sigma(Q_0)}\omega_3^{X_0}(Q_0)} = \frac{\sigma(E)}{\sigma(Q_0)} \geq \varepsilon.$$

We thus suppose that $\mathcal{F} \subseteq \mathbb{D}_{Q_0}\backslash\{Q_0\}$. For $t \ll 1$, recall that we define $\Sigma_t = \Sigma_{Q_0,t}$ in Lemma 2.7.38. Define $Q_t = Q_0\backslash\bigcup_{Q'\in\mathcal{I}_t} Q'$, where $\mathcal{I}_t = \{Q' \in \mathbb{D}_{Q_0} : t\ell(Q_0) < \ell(Q') \leq 2t\ell(Q_0), Q' \cap \Sigma_t \neq \varnothing\}$. It is easy to see that $\Sigma_t \subset \bigcup_{Q'\in\mathcal{I}_t} Q' \subset \Sigma_{Ct}$ for C a uniform constant. Then, for all $t = t(\varepsilon)$ small enough, we have that

$$\sigma(Q_0\backslash Q_t) \leq \sigma(\Sigma_{Ct}) \lesssim A_0(Ct)^{\zeta}\sigma(Q_0) \leq \frac{\varepsilon}{2}\sigma(Q_0),$$

where we have used Lemma 2.3.1 (vi). Letting $F = E \cap\, Q_t$, it follows that $\varepsilon\sigma(Q_0) \le \sigma(E) \le \sigma(F) + \frac{\varepsilon}{2}\sigma(Q_0)$, and therefore $\sigma(F)/\sigma(Q_0) \ge \varepsilon/2$. Using the conclusion of Step 2, we see that

$$\frac{\mathcal{P}_{\mathcal{F}}\omega_2^{X_0}(F)}{\mathcal{P}_{\mathcal{F}}\omega_2^{X_0}(Q_0)} \gtrsim \Big(\frac{\sigma(F)}{\sigma(Q_0)}\Big)^{\theta} \ge \Big(\frac{\varepsilon}{2}\Big)^{\theta}.$$

Now we claim that $\mathcal{P}_{\mathcal{F}}\omega_3^{X_0}(F) \ge c_\varepsilon \mathcal{P}_{\mathcal{F}}\omega_2^{X_0}(F)$. The point of our argument is that the region of discrepancy between A_2 and A_3 is uniformly far away (depending on $t = t(\varepsilon)$) from most of Q_0, allowing us to compare the Poisson kernels of ω_2 and ω_3 in the set $F \subseteq E$, which has been chosen so that it retains most of $\sigma(E)$ while staying far from the region of discrepancy between A_2 and A_3. Since $L_2 \equiv L_3$ in $R_{Q_0}\backslash(\Omega_{\mathcal{F}}\backslash\Omega_{\mathcal{F},Q_0})$, then we have by Lemma 2.7.38 that

$$k_2^{X_0}(y) \approx_t k_3^{X_0}(y) \quad \text{for } \sigma - \text{almost every } y \in Q_t \subset Q_0\backslash\Sigma_t,$$

where the implicit constants depends on t and hence on ε. It is then the case that $\omega_2^{X_0}(F\backslash(\cup_{Q_j\in\mathcal{F}}Q_j) \approx \omega_3^{X_0}(F\backslash(\cup_{Q_j\in\mathcal{F}}Q_j))$, and thus we observe the estimate

$$\begin{aligned}\mathcal{P}_{\mathcal{F}}\omega_3^{X_0}(F) &= \omega_3^{X_0}(F\backslash(\cup_{Q_j\in\mathcal{F}}Q_j)) + \sum_{Q_j\in\mathcal{F}} \frac{\sigma(F\cap Q_j)}{\sigma(Q_j)}\omega_3^{X_0}(Q_j)\\ &\ge c_\varepsilon \omega_2^{X_0}(F\backslash(\cup_{Q_j\in\mathcal{F}}Q_j)) + \sum_{Q_j\in\mathcal{F}} \frac{\sigma(F\cap Q_j)}{\sigma(Q_j)}\omega_3^{X_0}(Q_j). \qquad (2.9.17)\end{aligned}$$

It remains to estimate the last term. We need only consider the cubes in $\mathcal{F}$ that meet F. Let $Q_j \in \mathcal{F}$ be such a cube. If $Q_j \subset Q_t$, then again by Lemma 2.7.38 we have that $\omega_3^{X_0}(Q_j) \ge c_\varepsilon\omega_2^{X_0}(Q_j)$. Otherwise, $Q_j \cap (Q_0\backslash Q_t) \ne \varnothing$, whence there exists $Q' \in \mathcal{I}_t$ such that $Q' \subsetneq Q_j$ (since $Q_j \cap Q_t \ne \varnothing$). Accordingly, $\ell(Q_j) > t\ell(Q_0)$. Now let $\tilde{Q} \in \mathbb{D}_{Q_j}$ be a dyadic descendant of Q_j which contains x_{Q_j} and verifying $\ell(\tilde{Q}) = 2^{-M}\ell(Q_j)$ with $M = 2(1+\log_2(A_0 a_0^{-1})) \approx 1$, so that $\ell(\tilde{Q}) \approx \ell(Q_j)$. Let us see that $\tilde{Q} \subset Q_0\backslash\Sigma_{a_0t/2}$. Indeed, choose $x^* \in \Gamma\backslash Q_j$ and $y^* \in \tilde{Q}$ so that $\operatorname{dist}(\Gamma\backslash Q_j, \tilde{Q}) = |x^* - y^*|$, and reckon that

$$\begin{aligned}\operatorname{dist}(\Gamma\backslash Q_0, \tilde{Q}) \ge \operatorname{dist}(\Gamma\backslash Q_j, \tilde{Q}) &= |x^* - y^*| \ge |x^* - x_{Q_j}| - |x_{Q_j} - y^*|\\ &\ge a_0\ell(Q_j) - \operatorname{diam}\tilde{Q} \ge [a_0 - A_0 2^{-M}]\ell(Q_j) > \tfrac{a_0}{2}t\ell(Q_0),\end{aligned}$$

which does give our claim. We may then apply Lemma 2.7.38 one last time to see that $k_2^{X_0}(y) \approx_t k_3^{X_0}(y)$ for $\sigma-$almost every $y \in \tilde{Q}$, and therefore $\omega_3^{X_0}(Q_j) \geq \omega_3^{X_0}(\tilde{Q}) \approx_\varepsilon \omega_2^{X_0}(\tilde{Q}) \gtrsim \omega_2^{X_0}(Q_j)$, where we have used the doubling property of the elliptic measure on the dyadic cubes. We now plug this result back into (2.9.17) to obtain that

$$\begin{aligned}
&\mathcal{P}_{\mathcal{F}}\omega_3^{X_0}(F) \\
&\qquad \geq c_\varepsilon \omega_2^{X_0}(F \backslash (\cup_{Q_j \in \mathcal{F}} Q_j)) + c_\varepsilon \sum_{Q_j \in \mathcal{F}} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega_2^{X_0}(Q_j) = c_\varepsilon \mathcal{P}_{\mathcal{F}} \omega_2^{X_0}(F).
\end{aligned}$$

We have arrived at

Conclusion 2.9.18 (Step 3). There exists $\xi > 0$ for which the following statement holds: given $\varepsilon \in (0,1)$, there is $C_\varepsilon < \infty$ such that for every $Q_0 \in \mathbb{D}$, if $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0}$ is a disjoint family satisfying $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)} < \xi$, then for any Borel set $F \subset Q_0$,

$$\frac{\sigma(F)}{\sigma(Q_0)} \geq \varepsilon \implies \frac{\mathcal{P}_{\mathcal{F}} \omega_{L_\tau}^{X_0}(F)}{\mathcal{P}_{\mathcal{F}} \omega_{L_\tau}^{X_0}(Q_0)} \geq \frac{1}{C_\varepsilon},$$

and C_ε is uniform in τ.

2.9.6 Step 4: Fix the pole

The conclusion of Step 3 above almost looks like what is needed; but we ought to improve it so that its conclusion holds for any cube $Q \in \mathbb{D}_{Q_0}$ while keeping the pole X_0 fixed. Nevertheless, this is not difficult; the following result is immediate from the method of proof in [HM12].

Proposition 2.9.19 (Proposition 4.25 of [CHM19])**.** *There exists $\xi > 0$ for which the following statement holds: given $\varepsilon \in (0,1)$, there is $C_\varepsilon < \infty$ such that for every $Q_0 \in \mathbb{D}$ and for all $Q \in \mathbb{D}_{Q_0}$, if $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_Q$ is a disjoint family satisfying $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} < \xi$ (see Definition 2.6.2), then for any Borel set $F \subset Q$, we have that*

$$\frac{\sigma(F)}{\sigma(Q)} \geq \varepsilon \implies \frac{\mathcal{P}_{\mathcal{F}} \omega_{L_\tau}^{X_0}(F)}{\mathcal{P}_{\mathcal{F}} \omega_{L_\tau}^{X_0}(Q)} \geq \frac{1}{C_\varepsilon},$$

with C_ε uniform in τ. Consequently, by the extrapolation of Carleson measures, Theorem 2.6.3, it follows that $\omega_\tau^{X_0} \in A_\infty^{\text{dyadic}}(Q_0)$ uniformly in τ and Q_0. In particular, there *exists $1 < q < \infty$ such that $k_\tau^{X_0} \in RH_q^{\text{dyadic}}(Q_0)$ uniformly in $Q_0 \in \mathbb{D}$ and $\tau > 0$ (see Definition 2.3.14). Therefore, $k_\tau \in RH_q$ with RH_q characteristic independent of τ.*

Proof. We let ξ be the constant from Conclusion 2.9.18. Fix $\varepsilon \in (0,1)$, $Q_0 \in \mathbb{D}$, $Q \in \mathbb{D}_{Q_0}$, a disjoint family $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_Q$ with $\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} < \xi$, and a Borel set $F \subset Q$ verifying that $\sigma(F) \geq \varepsilon\sigma(Q)$. Then we may apply Conclusion 2.9.18 (with $Q_0 = Q$) to see that

$$\frac{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q}(F)}{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q}(Q)} \geq \frac{1}{C_\varepsilon}, \tag{2.9.20}$$

where X_Q is a Corkscrew point for the surface ball $\tilde{\Delta}_Q := \Delta(x_Q, 10c^{-1}\sqrt{n}A_2\ell(Q)) \supset Q$, and C_ε is uniform in τ, Q_0, and Q. To obtain our desired conclusion, we need only move the poles from X_Q to X_0. Note that

$$\begin{aligned}
\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q}(F) &= \omega_{L_\tau}^{X_Q}(F \setminus \cup_j Q_j) + \sum_j \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)}\omega_{L_\tau}^{X_Q}(Q_j) \\
&\lesssim \frac{\omega_{L_\tau}^{X_0}(F \setminus \cup_j Q_j)}{\omega_{L_\tau}^{X_0}(\tilde{\Delta}_Q)} + \sum_j \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \frac{\omega_{L_\tau}^{X_0}(Q_j)}{\omega_{L_\tau}^{X_0}(\tilde{\Delta}_Q)} = \frac{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(F)}{\omega_{L_\tau}^{X_0}(\tilde{\Delta}_Q)} \leq \frac{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(F)}{\omega_{L_\tau}^{X_0}(Q)} \\
&= \frac{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(F)}{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(Q)},
\end{aligned} \tag{2.9.21}$$

where we used the definition of $\mathcal{P}_\mathcal{F}$ in the first line, in the second line we used the change of poles for the elliptic measure (Lemma 2.7.30)[3], the definition of $\mathcal{P}_\mathcal{F}$, and the fact that $Q \subset \tilde{\Delta}_Q$, while in the last line we used that $\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(Q) = \omega_{L_\tau}^{X_0}(Q)$. On the other hand, note that $\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q} = \omega_{L_\tau}^{X_Q}(Q) \approx 1$ owing to the doubling property of $\omega_{L_\tau}^{X_Q}$ on $\frac{c}{2}\tilde{\Delta}_Q$ (Lemma 2.7.24) and the non-degeneracy of elliptic measure (Lemma 2.7.20). These observations together yield that

$$\frac{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(F)}{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_0}(Q)} \gtrsim \mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q}(F) \approx \frac{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q}(F)}{\mathcal{P}_\mathcal{F}\omega_{L_\tau}^{X_Q}} \geq \frac{1}{C_\varepsilon},$$

with uniform constants. This shows (2.9.20), which is the hypothesis (2.6.5) of Theorem 2.6.3 with $\mu = \omega_{L_\tau}^{X_0}$. Then, Theorem 2.6.3 allows us to conclude that $\omega_{L_\tau}^{X_0} \in A_\infty^{\text{dyadic}}(Q_0)$

[3]If $X_0 \in B(x_Q, 20c^{-1}\sqrt{n}A_2\ell(Q))$, then we may not directly apply Lemma 2.7.30. However, in this case we have that $\ell(Q) \approx \ell(Q_0)$, and since X_0, X_Q are Corkscrew points for $\tilde{\Delta}_{Q_0}$ and $\tilde{\Delta}_Q$ respectively, we may still obtain the second line of (2.9.21) via Harnack Chains and the Harnack inequality.

uniformly in τ and Q_0. By the characterization of A_∞^{dyadic} written in Definition 2.3.14, we may conclude that there exists $q \in (1, \infty)$ such that $k_\tau^{X_0} \in RH_q^{\text{dyadic}}(Q_0)$ uniformly in τ and Q_0. Lastly, by running over all $Q_0 \in \mathbb{D}$ and using Remark 2.7.45, we obtain that $k_\tau \in RH_q$ with RH_q characteristic independent of τ, as desired. $\square$

2.9.7 Step 5: Pass to the limit in τ

Proposition 2.9.19 is the desired conclusion for each operator L_τ, $\tau > 0$, with RH_q characteristic independent of τ. We now ought to pass to the limit as $\tau \to 0$ and argue that these quantitative absolute continuity properties are preserved. The required technology is the following result.

Lemma 2.9.22 (Limiting lemma)**.** *Let A_0, A be two matrices satisfying (2.7.1), and write $L_0 = -\operatorname{div} A_0 \nabla$, $L = -\operatorname{div} A\nabla$. For each $\tau \geq 0$ small enough, define $A_\tau = A_0$ in Γ_τ (see Definition 2.9.3) and $A_\tau = A$ in $\mathbb{R}^n \backslash \Gamma_\tau$. Accordingly, define the operator $L_\tau := -\operatorname{div} A_\tau \nabla$. Let $\{\omega_0^X\}, \{\omega^X\}, \{\omega_\tau^X\}$ be the families of elliptic measures associated to the operators L_0, L, L_τ respectively. Assume that there exists $q \in (1, \infty)$ such that $\frac{d\omega_{L_\tau}}{d\sigma} \in RH_q$ with the RH_q characteristic uniformly bounded in τ (see Definition 2.7.28). Then $\omega_L \ll \sigma$ and $\frac{d\omega_L}{d\sigma} \in RH_q$.*

Proof. Fix the surface ball $\Delta_0 \subset \Gamma$, let X_0 be a Corkscrew point for the ball Δ_0, and suppose that $\tau < \delta(X_0)/4$. We first show that $\omega_\tau^{X_0} \rightharpoonup \omega^{X_0}$ on Δ_0 as $\tau \searrow 0$. Define the functionals Φ and Φ_τ on $C_c(\Delta_0)$ by

$$\Phi(f) = \int_{\Delta_0} f \, d\omega^{X_0}, \qquad \Phi_\tau(f) = \int_{\Delta_0} f \, d\omega_\tau^{X_0}, \qquad f \in C_c(\Gamma).$$

Let u (respectively, u_τ) be the unique solution to the Dirichlet problem $Lu = 0$ in Ω, $u|_\Gamma = f$ (respectively, $Lu_\tau = 0$ in Ω, $u_\tau|_\Gamma = f$). By (2.7.14), we have that $\Phi(f) = u(X_0)$, $\Phi_\tau(f) = u_\tau(X_0)$. In this setting, using an elementary approximation argument, the Cauchy-Schwartz inequality, and Lemma 2.7.7 (iv), it is easy to show that the identity (2.7.11) holds for *all* $X \in \Omega \backslash \Gamma_{2\tau}$, since the pole X_0 lies far away from the support of $A_\tau - A$. Thus, according to Lemma 2.7.10, we may write

$$|\Phi(f) - \Phi_\tau(f)| = |u(X_0) - u_\tau(X_0)| \leq \left| \iint_{\Gamma_\tau} (A - A_\tau)^T(Y) \nabla_Y g_\tau^T(Y, X_0) \nabla u(Y) \, dY \right|$$

$$
\begin{aligned}
&\leq 2(C_A + C_{A_0})\Big(\iint_{\Gamma_\tau} |\nabla_Y g_\tau^T(Y,X_0)|^2\, dm(Y)\Big)^{\frac{1}{2}}\Big(\iint_{\Gamma_\tau} |\nabla u(Y)|^2\, dm(Y)\Big)^{\frac{1}{2}}\\
&\lesssim \Big(\iint_{\Omega\setminus B(X_0,\delta(X_0)/2)} |\nabla_Y g_\tau^T(Y,X_0)|^2\, dm(Y)\Big)^{\frac{1}{2}}\Big(\iint_{\Gamma_\tau} |\nabla u(Y)|^2\, dm(Y)\Big)^{\frac{1}{2}}\\
&\qquad\lesssim (\delta(X_0)/2)^{\frac{1-d}{2}}\Big(\iint_{\Gamma_\tau} |\nabla u(Y)|^2\, dm(Y)\Big)^{\frac{1}{2}}\\
&\qquad\qquad\longrightarrow 0 \text{ as } \tau\to 0,
\end{aligned}
$$

where we have employed Lemma 2.7.7 (iv), the fact that $u \in W$, and the absolute continuity of the integral. We have shown that $\Phi_\tau(f) \longrightarrow \Phi(f)$, for all $f \in C_c(\Delta_0)$, which gives the claimed weak convergence.

Now suppose that $f \in C_c(\Delta_0)$ and $\|f\|_{L^{q'}(\Delta_0,\sigma)} = 1$, where q' is the Hölder conjugate of q. Since $\omega_\tau^{X_0} \in RH_q(\Delta_0)$ uniformly in τ, then we have that (2.7.42) holds with Δ replaced by Δ_0 and X_Δ replaced by X_0. Consequently, we see that

$$
\begin{aligned}
|\Phi(f)| = \Big|\int_{\Delta_0} f\, d\omega^{X_0}\Big| = \Big|\lim_{\tau\to 0}\int_{\Delta_0} f\, d\omega_\tau^{X_0}\Big| \leq \sup_{\tau>0}\Big|\int_{\Delta_0} f k_\tau^{X_0}\, d\sigma\Big|\\
\leq \sup_{\tau>0}\Big(\|k_\tau^{X_0}\|_{L^q(\Delta_0,\sigma)}\|f\|_{L^{q'}(\Delta_0,\sigma)}\Big) \leq C_0\sigma(\Delta_0)^{1-q}.
\end{aligned}
$$

It follows that Φ is a bounded linear functional on $L^{q'}(\Delta_0,\sigma)$. Hence we must have that $\omega^{X_0} \ll \sigma$ and $k^{X_0} = \frac{d\omega^{X_0}}{d\sigma} \in L^q(\Delta_0,\sigma)$ satisfies the estimate (2.7.42), so that $k^{X_0} \in RH_q(\Delta_0)$. Since $\Delta_0 \subset \Gamma$ was arbitrary, we finally conclude that $\frac{d\omega}{d\sigma} \in RH_q$. □

Using the previous lemma in conjunction with Proposition 2.9.19 yields the desired conclusion for the operator L and Theorem 2.1.1 is shown. □

2.10 Proof of Theorem 2.1.4

In this section, we give the proof of Theorem 2.1.4; in fact, the proof of this theorem is very similar to but simpler than the proof of Theorem 2.1.1 in the previous section, as we will not need to use the projection lemma, Lemma 2.8.1, nor the extrapolation theorem, Theorem 2.6.3. As such, we mainly describe the set-up here and point out the differences with the proof of Theorem 2.1.4.

Let A_0, A be two matrices that satisfy (2.7.1), write $\mathcal{A}_0 = w^{-1}A_0$, $\mathcal{A} = w^{-1}A$, and suppose that $d\lambda(X) = \frac{\mathfrak{a}(X)^2}{\delta(X)^{n-d}}dX$ is a (continuous) Carleson measure with $|||\lambda|||_{\mathcal{C}} \leq \varepsilon_0$, where $\mathfrak{a}$ is defined in (2.1.2) and ε_0 is small and to be chosen later. As in Lemma 2.6.6, the natural discretization of the Carleson measure λ is the collection $\mathfrak{m} = \{\alpha_Q\}_{Q\in\mathbb{D}}$ with α_Q as defined in (2.6.7). By Lemma 2.6.6, we see that

$$\|\mathfrak{m}\|_{\mathcal{C}} \lesssim \varepsilon_0.$$

Let $L_0 = -\operatorname{div} A_0\nabla$, $L = -\operatorname{div} A\nabla$, and let ω_0, ω be the elliptic measures of L_0, L respectively.

Fix $Q_0 \in \mathbb{D}$, and recall that $R_{Q_0} \subset B(x_{Q_0}, 7\sqrt{n}A_0\ell(Q_0))$ (see (2.4.30)). Let

$$\hat{B}_0 := B(x_{Q_0}, 30c^{-1}\sqrt{n}A_2\ell(Q_0)), \tag{2.10.1}$$

and fix $X_0 \in \Omega$ as a Corkscrew point (with Corkscrew constant c) for the surface ball $\hat{B}_0 \cap \Gamma$. Then $X_0 \in \Omega\backslash B(x_{Q_0}, 30\sqrt{n}A_2\ell(Q_0)) \subset \Omega\backslash R^{**}_{Q_0} \subset \Omega\backslash R_{Q_0}$. Moreover, according to Corollary 2.7.26, we have that ω^{X_0} is dyadically doubling in Q_0, while we also have that $\delta(X_0) \approx \operatorname{dist}(X_0, Q_0) \approx \ell(Q_0)$.

Since $(\mathrm{D})_{p'}$ is solvable for L_0, then by Theorem 2.7.40, we have that $\omega^{X_0}_{L_0} = \omega^{X_0}_0 \in RH_p(Q_0)$ (with reverse Hölder characteristic independent of Q_0).

Step 0. Owing to our assumptions and Theorem 2.1.1, we a priori have that $\omega_L \ll \sigma$. However, we still ought to make the same qualitative reduction as in Step 0 of the proof of Theorem 2.1.1, to guarantee the local L^p integrability of the Poisson kernel for the fixed p in our hypothesis, which will be used in Step 1 below. Accordingly, we define A_τ as $A_\tau = A_0$ in the $\tau-$tube Γ_τ (see Definition 2.9.3), and $A_\tau = A$ in $\mathbb{R}^n\backslash\Gamma_\tau$, and write $L_\tau = -\operatorname{div} A_\tau\nabla$. Then $k^{X_0}_\tau = \frac{d\omega^{X_0}_{L_\tau}}{d\sigma} \in L^p(Q_0, \sigma)$ by Corollary 2.9.4. We will establish the conclusion of Theorem 2.1.4 for L_τ first, at the end of Step 2 (with constants uniform in τ), and in Step 3 we pass to the limit as $\tau \to 0$ using Lemma 2.9.22, very similarly as in Step 5 of the proof of Theorem 2.1.1.

Step 1: Exploit smallness of $\|\mathfrak{m}\|_{\mathcal{C}}$. Introduce the operator L_1 defined as $L_1 = L_\tau$ in $\hat{B}_0$ (see (2.10.1)), and $L_1 = L_0$ in $\Omega\backslash\hat{B}_0$. We write ω_1 for the elliptic measure associated to the operator L_1, and g_1 for the Green function associated to L_1. We have that $k^{X_0}_0 = \frac{d\omega^{X_0}_0}{d\sigma} \in RH_p(Q_0)$, and in particular by Theorem 2.7.40, Harnack Chains and the Harnack Inequality, we have that

$$\int_{Q_0} (k_0^{X_0})^p \, d\sigma \lesssim \sigma(Q_0)^{1-p}. \tag{2.10.2}$$

Our immediate goal in Step 1 is to show that (2.10.2) remains true when $k_0^{X_0}$ is replaced by $k_1^{X_0}$, the Poisson kernel for the operator L_1 defined above. The proof is essentially the same as that of Step 1 in the previous section, where $\mathcal{F} = \varnothing$ in our situation. Following the proof of Step 1 of Theorem 2.1.1 up to (2.9.12), we are able to show that

$$\begin{aligned} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} &\lesssim \|\mathfrak{m}\|_{\mathcal{C}}^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} + \|k_0^{X_0}\|_{L^p(Q_0,\sigma)} \\ &\lesssim \varepsilon_0^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0,\sigma)} + \|k_0^{X_0}\|_{L^p(Q_0,\sigma)}, \end{aligned} \tag{2.10.3}$$

uniformly over τ and Q_0, whence if ε_0 is small enough, we can hide the first term in the right-hand side of (2.10.3) to the left-hand side, and conclude that $\|k_1^{X_0}\|_{L^p(Q_0,\sigma)} \lesssim \|k_0^{X_0}\|_{L^p(Q_0,\sigma)}$ uniformly over τ and Q_0. This last estimate, together with (2.10.2), gives the desired result of this step.

Conclusion 2.10.4 (Step 1). The estimate (2.10.2) holds with $k_0^{X_0}$ replaced by $k_1^{X_0}$ with implicit constant independent of τ and Q_0.

We remark that the smallness of ε_0, used in (2.10.3), necessarily depends on the implicit constant in Theorem 2.7.47 applied to the operator L_0; hence, ε_0 depends on the RH_p characteristic of k_0.

Step 2: The desired result for L_τ, uniformly in τ. Now let $L_2 = L_\tau$ in $\Omega_{\hat{B}_0}$ and $L_2 = L_1$ in $\hat{B}_0$ (see (2.10.1)). Note that L_2 is exactly L_τ. According to our choice of $\hat{B}_0$ and Lemma 2.7.34, we have that $k_2^{X_0}(y) \approx k_1^{X_0}(y)$ for $\sigma-$almost every $y \in B(x_{Q_0}, 7\sqrt{n}A_0\ell(Q_0)) \supset Q_0$ (uniformly over τ and Q_0). As such, we reckon the estimate

$$\int_{Q_0} (k_2^{X_0})^p \, d\sigma \approx \int_{Q_0} (k_1^{X_0})^p \, d\sigma \lesssim \sigma(Q_0)^{1-p},$$

with implicit constants independent of τ and Q_0. Let $X_{Q_0} \in \Omega$ be a Corkscrew point for Q_0. Then by Lemma 2.7.33 and the doubling property of the elliptic measure, we have that $k_2^{X_0} \approx k_2^{X_{Q_0}}$ for $\sigma-$almost every $y \in Q_0$. Therefore, we conclude that $\int_{Q_0} (k_\tau^{X_{Q_0}})^p \, d\sigma \lesssim \sigma(Q_0)^{1-p}$ (with implicit constant independent of τ and Q_0). Since

$Q_0 \in \mathbb{D}$ was arbitrary, then the Poisson kernel k_τ for the operator L_τ satisfies the condition (2.7.46) in Remark 2.7.45. By this same remark, we know that then k_τ satisfies the condition (2.7.42) (with C_0 independent of τ), and finally by Theorem 2.7.40 we obtain the desired conclusion for L_τ with no dependence on τ.

Conclusion 2.10.5 (Step 2). We have that $\omega_\tau \ll \sigma$ and $k_\tau = \frac{d\omega_\tau}{d\sigma} \in RH_p$ (see Definition 2.7.28) with RH_p characteristic independent of τ.

Step 3: Pass to the limit in τ. Using the conclusion of the previous step and Lemma 2.9.22, we obtain that $k = \frac{d\omega_L}{d\sigma} \in RH_p$. Then we can use Theorem 2.7.40 to obtain that the Dirichlet problem with $L^{p'}$ data is solvable, where p' is the Hölder conjugate of p. This finishes the proof of the theorem. $\square$

Chapter 3

Generalized Carleson perturbations and applications

The research in this chapter was done in collaboration with J. Feneuil.

3.1 Introduction

In this section, we continue the introduction to this chapter, already begun in Section 1.2.2. Relevant literature review lies in Sections 1.3.4 and 1.3.5.

In this chapter, we study additive, scalar-multiplicative, and antisymmetric perturbations of Carleson type for the Dirichlet problem for real second-order divergence-form (possibly degenerate, not necessarily symmetric) elliptic operators on domains which admit an elliptic PDE theory. We call such domains *PDE friendly* (see Section 3.2 for our axioms and examples of PDE friendly domains). Roughly speaking, if L_0 and L_1 are two elliptic operators on such a domain, we seek conditions on the relative structure of L_1 to L_0 that preserve certain "good estimates" for the Dirichlet problem. In particular, we develop Carleson perturbations which allow for non-trivial differences at the boundary. As we have already discussed in the historical survey the adjacent literature (particularly, Sections 1.3.2, 1.3.4, and 1.3.5), and we have briefly motivated our results of this chapter in Section 1.2.2, let us proceed immediately to the preliminaries and statements of our main results.

3.1.1 Main results

The domains which we consider are described fully in Section 3.2.1, but here let us give a quick review. We assume that our domains $\Omega \subset \mathbb{R}^n, n \geq 2$ are 1-sided NTA domains (that is, they have the interior Corkscrew and the Harnack Chain properties, see Definition 3.2.8), and that they are paired with a positive doubling measure $dm = w\, dX$ on Ω such that $w \in L^1_{\text{loc}}(\Omega, dX)$ and such that (Ω, m) has an L^2-Poincaré inequality on interior balls. The weight w is tailored to the study of the boundary of Ω. We ask our operators $L = -\operatorname{div} A\nabla$ to satisfy an elliptic and boundedness condition that matches the behavior of w, that is, for almost every $X \in \Omega$, we require the existence of $C_L > 0$ such that

$$A(X)\xi \cdot \xi \geq (C_L)^{-1} w(X)|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n, \tag{3.1.1}$$

and

$$|A(X)\xi \cdot \zeta| \leq C_L w(X)|\xi||\zeta| \quad \text{for } \xi, \zeta \in \mathbb{R}^n. \tag{3.1.2}$$

If we write L as $-\operatorname{div}(w\mathcal{A}\nabla)$, we can remove the dependence on $w(X)$ in (3.1.3)–(3.1.4) and recover the classical elliptic and boundedness conditions

$$\mathcal{A}(X)\xi \cdot \xi \geq (C_L)^{-1}|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n \tag{3.1.3}$$

and

$$|\mathcal{A}(X)\xi \cdot \zeta| \leq C_L|\xi||\zeta| \quad \text{for } \xi, \zeta \in \mathbb{R}^n. \tag{3.1.4}$$

One may prefer to pick a second order operator L first, and then think of m as a way to describe the degeneracies of L. In particular, the case where L is uniformly elliptic and bounded in the classical sense means that m is the Lebesgue measure on Ω, and vice versa. Finally, we assume that on our domains (Ω, m) there is a robust theory for the "elliptic" operators in the sense of (3.1.1-3.1.2). The details of the theory that we assume are laid out in Definition 3.2.10; in summary, we require boundary Hölder continuity of solutions, the existence and uniqueness of doubling elliptic measures giving the appropriate representation formula for solutions to the continuous Dirichlet problem, and a weakly-defined Green's function.

The domains (Ω, m) described above are denoted as PDE-friendly domains. We mention several examples in Section 3.2.2, but a few of them are the 1-sided NTA domains satisfying the capacity density condition, the low-dimensional Ahlfors-David regular domains of [DFM19b], and mixed-dimensional sawtooth domains as in [MP21].

In the rest of the chapter, for any $X \in \Omega$, $\delta(X)$ is given as in (2.2.5) and B_X denotes $B(X, \delta(X)/4)$. When $x \in \partial\Omega$ and $r > 0$, we write $B(x,r)$ for the open ball in $\mathbb{R}^n$ and $\Delta(x,r)$ for the boundary ball $B(x,r) \cap \partial\Omega$. Note that the radius r and center x of a boundary ball are not necessary unique, but by a slight abuse of notation, a boundary ball Δ will mean either a triple $(x, r, \Delta(x,r))$, or just the set $\Delta(x,r)$. The truncated area integral $\mathscr{A}$ and the non-tangential maximal function N are constructed as follows:

$$\mathscr{A}^r(f)(x) = \Big(\iint_{\gamma^r(x)} |f(X)|^2 \frac{dm(X)}{m(B_X)} \Big)^{\frac{1}{2}} \qquad \text{for } r > 0,\ x \in \partial\Omega, \text{ and } f \in L^2_{\mathrm{loc}}(\Omega, m) \tag{3.1.5}$$

and

$$N^r(f)(x) = \sup_{\gamma^r(x)} |f| \qquad \text{for } r > 0,\ x \in \partial\Omega, \text{ and } f \in C(\Omega),$$

where

$$\gamma^r(x) = \{X \in \Omega,\ |X - x| \leq 2\delta(X) \leq 2r\}.$$

Definition 3.1.6 (Doubling family of measures)**.** We say that a family $\omega = \{\omega^X\}_{X \in \Omega}$ of Borel measures is *doubling* if there exists a constant $C > 0$ such that for $x \in \partial\Omega$, $r > 0$, and $X \in \Omega \setminus B(x, 4r)$, we have

$$\omega^X(B(x, 2r)) \leq C \omega^X(B(x,r)). \tag{3.1.7}$$

The measure σ is doubling if (3.1.7) is verified with σ instead of ω^X.

Below, we give a meaning to saying that an $L^2_{\mathrm{loc}}(\Omega, m)$ function satisfies a Carleson measure property.

Definition 3.1.8 (Carleson measure condition)**.** If $\omega = \{\omega^X\}_{X \in \Omega}$ is a doubling family of measures on $\partial\Omega$, we say that a function $f \in L^2_{\mathrm{loc}}(\Omega, m)$ satisfies the *ω-Carleson measure condition* if there exists $M > 0$ such that for any $x \in \partial\Omega$, any $r > 0$, and any

$X \in \Omega \backslash B(x, 2r)$[1], we have

$$\int_{\Delta(x,r)} |\mathscr{A}^r(f)(y)|^2 \, d\omega^X(y) \leq M \omega^X(\Delta(x,r)). \tag{3.1.9}$$

We write $f \in KCM(\omega)$ to say that f satisfies the ω-Carleson measure condition and $f \in KCM(\omega, M)$ if we want to refer to the constant in (3.1.9). We will often use M_f for the smallest admissible constant in (3.1.9), and we call it the *Carleson norm* of f. If σ is simply a measure on $\partial\Omega$, the σ-Carleson measure condition means (3.1.9) where ω^X is replaced by σ. At last, we shall also need the following variant of the Carleson measure condition. We say that

$$f \in KCM_{\sup}(\omega, M) \quad \text{if}_{def} \quad X \mapsto \sup_{B_X} |f| \in KCM(\omega, M).$$

We note that, in settings where the Green function $G(X,Y)$ is defined and can be compared to the elliptic measure (via a suitable estimate of the form (3.2.15)), our condition (3.1.9) is equivalent to the finiteness of the expression

$$\sup_{r>0} \sup_{x \in \partial\Omega} \sup_{X \in \Omega \backslash B(x,2r)} \frac{1}{\omega^X(\Delta(x,r))} \iint_{B(x,r) \cap \Omega} |f(Y)|^2 \frac{G(X,Y)}{\delta(Y)^2} \, dm(Y) \tag{3.1.10}$$

via Fubini's theorem. By comparing (3.1.10) to (1.3.9), we see that our Carleson measure condition is a reformulation of analogue Carleson measure conditions considered in [FKP91] and [AHMT]. Moreover, since $\mathscr{A}^r \leq \mathscr{A}$ (where $\mathscr{A}$ is the area integral with no truncation), our condition (3.1.9) readily captures the same results under (an analogue of) the stronger L^∞ assumption on the area integral (1.3.6); this last observation had essentially been made already in [AHMT, Chapter 3].

Now, we define $A_\infty-$absolute continuity among doubling families of measures.

Definition 3.1.11 (A_∞ for families of measures)**.** If $\omega_0 = \{\omega_0^X\}_{X\in\Omega}$ and $\omega_1 = \{\omega_1^X\}_{X\in\Omega}$ are two doubling families of measures on $\partial\Omega$, then we say that ω_1 is *A_∞-absolutely continuous with respect to* ω_0 - or $\omega_1 \in A_\infty(\omega_0)$ for short - if, for any $\xi > 0$, there exists $\zeta > 0$ such that for any boundary ball $\Delta := \Delta(x,r)$, any $X \in \Omega\backslash B(x,2r)$, and any

[1] Note that if $\operatorname{diam} \Omega < +\infty$ and r is large, then $\Omega \backslash B(x, 2r) = \emptyset$ and hence (3.1.9) is automatically true (by convention). If $\operatorname{diam} \partial\Omega < +\infty$ and r large, then $\Omega\backslash B(x,2r) \neq \emptyset$ and this definition also makes sense; if we want to weaken the definition to $r \in (0, \operatorname{diam} \partial\Omega)$, then we fall in the situation presented in Subsection 3.2.3.

Borel set $E \subset \Delta$, we have that

$$\frac{\omega_1^X(E)}{\omega_1^X(\Delta)} < \zeta \quad \text{implies} \quad \frac{\omega_0^X(E)}{\omega_0^X(\Delta)} < \xi. \tag{3.1.12}$$

If σ_0 or σ_1 are measures, then we replace ω_0^X by σ_0 or/and ω_1^X by σ_1 in (3.1.12).

Our first main theorem links a bound on the oscillations of bounded solutions to A_∞. The result is the analogue in our setting of [CHMT20, Theorem 1.1 (a) $\implies$ (b)] or [KKPT16, Theorem 4.1].

Theorem 3.1.13 (Weak-BMO solvability implies A_∞)**.** *Let (Ω, m) be a PDE friendly domain (see Definition 3.2.10). Let $L = -\operatorname{div} A\nabla$ be an elliptic operator satisfying* (3.1.1) *and* (3.1.2)*, and construct the elliptic measure $\omega := \{\omega^X\}_{X\in\Omega}$ as in* (3.2.11)*. Let σ be a doubling measure or doubling family of measures on $\partial\Omega$.*

If there exists $M > 0$ such that, for any Borel $E \subset \partial\Omega$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies

$$\delta\nabla u_E \in KCM(\sigma, M), \tag{3.1.14}$$

then $\omega \in A_\infty(\sigma)$.

In fact, we prove stronger local analogues; see Lemma 3.4.1 and Corollary 3.4.3.

Our second main theorem states that Carleson perturbations of an elliptic operator perserve the A_∞-absolute continuity, via an $S < N$ estimate. However, we give a much broader sense to Carleson perturbations than what was found previously in the literature, and that will be our contribution to the answer of Question 2 posed in Section 1.3.

Definition 3.1.15 (Generalized Carleson perturbations)**.** Let $L_0 = -\operatorname{div}(w\mathcal{A}_0\nabla)$ and $L_1 = -\operatorname{div}(w\mathcal{A}_1\nabla)$ be two operators satisfying (3.1.3)–(3.1.4), and let $\omega_0 = \{\omega_0^X\}_{X\in\Omega}$ be the elliptic measure of L_0 constructed in (3.2.11).

We say that L_1 is an **additive Carleson perturbation** of L_0 if

$$|\mathcal{A}_1 - \mathcal{A}_0| \in KCM_{\sup}(\omega_0).$$

We say that L_1 is a **scalar-multiplicative Carleson perturbation** of L_0 if there exists a scalar function b such that $C^{-1} \leq b \leq C$ for some $C > 0$ and

$$\mathcal{A}_1 = b\mathcal{A}_0, \qquad \text{and } \delta|\nabla b| \in KCM(\omega_0).$$

The operator L_1 is an **antisymmetric Carleson perturbation** of L_0 if there exists a bounded, antisymmetric matrix-valued function $\mathcal{T}$ such that

$$\mathcal{A}_1 = \mathcal{A}_0 + \mathcal{T}, \qquad \text{and } \delta w^{-1}|\operatorname{div} w\mathcal{T}| \in KCM(\omega_0) \tag{3.1.16}$$

where $\operatorname{div}(\mathcal{T})$ is the vector obtained by taking the divergence of each column of $\mathcal{T}$. At last, L_1 is a **(generalized) Carleson perturbation** of L_0 if there exists a matrix-valued function $\mathcal{C}$, a scalar function b, and an antisymmetric matrix-valued function $\mathcal{T}$ such that

$$|\mathcal{C}| \in KCM_{\sup}(\omega_0),\ \delta|\nabla b| + \delta w^{-1}|\operatorname{div}(w\mathcal{T})| \in KCM(\omega_0), \text{ and } \mathcal{A}_1 = b(\mathcal{A}_0 + \mathcal{C} + \mathcal{T}),$$

and the **norm of the Carleson perturbation** is the smallest value $K > 0$ such that $|\mathcal{C}| \in KCM_{\sup}(\omega_0, K)$ and $\delta|\nabla b|/b + \delta w^{-1}|\operatorname{div}(w\mathcal{T})| \in KCM(\omega_0, K)$.

Note that the additive Carleson perturbation is what was known as the Carleson perturbation in earlier articles, and so we extended the notion of Carleson perturbation to the 'scalar-multiplicative' and 'antisymmetric' perturbations. These last two types of perturbation can be seen (at least formally) as **drift Carleson perturbations** via the following well-known transformations:

$$L_1 := -\operatorname{div}(wb\mathcal{A}_0\nabla) = -b\operatorname{div}(w\mathcal{A}_0\nabla) - w(\mathcal{A}_0)^T\nabla b \cdot \nabla = bL_0 - w(\mathcal{A}_0)^T\nabla b \cdot \nabla \tag{3.1.17}$$

and

$$L_1 := -\operatorname{div}(w[\mathcal{A}_0 + \mathcal{T}]\nabla) = L_0 - \operatorname{div}(w\mathcal{T}) \cdot \nabla. \tag{3.1.18}$$

On the other hand, note that our perspective allows us to consider these perturbations without a priori constructing an elliptic theory for operators with drift terms. The scalar-multiplicative and antisymmetric perturbations are interesting because they are perturbations that can significantly change the coefficients of the initial matrix $\mathcal{A}_0$ in a neighborhood of the boundary. They also appeared naturally in previous works. In [DMb] and [Fen], the authors proved that, when the boundary is a uniformly rectifiable set of dimension $d < n-1$, the elliptic measure associated to the operators

$L_\beta = -\operatorname{div}[D_\beta]^{d+1-n}\nabla$ is A_∞-absolutely continuous with respect to the d-dimensional approach (see [DMb], [Fen] for the definitions of uniformly rectifiable and D_β); the proof in [Fen] relies on the fact L_β are scalar-multiplicative Carleson perturbations of each other. Theorem 1.6 in [CHMT20] states a particular case of the following assertion, which is an easy consequence of our Theorem 3.1.19 below: if L^* is an antisymmetric Carleson perturbation of L, then $\omega_{L^*} \in A_\infty(\omega_L)$, and the elliptic measure of the self-adjoint operator $L_s = (L+L^*)/2$ belongs to the same A_∞ class than ω_L and ω_{L^*}. The idea of taking Carleson perturbations in the drift term has also appeared before [HL01a, KP01], but to the best of our knowledge, the present chapter is the first time that drift Carleson perturbation are used to extend the class of transformations of the elliptic matrix A that perserves the A_∞-absolute continuity.

Theorem 3.1.19 ($S < N$ is preserved by Carleson perturbations)**.** *Let (Ω, m) be a PDE friendly domain (see Definition 3.2.10). Let $L_0 = -\operatorname{div} w\mathcal{A}_0\nabla$ and $L_1 = -\operatorname{div} w\mathcal{A}_1\nabla$ be two elliptic operators satisfying (3.1.3) and (3.1.4), and construct the elliptic measures $\omega_0 := \{\omega_0^X\}_{X\in\Omega}$ and $\omega_1 := \{\omega_1^X\}_{X\in\Omega}$ as in (3.2.11).*

If L_1 is a (generalized) Carleson perturbation of L_0, then for any $x \in \partial\Omega$, any $r \in (0, \operatorname{diam}\Omega)$, any Corkscrew point X associated to (x,r), and any weak solution u to $L_1 u = 0$, we have that

$$\int_{\Delta(x,r)} |\mathscr{A}^r(\delta\nabla u)|^2 d\omega_0^X \leq C \int_{\Delta(x,2r)} |N^{2r}(u)|^2 d\omega_0^X, \tag{3.1.20}$$

with a constant $C > 0$ that depends only on the dimension n, C_{L_0}, C_{L_1}, the norm of the Carleson perturbation, and the constants in the PDE friendly properties of (Ω, m).

In particular, (3.1.14) holds with $\sigma = \omega_0$, and hence $\omega_1 \in A_\infty(\omega_0)$.

Via different methods, a local $S < N$ result (which works even in more general L^q settings) has been obtained in [AHMT] for the 1-sided NTA domains satisfying the capacity density condition. We could also obtain the same result from [AHMT] by applying a good-λ argument to (3.1.20), but we do not need those bounds for the present results.

It is well know that A_∞ is an equivalence relationship (see [GR85a]), which means that Theorem 3.1.19 would also hold if we assume that L_0 is a Carleson perturbation of L_1 (which is *a priori* different from saying that L_1 is a Carleson perturbation of L_0, since the Carleson measure condition depends on the operator before perturbation). However, by combining Theorem 3.1.19 with the theorem below, we obtain that our notion of 'Carleson perturbations of elliptic operators' is actually an equivalence relationship, as expected.

Theorem 3.1.21 (A_∞ implies transitivity of CM)**.** *Let (Ω, m) be a PDE friendly domain (see Definition 3.2.10), and for $i \in \{0,1\}$, let μ_i be either an elliptic measure [$\mu_i = \{\omega_i^X\}_{X\in\Omega}$] or a doubling measure [$\mu_i = \sigma_i$] on $\partial\Omega$. If $\mu_1 \in A_\infty(\mu_0)$, then*

$$f \in KCM(\mu_0) \quad \textit{if and only if} \quad f \in KCM(\mu_1), \qquad \textit{for each } f \in L^2_{\rm loc}(\Omega, m). \tag{3.1.22}$$

For a local analogue of the above result, see Lemma 3.3.30. We actually can prove a characterization of A_∞ via the property (3.1.22), see Corollary 3.1.25 below. Theorem 3.1.21 can be seen as analogue of the John-Nirenberg lemma (which is for BMO functions) adapted to Carleson measures and A_∞ weights. The result is an extension to our setting of [HMM, Lemma 3.8], which itself is a modification of John-Nirenberg type inequalities proved in [HM09, Lemma 10.1], [AHLT01, Lemma 2.14], and [MMM20, Lemma A.1], although our method of proof is different. Since the condition $KCM_{\sup}(\omega_i)$ is only $KCM(\omega_i)$ applied to a transformation of f, we have in particular that if $\omega_1 \in A_\infty(\omega_0)$, then $f \in KCM_{\sup}(\omega_0) \Leftrightarrow f \in KCM_{\sup}(\omega_1)$. Lastly, see Lemma 3.3.30 for a local version.

Let us emphasize that none of our proofs rely on the construction of sawtooth domains on PDE friendly domains, nor do they rely on the extrapolation theory of Carleson measures. Indeed, it is not clear to us that sawtooth domains of PDE friendly domains are themselves PDE friendly. Even if they were, the construction of, and verification of PDE friendly axioms on sawtooth domains of some rough domains are long and difficult tasks [HM14, MP21]. Our method resembles loosely that of the recent paper [CHMT20], where an analogue of Theorem 3.1.13 is used to extend the FKP (additive) perturbation theory to the case of 1-sided chord-arc domains, but they also use sawtooth domains.

The rest of the chapter will be divided as follows. In the rest of the introduction, we give some applications of our three theorems (Theorem 3.1.13, Theorem 3.1.19, Theorem 3.1.21). In Section 3.2, we present the assumptions for the PDE friendly domains and examples of these domains. In Section 3.3, we recall the theory of A_∞-weights that we need for our proof, and moreover, we prove Theorem 3.1.21. Section 3.4 and Section 3.5 are devoted to the proofs of Theorem 3.1.13 and Theorem 3.1.19, respectively.

3.1.2 Applications of main results

Let us present several implications of our theorems.

First, a straightforward consequence of Theorems 3.1.13, 3.1.19, and 3.1.21 is the fact that if the elliptic measure ω_0 is already A_∞-absolutely continuous with respect to a doubling measure σ, and L_1 is a Carleson perturbation of L_0, then the $A_\infty(\sigma)$ absolute continuity is transmitted to ω_1. Thus, our results extend the FKP perturbation theory to PDE friendly domains, and hence, to the domains verifying the axioms of [DFM].

Corollary 3.1.23 (An extension of the FKP perturbation result to PDEF domains)**.** *Let (Ω, m) be a PDE friendly domain (see Definition 3.2.10), and let σ be a doubling measure on $\partial\Omega$. Consider an elliptic operator $L_0 := -\operatorname{div}(w\mathcal{A}_0\nabla)$ such that $\omega_0 \in A_\infty(\sigma)$. Assume that the elliptic operator $L_1 := -\operatorname{div}(w\mathcal{A}_1\nabla)$ is a Carleson perturbation of L_0 in the sense that there exist a matrix $\mathcal{C}$, a scalar b, and an antisymmetric matrix $\mathcal{T}$ such that $\mathcal{A}_1 = b(\mathcal{A}_0 + \mathcal{C} + \mathcal{T})$, and*

$$|\mathcal{C}| \in KCM_{\sup}(\sigma), \quad \text{and} \quad \delta|\nabla b| + \delta w^{-1}|\operatorname{div}(w\mathcal{T})| \in KCM(\sigma).$$

Then $\omega_1 \in A_\infty(\sigma)$.

Proof. Since $\omega_0 \in A_\infty(\sigma)$, Theorem 3.1.21 gives that L_1 is a generalized Carleson perturbation of L_0, as given in Definition 3.1.15, hence as needed for Theorem 3.1.19. Applying Theorem 3.1.19 and then Theorem 3.1.13 yields the desired $\omega_1 \in A_\infty(\sigma)$.

Moreover, our theory gives

Corollary 3.1.24 (Equivalence of A_∞ and weak-$BMO-$solvability)**.** *Let (Ω, m) be a PDE friendly domain (see Definition 3.2.10). Let L and ω as in Theorem 3.1.13, and take a doubling measure σ on $\partial\Omega$. The following are equivalent:*

(i) $\omega \in A_\infty(\sigma)$.

(ii) *the Dirichlet problem to $Lu = 0$ is weak-$BMO(\sigma)$ solvable; that is, there exists $M > 0$ such that, for any Borel $E \subset \partial\Omega$, the solution u_E constructed as $u_E(X) :=$*

$\omega^X(E)$ *satisfies*

$$\delta\nabla u_E \in KCM(\sigma, M).$$

Proof. The implication (ii) $\Rightarrow$ (i) is a consequence of Theorem 3.1.13. Since L is an ω-Carleson perturbation of itself, (i) $\Rightarrow$ (ii) follows from Theorems 3.1.19 and 3.1.21. □

After the first version of this work was posted online [FP], we learned that Cao, Domínguez, Martell, and Tradacete were about to finish an article (see [CDMT21]) with a result similar to our Corollary 3.1.24, using roughly the same techniques as the ones we used. They worked on the specific case of 1-sided NTA domains satisfying the capacity-density condition, but they gave many more characterizations of the A_∞ property of the elliptic measure than us. In particular, they prove that the full BMO solvability, and the $S < N$ estimate in some L^q (with S the conical square function), are equivalent to the A_∞ property among elliptic measures. Note however that the main interest of [CDMT21] differs from ours, in that they focused on criterions of A_∞, while we were primarily interested in extending the notion of Carleson perturbations that preserves A_∞, an issue which is not considered in [CDMT21].

Next, we show that our Theorems 3.1.13 and 3.1.19 yield a certain converse to Theorem 3.1.21, which gives a new characterization of A_∞ among elliptic measures, via the transitivity of the Carleson measure property. This characterization of A_∞ seems new to us, and it does not appear in [CDMT21] either.

Corollary 3.1.25 (A_∞ is equivalent to transitivity of CM, for elliptic measures)**.** *Let* (Ω, m) *be a PDE friendly domain (see Definition 3.2.10), and let* $\{\omega_0^X\}_{X\in\Omega}$, $\{\omega_1^X\}_{X\in\Omega}$ *be two elliptic measures on* $\partial\Omega$. *Then,* $\omega_1 \in A_\infty(\omega_0)$ *if and only if*

$$f \in KCM(\omega_1) \quad \textit{implies that } f \in KCM(\omega_0), \qquad \textit{for each } f \in L^2_{\mathrm{loc}}(\Omega, m). \tag{3.1.26}$$

Proof. The "only if" direction is immediate from Theorem 3.1.21. Now suppose that (3.1.26) holds. Let $E \subset \partial\Omega$ be an arbitrary Borel set, and write $u_1(X) := \omega_1^X(E)$. Note that $\delta|\nabla u_1| \in L^2_{\mathrm{loc}}(\Omega, m)$. According to Theorem 3.1.19, we have (3.1.20), which implies in particular that $\delta\nabla u_1 \in KCM(\omega_1)$. By hypothesis, it follows that $\delta\nabla u_1 \in KCM(\omega_0)$. Since E was arbitrary, then Theorem 3.1.13 allows us to conclude that $\omega_1 \in A_\infty(\omega_0)$. □

It seems to us that Corollary 3.1.25 has not been known even in the classical settings of the half-space or the unit ball. It is not clear that the FKP characterization of (classical)

A_∞ via a Carleson measure condition (1.3.8) immediately implies a suitable analogue of our Corollary 3.1.25. On the other hand, we emphasize that our characterization is proved only among elliptic measures; whether Corollary 3.1.25 holds for general doubling measures is an open question, even in the case of the half-space.

We also remark that the scalar-multiplicative Carleson perturbations contain the scalar subclass of Dahlberg-Kenig-Pipher operators (A1)-(A2). More precisely, it is easy to see that if $A = bI$ is a matrix satisfying the ellipticity and boundedness conditions (1.1.4) and the DKP conditions (A1)-(A2), then b verifies the assumptions

$$C^{-1} \leq b \leq C, \qquad \text{and} \qquad \delta \nabla b \in KCM(\sigma), \tag{3.1.27}$$

where σ is the surface measure; and on the other hand, if b verifies (3.1.27), then $A = bI$ satisfies (A2). By seeing this subclass as a scalar-multiplicative perturbation from the Laplacian $-\Delta$, we are able to obtain, for instance, alternate proofs of difficult results for the scalar subclass of DKP operators, which have recently been shown for the full generality of DKP operators. As a matter of fact, our result for the scalar operators goes slightly beyond that of the DKP operators, as we do not have to assume the boundedness condition on the gradient (A1). Pointedly, consider

Corollary 3.1.28 (A free boundary result for scalar DKP operators)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a uniform (that is, 1-sided NTA) domain with $(n-1)-$Ahlfors-David regular boundary (see Definition 3.2.17), and set $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$. Let b be a function on Ω verifying $C^{-1} \leq b \leq C$ and $\delta \nabla b \in KCM(\sigma)$. Then the following are equivalent.*

(i) The elliptic measure ω_L associated with the operator $L = -\operatorname{div} b \nabla$ is A_∞ with respect to σ.

(ii) $\partial\Omega$ is uniformly rectifiable.

(iii) Ω is a chord-arc domain.

Sketch of proof. For definitions of uniform rectifiability and chord-arc domain, see for instance [HMM+b]. For $L \equiv -\Delta$, then the above equivalences are known [AHM+17, AHM+]; in particular, (ii) and (iii) are equivalent, and either imply (i) with $L = -\Delta$.

We show (i) $\Longrightarrow$ (ii); the converse has a similar proof. Say that $L = -\operatorname{div} b\nabla$ and b has the described properties, and suppose that $\omega_L \in A_\infty(\sigma)$. Then b^{-1} also has the same properties as b; that is, $C^{-1} \leq \frac{1}{b} \leq C$ and $|\delta \nabla(1/b)| = \delta \nabla(b)/b^2 \in KCM(\sigma)$. Since

$-\Delta = -\operatorname{div}(b^{-1}b\nabla)$, then $-\Delta$ is a scalar-multiplicative Carleson perturbation of L, and by Corollary 3.1.23 it follows that $\omega_{-\Delta} \in A_\infty(\sigma)$. Thus we have that $\partial\Omega$ is uniformly rectifiable. $\square$

It is clear that Corollary 3.1.28 also holds with L being any generalized Carleson perturbation from $-\Delta$. The above result for functions b which also verify that $\delta\nabla b \in L^\infty(\Omega)$ is a particular case of the recent free boundary result for the DKP operators shown by Hofmann-Martell-Mayboroda-Toro-Zhao [HMM⁺a, HMM⁺b]; our method of Carleson perturbations allows us to dispense with the aforementioned boundedness condition.

Finally, let us consider an application of our theory to the study of elliptic measures on purely unrectifiable sets[2]. Let K be the Garnett-Ivanov Cantor set [Gar72, Iva84], also known as the 4-corner Cantor set, as defined in Section 3 of [DMa]. The set K is $1-$Ahlfors-David regular with surface measure $\sigma = \mathscr{H}^1$, and it is known to be purely unrectifiable, and such that the harmonic measure $\omega_{-\Delta}$ and surface measure $\sigma = \mathscr{H}^1$ are mutually singular. In [DMa], G. David and S. Mayboroda constructed an elliptic operator $L_a = -\operatorname{div} a\nabla$ on $\Omega := B\backslash K \subset \mathbb{R}^2$ such that $\omega_{L_a} \in A_\infty(\sigma)$ on $\partial\Omega$, where B is the unit ball in the plane centered at the origin (and $K \subset B$), and a is a certain scalar real-valued function with $1/C \leq a \leq C$ in Ω. It is not hard to show that the domain Ω is a 1-sided NTA domain (that is, it has interior Corkscrews and Harnack Chains). Hence $(\Omega, \mathscr{L}^2)$ is a PDE friendly domain, whence our perturbation theory applies. We can use our theory to study certain a priori properties of the function a constructed in [DMa].

First, any elliptic operator $L = -\operatorname{div} A\nabla$ which is a generalized Carleson perturbation of L_a verifies that $\omega_L \in A_\infty(\sigma)$ by Corollary 3.1.23[3]. On the other hand, since $\omega_{-\Delta} \perp \sigma$, it follows that the operator $-\Delta$ on Ω cannot be a generalized Carleson perturbation of L_a; in particular, it cannot be a scalar-multiplicative Carleson perturbation. When we put this realization together with Corollary 3.1.24 and Fubini's theorem, we conclude that

$$\sup_{x\in K}\sup_{r\in(0,2)} \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r)} \delta(Y)|\nabla a(Y)|^2\, dY = +\infty.$$

In other words, the measure with density $\delta|\nabla a|^2$ is not a Carleson measure.

[2]The authors would like to thank Max Engelstein for pointing this application out.

[3]In the case of the classical *additive* Carleson perturbations, this consequence already follows from the perturbation theory in [AHMT].

3.2 Hypotheses and elliptic theory

Throughout, our ambient space is $\mathbb{R}^n$, $n \geq 2$.

3.2.1 PDE friendly domains

In this section we describe the PDE friendly domains and present several examples. First, let us set up some background definitions. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be open.

Definition 3.2.1 (The doubling measure m on the domain)**.** For the remainder of the chapter, we denote by m a measure on Ω that satisfies the following properties:

(i) The measure m is absolutely continuous with respect to the Lebesgue measure; that is, there exists a non-negative weight $w \in L^1_{\text{loc}}(\Omega)$ such that for each Borel set $E \subset \Omega$, $m(E) = \iint_E w(X)\, dX$.

(ii) The measure m is doubling, meaning that there is a constant $C_m \geq 1$ such that

$$m(B(X, 2r) \cap \Omega) \leq C_m m(B(X, r) \cap \Omega) \qquad \text{for each } X \in \overline{\Omega} \text{ and } r > 0. \tag{3.2.2}$$

(iii) For any open set D compactly contained in Ω, and any sequence $\{u_i\}_i \subset C^\infty(\overline{D})$ verifying that $\iint_D |u_i|\, dm \to 0$ and $\iint_D |\nabla u_i - v|^2\, dm \to 0$ as $i \to \infty$, where v is a vector-valued function in $L^2(D, m)$, we have that $v \equiv 0$.

(iv) We assume an L^2-Poincaré inequality on interior balls: there exists C_P such that for any ball B satisfying $2B \subset \Omega$ and any function $u \in W^{1,2}(B, m)$, one has

$$\fint\!\!\fint_B |u - u_B|^2\, dm \leq C_P r \Big(\fint\!\!\fint_B |\nabla u|^2\, dm \Big)^{\frac{1}{2}}, \tag{3.2.3}$$

where u_B stands for $\fint_B u\, dm$ and r is the radius of B.

Let us briefly discuss our assumptions on m. The space $L^2_{\text{loc}}(\Omega, m)$ is not necessarily a space of distributions, meaning that we may not access the notion of a distributional gradient. However, as in [HKM06] and [DFM], the assumption (iii) allows us to construct a notion of gradient ∇ for functions in $L^2_{\text{loc}}(\Omega, m)$, and then we let $W^{1,2}_{\text{loc}}(\Omega, m)$ be the space of functions in $L^2_{\text{loc}}(\Omega, m)$ whose gradient is also in $L^2_{\text{loc}}(\Omega, m)$. It is in this sense that we take the gradient in (3.2.3).

Remark 3.2.4. As long as the weight w that defines m satisfies the slowly varying property

$$\sup_B w \leq C \inf_B w \qquad \text{for any ball } B \text{ such that } 2B \subset \Omega, \tag{3.2.5}$$

then $L^2(\Omega, m)$ is a space of distributions, and the gradient on $L^2(\Omega, m)$ is the gradient in the sense of distribution. In addition, (3.2.3) is true. So as long as (3.2.5) is verified, we just need to take m such that (3.2.2) is true.

From there, we can consider the operator $L = -\operatorname{div}(w\mathcal{A}\nabla)$ that satisfies (3.1.3) and (3.1.4). We say that u is a *weak solution* to $Lu = 0$ if $u \in W^{1,2}_{\text{loc}}(\Omega, m)$ and satisfies

$$\iint_\Omega \mathcal{A}\nabla u \cdot \nabla \varphi \, dm = 0 \qquad \text{for each } \varphi \in C^\infty_c(\Omega).$$

We can deduce the Harnack inequality.

Lemma 3.2.6 (Harnack inequality, Theorem 11.35 in [DFM]). *Let $\Omega \subset \mathbb{R}^n$ and m be as in Definition 3.2.1, and $L = -\operatorname{div}(w\mathcal{A}\nabla)$ satisfy (3.1.3) and (3.1.4). If B is a ball such that $2B \subset \Omega$, and if $u \in W^{1,2}_{\text{loc}}(\Omega, m)$ is a non-negative solution to $Lu = 0$ in $2B$. Then*

$$\sup_B u \leq C \inf_B u, \tag{3.2.7}$$

where C depends only on n, C_m, C_P, and C_L.

Our results are about boundaries, more exactly measures and elliptic measures on the boundary. So, in order to link solutions in Ω and properties of $\partial\Omega$, we require the domain Ω to have enough access to the boundary.

Definition 3.2.8 (1-sided NTA). We say that (Ω, m) is a *1-sided* NTA domain if the following two conditions holds.

Corkscrew point condition (quantitative openness). There exists $c_1 \in (0, 1)$ such that for any $x \in \partial\Omega$ and any $r \in (0, \operatorname{diam}\Omega)$ we can find X such that $B(X, c_1 r) \subset B(x, r) \cap \Omega$.
For $x \in \partial\Omega$ and $r > 0$, we say that X is a *Corkscrew point* associated to the couple (x, r) if $c_1 r/100 \leq \delta(X) \leq |X - x| \leq 100r$.
Harnack chain condition (quantitative path-connectedness). For any $\Lambda \geq 1$, there exists N_Λ such that if $X, Y \in \Omega$ satisfy $\delta(X) > r$, $\delta(Y) > r$, and $|X - Y| \leq \Lambda r$,

then we can find N_Λ balls $B_1, \dots, B_N$ such that $X \in B_1$, $Y \in B_{N_\Lambda}$, $2B_i \subset \Omega$ for $i \in \{1, N_\Lambda\}$, and $B_i \cap B_{i+1} \geq 0$ for $i \in \{1, N_\Lambda - 1\}$.

Remark 3.2.9. In the Harnack chain condition, we can assume without loss of generality that X is the center of B_1, that Y is the center of B_{N_Λ}, and that $20B_i \subset \Omega$. We may have to increase the value of N_Λ but it will still be independent of X, Y, and r.

At last, for our results to hold, we need a nice elliptic theory. For the purpose of the chapter, we shall state the results that we need here, and some geometric settings where they hold.

Definition 3.2.10 (PDE friendly domains)**.** We say that (Ω, m) is PDE friendly if Ω is 1-sided NTA, if m is as in Definition 3.2.1, and if we have the following elliptic theory.

Let $L = -\operatorname{div}(w\mathcal{A}\nabla)$ be any second order divergence order operator, where w is the weight in Definition 3.2.1 and where A is matrix with measurable coefficients which satisfies the ellipticity and boundedness conditions (3.1.3)–(3.1.4).

Existence and uniqueness of elliptic measure. There exist an elliptic measure associated to L, which is the only family of probability measures $\omega_L = \{\omega_L^X\}_{X \in \Omega}$ on $\partial\Omega$ such that, for any function $f \in C_c^\infty(\mathbb{R}^n)$, the function u_f constructed as

$$u_f(X) = \int_{\partial\Omega} f(y) d\omega_L^X(y), \qquad \text{for } X \in \Omega, \tag{3.2.11}$$

is continuous on $\overline{\Omega}$, satisfies $u_f = f$ on $\partial\Omega$, and is a weak solution to $Lu = 0$.
Doubling measure property. For $x \in \partial\Omega$ and $r > 0$, we have that

$$\omega_L^X(\Delta(x, 2r)) \leq C\omega_L^X(\Delta(x, r)) \quad \text{for } X \in \Omega \setminus 3B, \tag{3.2.12}$$

where $\Delta(x, r) := B(x, r) \cap \partial\Omega$, and $C > 0$ is independent of x, r, and X, and depends on L only via C_L.
Change of pole. Let $x \in \partial\Omega$, $r > 0$, and X be a Corkscrew point associated to (x, r). If $E \subset \Delta(x, r)$ is a Borel set, then

$$C^{-1}\omega_L^X(E) \leq \frac{\omega_L^Y(E)}{\omega_L^Y(\Delta(x, r))} \leq C\omega_L^X(E), \quad \text{for } Y \in \Omega \setminus B(x, 2r), \tag{3.2.13}$$

where $C > 0$ is independent of x, r, X, E and Y, and depends on L only via C_L.

Hölder regularity at the boundary. For any $X \in \Omega$ and any Borel set $E \subset \partial\Omega$, we have

$$\omega_L^X(E) \leq C\Big(\frac{\delta(X)}{\mathrm{dist}(X,E)}\Big)^\gamma. \tag{3.2.14}$$

where $C > 0$ and $\gamma \in (0,1)$ are independent of X and E, and depend on L only via C_L.

Comparison with the Green function. Let $X \in \Omega$, write r for $\delta(X)/2$, and take $x \in \partial\Omega$ such that $|X - x| = 2r$. That is, X is a Corkscrew point associated to $(x, 2r)$. There exists a weak solution G_X^* to $L^*u = -\operatorname{div}(w\mathcal{A}^T\nabla) = 0$ in $B(x,r) \cap \Omega$ such that if $y \in \Delta(x,r)$, $s \in (0,r)$, and $Y \in B(x,r) \cap \Omega$ is a Corkscrew point associated to (y,s), we have

$$C^{-1}\frac{m(B(y,s)\cap\Omega)}{s^2}G_X^*(Y) \leq \omega_L^X(\Delta(y,s)) \leq C\frac{m(B(y,s)\cap\Omega)}{s^2}G_X^*(Y). \tag{3.2.15}$$

where $C > 0$ is independent of X, y, s and Y, and depends on L only via C_L.

Of course, when we write $G_X^*(Y)$, we think of the Green function associated to L^* with pole at X. Indeed, in a setting where the notion of the Green function has been developed, like in [DFM19b], we write $g(X,Y)$ for the Green function associated to L with pole at Y, and we set $G_X^*(Y) := g(X,Y)$. In this case, the bounds (3.2.15) are a consequence of [DFM19b, Lemma 15.28] and the fact that G_X^* is a weak solution to L^*u comes from [DFM19b, Lemma 14.78]. However, the notion of Green function has not been properly introduced here, and we do not want to do so, since the only property of the Green function that we really need is the fact that there exists a solution to $L^*u = 0$ in $B(x,r)\cap\Omega$ that satisfies the bounds (3.2.15).

The combination of (3.2.14) and (3.2.6) gives the existence of $c_2 > 0$ such that, for any $x \in \partial\Omega$, any $r > 0$, any Corkscrew point associated to (x,r), and any Borel set $E \supset \Delta(x,r)$, one has

$$\omega_L^X(E) \geq c_2, \tag{3.2.16}$$

where $c_2 > 0$ is independent of x, r, X and E, and depends on L only via C_L. Indeed, (3.2.14) gives that

$$\omega_L^{X'}(\partial\Omega \setminus E) \leq C\Big(\frac{\delta(X')}{\mathrm{dist}(X',E)}\Big)^\alpha \leq C\Big(\frac{|X'-x|}{|X'-x|-r}\Big)^\alpha \leq \frac{1}{2}$$

as long as $|X' - x| \leq c'r$ with a constant c' that depends only on C and α. So if $|X' - x| \leq c'r$ but is still a Corkscrew point associated to $(x, c'r)$, since $\omega^{X'}$ is a probability measure, we have $\omega^{X'}(E) \geq \frac{1}{2}$. We conclude (3.2.16) by linking X' and X by a (uniformly finite) Harnack chain of balls and using the Harnack inequality (Lemma 3.2.6) on each of the balls in the chain.

3.2.2 Examples of PDE friendly domains

Let us first state precisely some definitions of boundary conditions which we have alluded to in previous sections.

Definition 3.2.17 (Ahlfors-David regular set)**.** Fix $d \in (0, n-1]$. We say that $\Gamma \subset \mathbb{R}^n$ is a *d-Ahlfors-David regular set* (or $d-$ADR) if there exists $C_d > 0$ and a measure σ on Γ such that

$$C_d^{-1} r^d \leq \sigma(B(x,r) \cap \Gamma) \leq C_d r^d \quad \text{for each } x \in \Gamma, \qquad 0 < r \leq \operatorname{diam} \Gamma. \tag{3.2.18}$$

If (3.2.18) is verified, maybe to the price of taking a larger C_d, we can always choose σ to be the d-dimensional Hausdorff measure on Γ.

Definition 3.2.19 (Capacity and capacity density condition)**.** Given an open set $D \subset \mathbb{R}^n$, $n \geq 2$, and a compact set $K \subset D$, we define the *capacity* of K relative to D as

$$\mathrm{Cap}_2(K, D) = \inf \Big\{ \iint_D |\nabla v(X)|^2 \, dX : v \in C_c^\infty(D),\ v(x) \geq 1 \text{ on } K \Big\}.$$

An open set Ω is said to satisfy the *capacity density condition* (CDC) if there exists a uniform constant $c_1 > 0$ such that

$$\frac{\mathrm{Cap}_2(\overline{B(x,r)} \setminus \Omega, B(x, 2r))}{\mathrm{Cap}_2(\overline{B(x,r)}, B(x,2r))} \geq c_1,$$

for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$.

We now describe several examples of PDE friendly domains (Ω, m).

(i) Ω is a 1-sided NTA domain satisfying the capacity density condition, and $dm = dX$. The elliptic theory for these operators may be found in [HMT], and see [AHMT] for an (additive) perturbation theory in this context. These domains include, in particular, the 1-sided chord-arc (that is, 1-sided NTA and $(n-1)-$ADR) domains, and the $(n-1)-$ADR domains with uniformly rectifiable boundaries.

(ii) The case where the boundary is low dimensional; fix $d \in (0, n-1)$ and assume that $\Gamma \subset \mathbb{R}^n$ is a $d-$ADR closed set. Set $\Omega = \mathbb{R}^n \backslash \Gamma$ and $dm = \delta(X)^{d+1-n}\, dX$. In this situation, the Harnack Chain condition and the Corkscrew point condition are always true, and the elliptic theory was constructed in [DFM19b]. An additive perturbation theory was written in [MP21], for $d \geq 1$. Our perturbation theory works for $d \in (0,1)$ as well.

(iii) Given (Ω, m) as in the previous case (and assume that $d \geq 1$), given a family $\mathcal{F}$ of pairwise disjoint "dyadic cubes" (these are the David-Christ cubes; see Section 3.3 for the definition) on the boundary $\Gamma = \partial\Omega$, we may construct a sawtooth domain $(\Omega_{\mathcal{F}}, m_{\mathcal{F}})$ (with $m_{\mathcal{F}} = m|_{\Omega_{\mathcal{F}}}$) that "hides" $\mathcal{F}$, via the procedure in [HM14] using Whitney cubes; see Sections 3 and 4 of [MP21] for the details. The boundary $\partial\Omega_{\mathcal{F}}$ has pieces of dimension d, and other pieces of dimension $n-1$, and thus is mixed-dimensional. In Section 5 of [MP21], it is shown that we may define a Borel regular measure $\sigma_\star$ on $\partial\Omega_{\mathcal{F}}$ (a "surface measure") so that the triple $(\Omega_{\mathcal{F}}, m_{\mathcal{F}}, \sigma_\star)$ satisfies the assumptions (H1)-(H6) of the recent axiomatic elliptic theory in [DFM]. Briefly, the assumptions (H1)-(H6) include the same conditions on $m_{\mathcal{F}}$ that we have placed on m in the previous section, the interior Corkscrew point condition and the Harnack Chain condition in $\Omega_{\mathcal{F}}$, a doubling property of $\sigma_\star$, and a slow growth condition on $m_{\mathcal{F}}$ with respect to $\sigma_\star$. With these assumptions verified, the elliptic theory of [DFM] gives us a sufficiently robust elliptic theory on these mixed-dimensional sawtooth domains. Thus $(\Omega_{\mathcal{F}}, m_{\mathcal{F}})$ is a PDE friendly domain.

(iv) For that matter, any triple (Ω, m, μ) (with μ a measure on $\partial\Omega$) satisfying (H1)-(H6) of [DFM] is a PDE friendly domain. This includes the 1-sided chord-arc domains, the domains with low-dimensional boundaries as in (ii), and the mixed-dimensional sawtooth domains as in (iii), but there are many more examples, including some domains with boundaries having atoms, and t-independent degenerate operators on $\mathbb{R}^n_+$ that can be written as $L = -\operatorname{div} A(x)\nabla$ with a matrix $A(x)$ that lies in the Muckenhoupt class $A_2(\mathbb{R}^{n-1})$; see Section 3 of [DFM] for more details and examples. *Proof.* I deleted the mention of Caffarelli-Silvestre fractional operators, because upon looking at Section 3 of [DFM], it is not clear that these are considered with enough details. Mentioning them here might mislead the reader

3.2.3 Local theory

An important remark is that, even if our main results (Theorems 3.1.13, 3.1.19, and 3.1.21) are stated in their global form, all our theory is intrinsically local. The local versions of our theorems are Corollary 3.4.3, Lemma 3.5.1, and Lemma 3.3.30.

A consequence of the locality of our results and proofs is the fact that, if we are only interested in local results, then we only need to assume a local version of the fact that (Ω, m) is PDE friendly.

Given $x_0 \in \partial\Omega$ and $r_0 > 0$, we say that (Ω, m) is *locally PDE friendly* in $B_0 := B(x_0, r_0)$ if the following assumptions hold.

(a) In Definition 3.2.8, we only require the existence of Corkscrew points associated to $y \in \frac{3}{4}B_0$ and $r < r_0/2$. We want Harnack chains between X and Y only when $X, Y \in \frac{3}{4}B_0 \cap \Omega$. But the Harnack chains might go out of B_0, which complicate the definitions. In the sequel, we write T_0 for the union of $B_0 \cap \Omega$ with all the Harnack chains linking two points $X, Y \in \frac{3}{4}B_0 \cap \Omega$.

(b) The measure m needs to be defined only on T_0 (as given in (a)). We only need the doubling property (3.2.2) when $B(X, 2r) \cap \Omega \subset T_0$, and we only need the Poincaré inequality (3.2.3) when $B \subset T_0$ and $2B \subset \Omega$.

(c) We need a notion of elliptic measure associated to our elliptic operator L. We want a collection $\{\omega_L^X\}_{X \in \Omega \cap B_0}$ of measures on $\Delta_0 := B_0 \cap \partial\Omega$ that satisfy $c_2 \leq \omega_L^X(\Delta_0) \leq 1$ for any $X \in T_0$, and for which $u_E(X) := \omega_L^X(E)$ are solutions to $Lu = 0$ in T_0 whenever $E \subset \Delta_0$ are Borel sets.

(d) On the "elliptic measure" defined on the previous point, we ask for the doubling property (3.2.12) only when $\Delta(x, 2r) \subset \Delta_0$ and $X \in T_0$, for the change of pole property (3.2.13) only when the two poles X, Y are in T_0, for the Hölder continuity (3.2.14) only when $X \in B_0 \cap \Omega$, and for the comparison with a "Green function" (3.2.15) only when $X \in B_0 \cap \Omega$ and $\Delta(y, s) \subset \Delta_0$.

Under the above assumptions, local forms of our results hold, and we have that if L_1 and L_0 differs by a Carleson perturbation in T_0 (the Carleson perturbations are defined with a local analogue of KCM or $KCM_{\sup}$), then the A_∞ absolute continuity of the elliptic measure is preserved from L_0 to L_1 in $\frac{1}{2}B_0 \cap \partial\Omega$. Let us now give an example.

Assume that $\Omega \subset \mathbb{R}^n$ is the unbounded connected component of an $(n-1)$-Ahlfors regular set (we call σ the Ahlfors regular measure), and assume that Ω is 1-sided NTA.

For instance, Ω can be $\mathbb{R}^n \setminus B(0,1)$. We choose then $m = \mathscr{L}^n$ to be the Lebesgue measure on Ω. Then Definitions 3.2.1 and 3.2.8 are verified. So in order to be able to apply our theory, we need to check whether $(\Omega, \mathscr{L}^n)$ is PDE friendly, and in particular, whether it has a nice elliptic theory.

However, the harmonic measure $\{\omega^X\}$ associated to this domain - which is the elliptic measure associated to the Laplacian - is not a probability measure, because the Brownian motion has a non-zero probability to escape at infinity. Even worse, we have that $\omega^X(\partial\Omega)$ tends to 0 as $|X| \to \infty$. So we deduce that such $(\Omega, \mathscr{L}^n)$ are not PDE friendly. On the other hand, by considering local perspectives, our theory can still be applied.

The first local perspective is to stick to local estimates. Let x_0 be any point in $\partial\Omega$ and for instance $r_0 = 100\,\mathrm{diam}(\partial\Omega)$. Then one can check that $(\Omega, \mathscr{L}^n)$ is locally PDE friendly in $B_0 := B(x_0, r_0)$. Take L_0 a uniformly elliptic operator and we assume that its elliptic measure satisfies $\omega_0 \in A_\infty(\sigma, \Delta_0)$, that is, for any $\xi > 0$, there exists $\zeta > 0$ such that for any boundary ball $\Delta := \Delta(x,r) \subset \Delta_0 := \Delta(x_0, r_0)$ and any Borel set $E \subset \Delta$, we have

$$\frac{\omega_0^X(E)}{\omega_0^X(\Delta)} < \zeta \quad \text{implies} \quad \frac{\sigma(E)}{\sigma(\Delta)} < \xi \qquad \text{for } X \in [\Omega \setminus B(x,2r)] \cap B_0. \tag{3.2.20}$$

We say that $f \in KCM_{\Delta_0}(\sigma)$ if there exists $M > 0$ such that for $x \in \partial\Omega$ *and* $r < r_0$, we have

$$\int_{\Delta(x,r)} |\mathscr{A}^r(f)(y)|^2 \, d\sigma(y) \leq M\sigma(\Delta(x,r)). \tag{3.2.21}$$

Choose $L_1 = -\operatorname{div}[b(\mathcal{A}_0 + \mathcal{C} + \mathcal{T})\nabla]$ a Carleson perturbation of L_0 in B_0, that is

$$|\mathcal{C}| \in KCM_{\sup,\Delta_0}(\sigma) \quad \text{and} \quad \delta|\nabla b| + \delta|\operatorname{div}\mathcal{T}| \in KCM_{\Delta_0}(\sigma)$$

By applying Lemma 3.5.1 and Corollary 3.4.3, we obtain that $\omega_1 \in A_\infty(\sigma, \Delta_0)$.

An alternative perspective is to change $(\Omega, \mathscr{L}^n)$ so that it becomes PDE friendly. We do not want to change Ω, and we want to keep $m = \mathscr{L}^n$ when we are close to $\partial\Omega$. We need to pay a price on the part of the measure m (and thus the elliptic operators under consideration) which is far from the boundary. We usually do not mind to change the operator far from the boundary, because we can use a comparison principle. We take $m = \mathscr{L}^n$ on $B_0 \cap \Omega$ and $dm = \delta^{1-n}dx$ on $\mathbb{R}^n \setminus B_0$. The reader can then check that the triple (Ω, m, σ) satisfies the assumptions (H1)–(H6) in [DFM] and hence (Ω, m) is PDE friendly.

3.3 Theory of A_∞-weights.

In this section, we gather the properties of $\partial\Omega$. We do not really need to know that Ω is PDE friendly, because the incoming results hold on $\partial\Omega$ as a set, except of course when we ultimately apply the theory for elliptic measures in the proof of Theorem 3.1.21. The measure m will be mentioned through its appearance in the definition of the area integral $\mathscr{A}^r$ and hence in the definition of Carleson measure, but the reader can check that none of the properties of m matter.

As before, $\Delta(x,r)$ stands for the boundary ball $B(x,r)\cap\partial\Omega$. The results on this section will be stated in a "local" form, so that they can be applied when σ is either a single doubling Borel measures or an elliptic measure (i.e. a collection of measures).

Let $\Delta_0 = \Delta(x_0, r_0)$ be a boundary ball with $r_0 \in (0, \operatorname{diam}\partial\Omega)$. We say that σ is doubling in Δ_0 if

$$\sigma(\Delta(x,2r)\cap\Delta_0) \leq C_\sigma \sigma(\Delta(x,r)\cap\Delta_0) \qquad \text{for } x\in\partial\Omega,\ r>0,$$

and we say that σ is locally doubling if σ is doubling in all the boundary balls Δ (but the constant C_σ might depend on Δ). We say that $f \in KCM_{\Delta_0}(\sigma, M)$ if for any boundary ball $\Delta = \Delta(x,r)\subset\Delta_0$, we have that $\int_\Delta |\mathscr{A}^r(f)(y)|^2 d\sigma(y) \leq M\sigma(\Delta)$. At last, we say that σ_0 is A_∞-absolutely continuous with respect to σ_0 on Δ_0 - or $\sigma_1\in A_\infty(\sigma_0,\Delta_0)$ for short - if Definition 3.1.11 holds when we assume that $\Delta\subset\Delta_0$, that is, for any $\xi>0$, there exists $\zeta>0$ such that for any boundary ball $\Delta\subset\Delta_0$ and any Borel set $E\subset\Delta$, we have

$$\frac{\sigma_1(E)}{\sigma_1(\Delta)} < \zeta \quad \text{implies} \quad \frac{\sigma_0(E)}{\sigma_0(\Delta)} < \xi. \tag{3.3.1}$$

We begin the section with some preliminary work on the functional $\mathscr{A}^r$ introduced in (3.1.5). For $\alpha\geq 2$ and $x\in\partial\Omega$, define the cone with larger aperture

$$\gamma_\alpha^r(x) := \{X\in\Omega,\ |X-x|\leq\alpha\delta(X)\leq\alpha r\}$$

and corresponding area integral

$$\mathscr{A}_\alpha^r(f)(x) := \Big(\iint_{\gamma_\alpha^r(x)} |f(X)|^2 \frac{dm(X)}{m(B_X)} \Big)^{\frac{1}{2}}, \qquad f \in L^2_{\mathrm{loc}}(\Omega, m).$$

Our first result compares $\mathscr{A}_\alpha^r$ and $\mathscr{A}^r$, and is a classical consequence of Fubini's theorem.

Lemma 3.3.2 (Comparison of area integrals with different apertures). *Let $\alpha \geq 2$, $\Delta_0 := \Delta(x_0, r_0)$ be a boundary ball with $r_0 > 0$, and let σ be a doubling measure on $(2+\alpha)\Delta_0$. For any $\Delta_r \subset \Delta_0$ and any $f \in L^2_{\mathrm{loc}}(\Omega, m)$, we have that*

$$⨍_{\Delta_r} |\mathscr{A}_\alpha^r(f)|^2 \, d\sigma \leq C_\alpha ⨍_{(2+\alpha)\Delta_r} |\mathscr{A}^r(f)|^2 \, d\sigma, \tag{3.3.3}$$

where C_α depends only on α and C_σ. Thus, if $f \in KCM_{(2+\alpha)\Delta_0}(\sigma, M_f)$, then

$$M_{\alpha,f} := \sup_{\Delta_r \subset \Delta_0} \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} |\mathscr{A}_\alpha^r(f)|^2 \, d\sigma \leq C_\alpha M_f. \tag{3.3.4}$$

Proof. The bound (3.3.4) is a straightforward consequence of (3.3.3), which is the only inequality that we need to prove. Fix a surface ball $\Delta_r = \Delta(x, r) \subset \partial\Omega$, and write T_{Δ_r} for $\bigcup_{y \in \Delta_r} \gamma_\alpha^r(y)$. Fubini's lemma entails that

$$\begin{aligned} \int_{\Delta_r} |(\mathscr{A}_\alpha^r f)|^2 \, d\sigma &= \int_{\partial\Omega} \iint_{\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma_\alpha^r(y)}(Y) |f(Y)|^2 \, \frac{dm(Y)}{m(B_Y)} \, d\sigma(y) \\ &= \iint_{T_{\Delta_r}} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} \Big(\int_{\partial\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma_\alpha^r(y)}(Y) \, d\sigma(y) \Big). \end{aligned} \tag{3.3.5}$$

However, $Y \in \gamma_\alpha^r(y)$ if and only if $\delta(Y) \leq r$ and $y \in 4\alpha B_Y$. We deduce that

$$\int_{\partial\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma_\alpha^r(y)}(Y) \, d\sigma(y) = \sigma(\Delta_r \cap 4\alpha B_Y)$$

Let z be such that $|Y - z| = \delta(Y) \leq r$. Then the doubling property of σ yields that

$$\sigma(\Delta_r \cap 4\alpha B_Y) \leq \sigma(\Delta(z, 2\alpha\delta(Y))) \lesssim \sigma(\Delta(z, \delta(Y))) \lesssim \sigma(\partial\Omega \cap 8B_Y).$$

The last two computations give that

$$\int_{\partial\Omega} \mathbb{1}_{\Delta_r}(y)\mathbb{1}_{\gamma^r_\alpha(y)}(Y)\, d\sigma(y) \lesssim \sigma(\partial\Omega \cap 8B_Y) = \int_{\partial\Omega} \mathbb{1}_{\gamma^r(y)}(Y)\, d\sigma(y).$$

We reinject the last bound in 3.3.5 to get

$$\begin{aligned} \int_{\Delta_r} |(\mathscr{A}^r_\alpha f)|^2\, d\sigma &\lesssim \iint_{T_{\Delta_r}} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} \Big(\int_{\partial\Omega} \mathbb{1}_{\gamma^r(y)}(Y)\, d\sigma(y) \Big) \\ &= \int_{\partial\Omega} \iint_{T_{\Delta_r}} \mathbb{1}_{\gamma^r(y)}(Y) |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} d\sigma(y). \end{aligned} \tag{3.3.6}$$

Observe that for each $y \in \partial\Omega$, $T_{\Delta_r} \cap \gamma^r(y) \neq \varnothing$ precisely when we can find $Y \in \Omega$ and $z \in \Delta_r$ such that $Y \in \gamma^r_\alpha(z) \cap \gamma^r(y)$, and hence

$$|y - x| \leq |y - Y| + |Y - z| + |z - x| < \delta(Y) + \alpha\delta(Y) + r \leq (2 + \alpha)r.$$

Consequently, (3.3.6) becomes

$$\begin{aligned} \int_{\Delta_r} |(\mathscr{A}^r_\alpha f)|^2\, d\sigma &\lesssim \iint_{T_{\Delta_r}} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} \Big(\int_{\partial\Omega} \mathbb{1}_{\gamma^r(y)}(Y)\, d\sigma(y) \Big) \\ &= \int_{(2+\alpha)\Delta_r} \iint_{\gamma^r(y)} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} d\sigma(y) = \int_{(2+\alpha)\Delta_r} |\mathscr{A}^r(f)|^2 d\sigma. \end{aligned}$$

We conclude that

$$\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} |\mathscr{A}^r_\alpha(f)|^2\, d\sigma \leq C_\alpha \frac{1}{\sigma((2+\alpha)\Delta_r)} \int_{(2+\alpha)\Delta_r} |\mathscr{A}^r(f)|^2 d\sigma$$

by using the doubling property of σ again. The lemma follows. $\square$

We use the dyadic decomposition of $\partial\Omega$ by Christ, which is a consequence of the metric structure of $\partial\Omega$ induced by $\mathbb{R}^n$.

Lemma 3.3.7 (Dyadic cubes for a space of homogeneous type [Chr90])**.** *There exists a universal constant* a_0 *such that for each* $k \in \mathbb{Z}$*, there is a collection of sets* (the sets are called "dyadic cubes")

$$\mathbb{D}^k = \mathbb{D}^k(\partial\Omega) := \{Q^k_j \subset \partial\Omega \,:\, j \in \mathscr{J}^k\},$$

satisfying the following properties.

(i) $\partial\Omega = \bigcup_{j \in \mathscr{J}^k} Q_j^k$ *for each* $k \in \mathbb{Z}$*,*

(ii) If $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \varnothing$*.*

(iii) For each pair (j, k) *and each* $m < k$*, there is a unique* $i \in \mathscr{J}^m$ *such that* $Q_j^k \subset Q_i^m$*. When* $m = k - 1$*, we call* Q_i^m *the* dyadic parent *of* Q_j^k*, and we say that* Q_j^k *is a* dyadic child *of* Q_i^m*.*

(iv) diam $Q_j^k < 2^{-k}$*.*

(v) Each Q_j^k *contains some surface ball* $\Delta(x_j^k, a_0 2^{-k}) = B(x_j^k, a_0 2^{-k}) \cap \partial\Omega$*.*

Remark 3.3.8. The result of Christ assumes a doubling measure on $\partial\Omega$. However, we note that the collections $\mathbb{D}^k$ themselves are defined only through the quasi-metric structure of the space, and with no dependence on a doubling measure.

The underlying doubling measure in the result of Christ is used to prove an extra property that imposes a thin boundary (in a quantitative way) on dyadic cubes. Since we do not need thin boundaries for our proofs, and since the result is a bit technical, we avoided to write the full statement. But note that Christ proved, in particular, that for any locally doubling measure σ and any dyadic cube Q_j^k, we have that $\sigma(\partial Q_j^k) = 0$.

At last, Christ's result only provides the existence of a small $\tau > 0$ such that the properties (iv) and (v) holds for τ instead of the $1/2$ of our statement. However, it is easy to see that by repeating the collection $\mathbb{D}^k$ over several generations and by taking a smaller a_0, it is always possible to take $\tau = 1/2$.

We shall denote by $\mathbb{D} = \mathbb{D}(\partial\Omega)$ the collection of all relevant Q_j^k; that is,

$$\mathbb{D} = \mathbb{D}(\partial\Omega) := \bigcup_{k \in \mathbb{Z}} \mathbb{D}^k(\partial\Omega).$$

Henceforth, we refer to the elements of $\mathbb{D}$ as *dyadic cubes*, or *cubes*. For $Q \in \mathbb{D}$, we write

$$\mathbb{D}_Q := \{Q' \in \mathbb{D} \,:\, Q' \subseteq Q\} \text{ and } \mathbb{D}_Q^k = \mathbb{D}^k(\partial\Omega) \cap \mathbb{D}_Q.$$

Note carefully that if Q_i^{k+1} is the dyadic parent of Q_j^k, then it is possible that, *as sets*, $Q_i^{k+1} = Q_j^k$. In fact, some dyadic cubes may consist of single points (*atoms*), that is a dyadic cube can be equal (as sets) to all of its dyadic descendants. Even if there are no atoms, a dyadic cube could still equal (as sets) an arbitrarily large number of its descendants. Dyadic cubes which are of different generations but are equal as sets, will always be considered distinct. Hence, for $Q \in \mathbb{D}$, we write $\ell(Q) = 2^{-k}$ (the *length* of Q) for the *only* $k \in \mathbb{Z}$ such that $Q \in \mathbb{D}_k$.

Properties (iv) and (v) imply that for each $Q \in \mathbb{D}$, there is a point $x_Q \in \partial\Omega$ such that

$$\Delta(x_Q, a_0\ell(Q)) \subset Q \subseteq \Delta(x_Q, \ell(Q)). \tag{3.3.9}$$

We call x_Q the *center* of Q.

We redefine our notions using the dyadic cubes instead of the surface balls.

Definition 3.3.10 (Dyadically doubling measures)**.** We say that a Borel measure σ on $Q_0 \in \mathbb{D}$ is *dyadically doubling* in Q_0 if $0 < \sigma(Q) < \infty$ for every $Q \in \mathbb{D}_{Q_0}$ and there exists a constant $C \geq 1$ such that $\sigma(Q) \leq C\sigma(Q')$ for every $Q \in \mathbb{D}_{Q_0}$ and for every dyadic child Q' of Q.

We let the reader check that if σ is a doubling measure in Δ_0 and $Q_0 \subset \Delta_0$, then σ is dyadically doubling in Q_0.

Definition 3.3.11 (Dyadic A_∞ for families of measures)**.** Fix $Q_0 \in \mathbb{D}$. If σ_0 and σ_1 are two doubling measures on Q_0, then we say that $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q)$ if, for any $\xi > 0$, there exists $\zeta > 0$ such that any $Q \in \mathbb{D}_{Q_0}$, and any Borel set $E \subset Q$, we have that

$$\frac{\sigma_1(E)}{\sigma_1(Q)} < \zeta \text{ implies } \frac{\sigma_0(E)}{\sigma_0(Q)} < \xi.$$

We define the truncated area integrals adapted to a dyadic cube $Q \in \mathbb{D}$ as

$$\mathscr{A}^Q := \mathscr{A}^{\ell(Q)} \text{ and } \mathscr{A}_\alpha^Q := \mathscr{A}_\alpha^{\ell(Q)}.$$

Definition 3.3.12 (Dyadic Carleson measure condition)**.** Fix $Q_0 \in \mathbb{D}$. If σ is a doubling measure on Q_0, we say that a function $f \in L^2_{\text{loc}}(\Omega, m)$ satisfies the *dyadic σ-Carleson measure condition on* Q_0, written $f \in KCM_{Q_0}(\sigma)$, if there exists $M > 0$ such that

$$\int_Q |(\mathscr{A}^Q f)(y)|^2 \, d\sigma(y) \leq M\sigma(Q), \qquad \text{for each } Q \in \mathbb{D}_{Q_0}.$$

We write $f \in KCM_{Q_0}(\sigma, M)$ if we want to refer to the constant in the above inequality.

Due to (3.3.9), one can see that the dyadic versions of the doubling measure property, the A_∞ absolute continuity, and the Carleson measure condition are *a priori* a bit weaker than the general version on balls. However, we can recover the general statement on balls from the dyadic statement, and this is essentially because of the next lemma, which is a slightly refined variant of Lemma 19 in [Chr90].

Lemma 3.3.13 (Covering lemma for boundary balls [Chr90])**.** *Fix a boundary ball $\Delta := \Delta(x,r)$, an integer $k \in \mathbb{Z}$ such that $a_0 2^{-k} > r$, and let σ be a doubling measure in $\Delta(x, 2^{4-k})$. Then there exists $N \in \mathbb{N}$ (depending only on C_σ and not on x, r, k) such that there exist at most N cubes $Q_1^k, \dots, Q_{N_\Delta}^k$ of $\mathbb{D}^k$ that intersect Δ.*

Consequently, the property (i) of the dyadic decomposition entails that $\Delta \subset \bigcup_{j=1}^{N_\Delta} Q_j$.

Proof. Let $\Delta := \Delta(x,r)$ and k be as in the lemma, and let $\{Q_j^k\}_{j\in J}$ be the collection of dyadic cubes in $\mathbb{D}^k$ that intersect Δ. Since the number of dyadic cubes is countable, we can identify J to $\{0, \dots, N_\Delta - 1\}$ or $\mathbb{N}_0$. Due to (3.3.9), for each $j \in J$, the center x_j of Q_j^k necessarily satisfies $|x - x_j| \leq r + 2^{-k} \leq 2^{1-k}$, and hence $|x_j - x_0| \leq 2^{2-k}$. We deduce, again thanks to (3.3.9), that

$$\Delta(x_j, a_0 2^{-k}) \subset Q_j^k \subset \Delta(x_0, 2^{3-k}) \subset \Delta(x_j, 2^{3-k}) \qquad \text{for } j \in J.$$

The doubling property of σ entails that the smallest and the biggest sets in the inclusion above have similar measure, hence we also have that $C'_\sigma \sigma(Q_j^k) \geq \sigma(\Delta(x_0, 2^{3-k}))$ with C'_σ depending only on the doubling constant of σ on $\Delta(x, 2^{4-k})$. We conclude that

$$C'_\sigma \sigma(Q_j^k) \geq \sigma(\Delta(x_0, 2^{3-k})) \geq \sigma\Big(\bigcup_{j\in J} Q_j^k\Big) = \sum_{i\in J} \sigma(Q_i^k) \qquad \text{for } j \in J,$$

which means that the cardinality of J is finite and bounded by C'_σ, as desired. □

Let us state a local equivalence of the A_∞ conditions studied in this chapter.

Proposition 3.3.14 (Local interplay of A_∞ and A_∞^{dyadic})**.** *Let σ_0 and σ_1 be two locally doubling Borel measures on $\partial\Omega$. The following statements hold.*

(a) Fix $\Delta = \Delta(x,r) \subset \partial\Omega$ and $k \in \mathbb{Z}$ such that $a_0 2^{-k} > r$. If $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q_j^k)$ for each $Q_j^k \in \mathbb{D}^k$ that intersects Δ, then $\sigma_1 \in A_\infty(\sigma_0, \Delta)$.

(b) Fix $Q \in \mathbb{D}(\partial\Omega)$. If for some $r > a_0 \ell(Q)$ there exists a cover of Q by a family $\{\Delta_j\}_j$ of surface balls of radius r for which $\sigma_1 \in A_\infty(\sigma_0, \Delta_j)$ for each Δ_j, then

$\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q)$.

(c) If σ_0 and σ_1 are both doubling, $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0)$ if and only if $\sigma_1 \in A_\infty(\sigma_0)$.

Remark 3.3.15. In (a), the constants in $\sigma_1 \in A_\infty(\sigma_0, \Delta)$ depend only on the doubling constants of σ_0 and σ_1 in $\Delta(x, 2^{4-k})$, and the constants in $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q_j^k)$. Of course, a similar property holds for (b).

Proof. We prove (a); for the other statements we mention only that the proof of (b) is entirely analogous to that of (a), and (c) follows from (a), (b), and Remark 3.3.15.

Fix $\Delta := \Delta(x, r) \subset \partial\Omega$ and $k \in \mathbb{Z}$ such that $a_0 2^{-k} > r$. Let $\{Q_j\}_j \subset \mathbb{D}^k$ be the collection of cubes in $\mathbb{D}^k$ that intersect Δ. Now let $\Delta' = \Delta(x', r') \subset \Delta$ be a surface ball, fix $\xi > 0$, and let $k' \in \mathbb{Z}$ satisfy $r' < a_0 2^{-k'} \leq 2r'$. We take $\{Q_j'\}_{j\in J} \subset \mathbb{D}^{k'}$ to be the cover for Δ' afforded by Lemma 3.3.13, and since $k' \geq k$, it is easy to see that $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q_j')$.

Let $\zeta > 0$ be small to be chosen later, and suppose that $E \subset \Delta'$ is a Borel set that satisfies $\sigma_1(E) < \zeta\sigma_1(\Delta')$, and we want to prove that $\sigma_0(E) < C\xi\sigma_0(\Delta')$ for a constant $C > 0$ independent of Δ' and E. For each $j \in J$, we have $\Delta' \cap Q_j' \neq \varnothing$, and therefore

$$\Delta' \subset \Delta(x_{Q_j'}, 2^{-k'} + 2r') \subset \Delta(x_{Q_j'}, 4\ell(Q_j')).$$

Since σ_1 is locally doubling, then $\sigma_1(\Delta') \lesssim \sigma_1(\Delta(x_{Q_j'}, a_0\ell(Q_j'))) \leq \sigma_1(Q_j')$, and thus

$$\sigma_1(E \cap Q_j') \leq C\zeta\sigma_1(Q_j') \qquad \text{for each } j \in J,$$

where $C > 0$ depends only of the doubling constant of σ_1 in $\Delta(x, 2^{4-k})$. By the A_∞ property on Q_j', there exists ζ_j small enough (and independent of E and Δ') such that $\sigma_0(E \cap Q_j') < \xi\sigma_0(Q_j')$ whenever $\zeta \leq \zeta_j$. We take $\zeta = \min_j\{\zeta_j\}$, which is positive since the number of Q_j' is uniformly bounded, and we obtain

$$\sigma_0(E) = \textstyle\sum_j \sigma_0(E \cap Q_j') \leq \xi\sigma_0(\cup_j Q_j') \leq \xi\sigma_0(\Delta(x', r' + \ell(Q_j'))) \leq C\xi\sigma_0(\Delta'),$$

where we used the doubling property of σ_0 in $\Delta(x, 2^{4-k})$ and $\ell(Q_j') \lesssim r'$. $\square$

Lemma 3.3.16 (Dyadic cubes as a base for the Carleson measure test)**.** *Let $\alpha \geq 2$ and $Q_0 \in \mathbb{D}$, and let Δ_0 be a boundary ball that contains $\Delta(x_Q, \ell(Q))$ for every $Q \in \mathbb{D}_{Q_0}$. Take a doubling measure σ on $(2+\alpha)\Delta_0$, and suppose that $f \in KCM_{(2+\alpha)\Delta_0}(\sigma, M_f)$.*

Then
$$M_{\alpha,f}^{\text{dyadic}} := \sup_{Q\in\mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \int_Q |\mathscr{A}_\alpha^Q(f)|^2 \, d\sigma \le CM_f,$$
where $C > 0$ *depends only on the doubling constant of* σ.

Proof. We use (3.3.9) to change the integration on cubes to integration on balls, and then we conclude using Lemma 3.3.2. □

We focus now our efforts on the proof of Theorem 3.1.21. We first need a Calderón-Zygmund decomposition. Its proof is standard, and is left to the reader.

Lemma 3.3.17 (Calderón-Zygmund decomposition)**.** *Take* $Q_0 \in \mathbb{D}$ *and* σ *a dyadically doubling measure on* Q_0 *with doubling constant* C_σ. *For any function* $f \in L^1(Q_0, \sigma)$ *and any level* $\lambda > \frac{1}{\sigma(Q_0)} \int_{Q_0} |f| \, d\sigma$, *there exists a collection of maximal and therefore disjoint dyadic cubes* $\{Q_j\}_j \subset \mathbb{D}_{Q_0}$ *such that*
$$f(x) \le \lambda, \quad \textit{for } \sigma - \textit{a.e. } x \in Q_0 \setminus \bigcup_j Q_j,$$
$$\lambda < \frac{1}{\sigma(Q_j)} \int_{Q_j} f \, d\sigma \le C_\sigma \lambda.$$

Our next objective is a John-Nirenberg inequality for Carleson measures.

Lemma 3.3.18 (John-Nirenberg Lemma for Carleson measures)**.** *Let* $\Delta_0 \subset \partial\Omega$ *be a boundary ball, and let* σ *be a doubling measure on* $30\Delta_0$ *with doubling constant* C_σ. *Suppose that* $f \in KCM_{30\Delta_0}(\sigma, M_f)$. *Then for each boundary ball* $\Delta = \Delta(x,r) \subset \Delta_0$, *we have that*
$$\sigma\big(\{y \in \Delta : |(\mathscr{A}^r f)(y)|^2 > t\}\big) \le Ce^{-ct/M_f} \sigma(\Delta), \qquad \textit{for } t > 0, \tag{3.3.19}$$
where $c, C > 0$ *depend only on* C_σ.

As a consequence, for any $p \in (0, +\infty)$, *we have that*
$$\Big(\frac{1}{\sigma(\Delta)} \int_\Delta |\mathscr{A}^r(f)|^p \, d\sigma\Big)^{\frac{1}{p}} \le C_p (M_f)^{\frac{1}{2}}, \tag{3.3.20}$$
where C_p *depends only on* C_σ *and* p.

Proof. The second estimate (3.3.20) is a easy consequence of Hölder's inequality (when

$p < 2$) or (3.3.19) (when $p > 2$). So we only need to prove (3.3.19).

We take $f \in KCM_{30\Delta_0}(\sigma, M_f)$ and $\alpha := 4$. Fix $\Delta = \Delta(x,r) \subset \Delta_0$. Let $k \in \mathbb{N}$ such that $r < 2^{-k} \leq 2r$, and $\{R_j\}_{j\in J}$ be the collection of dyadic cubes in $\mathbb{D}^k$ that intersects Δ. Observe that for $j \in J$, the center x_j of R_j verifies $|x_j - x| \leq 2^{-k} + r \leq 3r$, that is

$$R_j \subset B(x_j, 2^{-k}) \subset 5\Delta \subset 5\Delta_0. \tag{3.3.21}$$

We can easily check that the above inclusions are also true for every descendant of the R_j's. So for any $R \in \bigcup_j \mathbb{D}_{R_j}$, we have $R \subset B(x_R, \ell(R)) \subset 5\Delta_0$. Lemma 3.3.16 entails that

$$M^{\text{dyadic}}_{\alpha,f} := \sup_{j\in J} \sup_{R\in\mathbb{D}_{R_j}} \frac{1}{\sigma(R)} \int_R |\mathscr{A}^R_\alpha(f)|^2 \, d\sigma \leq C' M_f < +\infty, \tag{3.3.22}$$

for a C depends only on C_σ (recall that $\alpha = 4$, so we have no dependence on α).

Fix now $t > 0$. By property (i), the $\{R_j\}_j$ covers Δ, and by (3.3.21), the Q_j's stay within 5Δ. Those two facts, combined with the fact that σ is doubling, entail that the desired estimate (3.3.19) is a consequence of

$$\sigma\big(\big\{y \in R_j : |(\mathscr{A}^{R_j} f)(y)|^2 > t\big\}\big) \leq Ce^{-ct/M_f}\sigma(R_j), \qquad \text{for } j \in J,$$

where $c, C > 0$ depends only on C_σ.

The index j does not matter anymore, so we drop it and we write Q_0 for any of the R_j. We also write M'_f for $M^{\text{dyadic}}_{\alpha,f}$ to lighten the notation. The problem is now purely dyadic. Since σ is doubling, σ is also dyadically doubling with a constant C'_σ that depends only on C_σ. By (3.3.22), we have that

$$\sup_{Q\in\mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \int_Q |\mathscr{A}^Q_\alpha|^2 \, d\sigma \leq M'_f, \tag{3.3.23}$$

and we want to prove that

$$\sigma\big(\big\{y \in Q_0 : |(\mathscr{A}^{Q_0} f)(y)|^2 > t\big\}\big) \leq Ce^{-ct/M'_f}\sigma(Q_0). \tag{3.3.24}$$

Note that the area integral has different aperture in (3.3.23) (big aperture) and (3.3.24) (small aperture), and it will become important later in the proof.

Perform the Calderón-Zygmund decomposition of the area integral with large aperture $|\mathscr{A}_\alpha^{Q_0}(f)|^2$ on Q_0, at height $2M'_f$. Since $2M'_f > \fint_{Q_0} |\mathscr{A}_\alpha^{Q_0}(f)|^2\, d\sigma$, according to Lemma 3.3.17 we may furnish a maximal family $\{Q_{1,j}\} \subset \mathbb{D}_{Q_0}$ for which we have

$$\begin{aligned} |(\mathscr{A}_\alpha^{Q_0}(f))(y)|^2 \leq 2M'_f \ \text{ for } \sigma - a.e.\ y \in Q_0 \backslash \cup_j Q_{1,j}, \\ 2M'_f < \frac{1}{\sigma(Q_{1,j})} \int_{Q_{1,j}} |\mathscr{A}_\alpha^{Q_0}(f)|^2\, d\sigma \leq 2C'_\sigma M'_f. \end{aligned} \tag{3.3.25}$$

Note that the last line above gives that

$$\sigma(\cup_j Q_{1,j}) < \frac{1}{2M'_f} \sum_j \int_{Q_{1,j}} |\mathscr{A}_\alpha^{Q_0}(f)|^2\, d\sigma \leq \frac{1}{2M'_f} \int_{Q_0} |\mathscr{A}_\alpha^{Q_0}(f)|^2\, d\sigma \leq \frac{1}{2}. \tag{3.3.26}$$

Let us study the difference of the area integral with small aperture on the cube $Q_{1,j}$.

$$|\mathscr{A}^{Q_0}(f)(y)|^2 - |\mathscr{A}^{Q_{1,j}}(f)(y)|^2 = \iint_{\gamma^{\ell(Q_0)}(y) \backslash \gamma^{\ell(Q_{1,j})}(y)} |f(Y)|^2\, \tfrac{dm(Y)}{m(B_Y)}, \quad y \in Q_{1,j}.$$

First, say that $Q'_{1,j} \in \mathbb{D}_Q$ is the dyadic parent of $Q_{1,j}$, and let us show that $\sigma(Q'_{1,j} \backslash \cup_k Q_{1,k}) \neq 0$. Indeed, otherwise there is a (possibly finite) subsequence Q_{1,k_m} such that $Q'_{1,j} = \cup_m Q_{1,k_m} \cup Z$, where $\sigma(Z) = 0$. By the dyadic nature

$$\begin{aligned} \frac{1}{\sigma(Q_{1,j})} \int_{Q'_j} |\mathscr{A}_\alpha^{Q}(f)|^2\, d\sigma &= \frac{1}{\sigma(Q'_{1,j})} \sum_m \int_{Q_{1,k_m}} |\mathscr{A}_\alpha^{Q}(f)|^2\, d\sigma \\ &> \frac{1}{\sigma(Q'_{1,j})} \sum_m 2M'_f \sigma(Q_{1,k_m}) = 2M'_f, \end{aligned}$$

but this is a contradiction to the maximality of the collection $\{Q_{1,j}\}$. The claim is established. Now let $y' \in Q'_j \backslash \cup_k Q_{1,k}$ be arbitrary, and observe that for all $y \in Q_j$,

$$\gamma^{Q_0}(y) \backslash \gamma^{Q_{1,j}}(y) \subset \gamma_\alpha^{Q_0}(y'),$$

where $\gamma_\alpha^{Q_0}$ is the wider cone and $\alpha = 4$. Indeed, if $Y \in \Omega$ belongs to the left-hand side above, then $\ell(Q_{1,j}) < \delta(Y) \leq \ell(Q_0)$ for free, and furthermore,

$$|Y - y'| \leq |Y - y| + |y - y'| \leq 2\delta(Y) + \ell(Q'_{1,j}) < 4\delta(Y).$$

We have thus deduced that

$$|(\mathscr{A}^{Q_0}f)(y)|^2-|(\mathscr{A}^{Q_{1,j}}f)(y)|^2 \leq |(\mathscr{A}_\alpha^{Q_0}f)(y')|^2 \text{ for each } y\in Q_{1,j} \text{ and any } y'\in Q'_{1,j}.$$

Since we may fix $y'\in Q'_{1,j}\setminus\cup_k Q_{1,k}$ such that (3.3.25) holds at y', we have that

$$|(\mathscr{A}^{Q_0}f)(y)|^2-|(\mathscr{A}^{Q_{1,j}}f)(y)|^2 \leq 2M'_f, \qquad \text{for each } y\in Q_{1,j}. \tag{3.3.27}$$

We repeat this process. For each $Q_{1,j}$, we apply the Calderón-Zygmund decomposition of $|\mathscr{A}_\alpha^{Q_{1,j}}(f)|^2$ on $Q_{1,j}$, at height $2M_\alpha$. Thus there exists a sequence of maximal cubes $\{Q_{2,j}\}$ in $\cup_j Q_{1,j}$ such that

$$\sigma(\cup_j Q_{2,j}) \leq \frac{1}{2M'_f}\sum_j \int_{Q_{1,j}} |\mathscr{A}_\alpha^{Q_{1,j}}(f)|^2\,d\sigma \leq \frac{1}{2M'_f}\sum_j M'_f\sigma(Q_{1,j}) < 2^{-2}\sigma(Q).$$

Moreover, on $Q_0\setminus\cup_j Q_{1,j}$, we have that $|(\mathscr{A}^{Q_0}f)(y)|^2 \leq |(\mathscr{A}_\alpha^{Q_0}f)(y)|^2 \leq 2M'_f$ for $\sigma-$a.e. y; while on $\cup_j Q_{1,j}\setminus\cup_i Q_{2,i}$, thanks to (3.3.27), for $\sigma-$a.e. y we have that

$$\begin{aligned}|(\mathscr{A}^{Q_0}f)(y)|^2 \leq |(\mathscr{A}^{Q_0}f)(y)|^2-|(\mathscr{A}^{Q_{1,j}}f)(y)|^2+|(\mathscr{A}_\alpha^{Q_{1,j}}f)(y)|^2&\\ \leq 2M'_f+2M'_f = 2(2M'_f).&\end{aligned}$$

Consequently, $|(\mathscr{A}^{Q_0}f)(y)|^2 \leq 2(2M'_f)$ $\sigma-$a.e. on $Q_0\setminus\cup_k Q_{2,k}$.

We may now iterate this process. As such, for each integer $k\in\mathbb{N}$, there exists a sequence of maximal cubes $\{Q_{k,j}\}$ such that $\sigma(\cup_k Q_{k,j}) \leq 2^{-k}\sigma(Q)$, and (via an easy telescoping argument)

$$|(\mathscr{A}^{Q_0}f)(y)|^2 \leq 2kM'_f, \qquad \text{for } \sigma-\text{a.e. } y\in Q\setminus\cup_k Q_{k,j}.$$

Therefore, we have shown that $\sigma(\{y\in Q_0 : |(\mathscr{A}^{Q_0}f)(y)|^2 > 2kM'_f\}) \leq 2^{-k}\sigma(Q_0)$ for each integer $k\geq 0$, whence (3.3.24) easily follows. $\square$

We recall here a classical characterization of A_∞ via reverse Hölder estimates.

Proposition 3.3.28 (RH characterization of A_∞ [GR85a]). *Let σ_0 and σ_1 be two doubling measures on a boundary ball $\Delta\subset\partial\Omega$. Then the following are equivalent:*

(i) $\sigma_1\in A_\infty(\sigma_0,\Delta)$,

(ii) $\sigma_0 \in A_\infty(\sigma_1, \Delta)$,

(iii) $\sigma_1 \ll \sigma_0$ *and the Radon-Nikodym derivative* $k := d\sigma_1/d\sigma_0$ *satisfies a reverse Hölder bound on* (Δ, σ_0)*. More precisely, there exists* $q > 1$ *and* $C > 0$ *such that*

$$\Big(\frac{1}{\sigma_0(\Delta')}\int_{\Delta'} k^q\,d\sigma_0\Big)^{\frac{1}{q}} \leq C\frac{1}{\sigma_0(\Delta')}\int_{\Delta'} k\,d\sigma_0 \text{ for any boundary ball } \Delta' \subset \Delta. \tag{3.3.29}$$

If k *satisfies (3.3.29), we say that* $k \in RH_q(\Delta, \sigma_0)$.

The only time when we need the powerful characterization of A_∞ given above is to prove the following transitivity of Carleson measures.

Lemma 3.3.30 (Local A_∞ implies the transference of the Carleson measure condition)**.** *Let* $\Delta \subset \partial\Omega$ *be a boundary ball, and let* σ_0, σ_1 *be two doubling measures on* 30Δ*. If* $\sigma_1 \in A_\infty(\sigma_0, \Delta)$*, then for each* $f \in L^2_{\mathrm{loc}}(\Omega, m)$*,*

$$\text{if} \qquad f \in KCM_{30\Delta}(\sigma_0, M_f), \qquad \text{then} \qquad f \in KCM_\Delta(\sigma_1, CM_f),$$

where $C > 0$ *depends only on the doubling constant of* σ_0 *and the constants* C, q *in the characterization of* $\sigma_1 \in A_\infty(\sigma_0, \Delta)$ *given in Proposition 3.3.28.*

Proof. Let Δ, σ_0, and σ_1 be as in the assumption of the lemma, fix $f \in KCM_{30\Delta}(\sigma_0, M_f)$ and $\Delta' \subset \Delta$. We want to prove that $\frac{1}{\sigma_1(\Delta')}\int_{\Delta'} |\mathscr{A}^r(f)|^2\,d\sigma_1 \leq CM_f$. Since $\sigma_1 \in A_\infty(\sigma_0, \Delta')$, writing $k = d\sigma_1/d\sigma_0$ and Hölder inequality gives that

$$\begin{aligned}
\fint_{\Delta'} |\mathscr{A}^r(f)|^2\,d\sigma_1 &= \frac{\sigma_0(\Delta')}{\sigma_1(\Delta')}\fint_{\Delta'} |\mathscr{A}^r(f)|^2\,k\,d\sigma_0 \\
&\leq \frac{\sigma_0(\Delta')}{\sigma_1(\Delta')}\Big(\fint_{\Delta'} k^q\,d\sigma_0\Big)^{\frac{1}{q}}\Big(\fint_{\Delta'} |\mathscr{A}^r(f)|^{2p}\,d\sigma_0\Big)^{\frac{1}{p}}
\end{aligned}$$

where $q > 1$ is the parameter given by Proposition 3.3.28 and $\frac{1}{p} + \frac{1}{q} = 1$. Using (3.3.29) and (3.3.20) allows us to deduce

$$\fint_{\Delta'} |\mathscr{A}^r(f)|^2\,d\sigma_1 \lesssim \frac{\sigma_0(\Delta')}{\sigma_1(\Delta')}\Big(\fint_{\Delta'} k\,d\sigma_0\Big)M_f = M_f.$$

The lemma follows. $\square$

Proof of Theorem 3.1.21. We shall only consider the case where $\mu_0 = \sigma_0$ is a doubling measure and $\mu_1 = \{\omega_1^X\}_{X\in\Omega}$ is an elliptic measure, and we shall only prove the implication

$$f \in KCM(\sigma_0) \implies f \in KCM(\omega_1), \qquad \text{for each } f \in L^2_{\mathrm{loc}}(\Omega, m).$$

All the other situations are analogous to this one with obvious modifications.

So take $f \in L^2_{\mathrm{loc}}(\Omega, m)$ that verifies $f \in KCM(\sigma_0, M_f)$. We will show that $f \in KCM(\omega_1, CM_f)$. Thus fix $x \in \partial\Omega$, $r \in (0, \operatorname{diam}\Omega)$, and let $Y \in \Omega$ be a Corkscrew point for $\Delta := \Delta(x, r)$. There exists $c_1 > 0$ such that $\delta(Y) \geq 60c_1 r$, so ω_1^Y is doubling on $c_1\Delta = \Delta(x, c_1 r)$ by (3.2.12). However, ω_1^Y is also doubling on 30Δ. Indeed, we can cover 30Δ by a uniformly finite number of small balls $\{\Delta_i = \Delta(x_i, r')\}$ of radius $r' = c_1 r/2$ by the Vitali covering lemma, then we pick corkscrew points Y_i associated to (x_i, r), and the same argument yields that ω^{Y_i} is doubling on $2\Delta_i$. The Harnack chain condition allows us to connect Y_i and Y by Harnack chains, and the Harnack inequality (Lemma 3.2.6) yields that ω_1^Y is doubling on each ball $2\Delta_i$ and then on Δ.

Of course, by assumption, we also have $f \in KCM_{30\Delta}(\omega_0^Y)$, and that $\omega_1^Y \in A_\infty(\omega_0^Y, \Delta)$, so by Lemma 3.3.30, we deduce that $f \in KCM_\Delta(\omega_1^Y, C'M_f)$, and C' is independent of Δ and Y.

We conclude by the change of pole property (3.2.13), which shows without difficulty that

$$f \in KCM_\Delta(\omega_1^Y, C'M_f) \implies f \in KCM_\Delta(\omega_1^X, C''M_f), \qquad \text{for } X \in \Omega \setminus B(x, 2r)$$

for a constant C'' independent of Δ and X. The theorem follows. □

3.4 Proof of Theorem 3.1.13

Our proof method is analogous to that of [DFM19a, Theorem 8.9]; see also [KKPT16] and [CHMT20]. In particular, we remark that our method of proof for Theorem 3.1.13 differs from that of [CHMT20, Theorem 1.1$(a) \implies (b)$] in that we do not (and cannot, because it is not true in our more general setting of PDE-friendly domains) use that every dyadic cube will have a proper descendant after a uniform number of dyadic generations, nor do we use (and cannot use) the largeness of the elliptic measure of the complement of a surface ball.

We also want to thank José-María Martell for pointing out to us that we do not need to assume in our proof that the elliptic measure is a probability measure, but only that the full measure of the elliptic measure is uniformly bounded from below by a constant $c_2 > 0$. We changed our proof to match this case.

We will prove the following local result, that implies Theorem 3.1.13.

Lemma 3.4.1 (Local $KCM \implies$ local A_∞, dyadic version)**.** *Let (Ω, m) be PDE friendly. Let $L = -\operatorname{div} A\nabla$ be an elliptic operator satisfying (3.1.1) and (3.1.2), and construct the elliptic measure $\omega := \{\omega^X\}_{X\in\Omega}$ as in (3.2.11).*

There exists $\alpha \geq 2$ that depends only on the constants in the Corkscrew point condition, the Harnack chain condition, and the Hölder continuity (3.2.14) such that the following holds. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. If there exists a constant $M > 0$ and a dyadically doubling measure σ on Q_0 such that, for any Borel $E \subset Q_0$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies

$$\sup_{Q\in\mathbb{D}_{Q_0}} \fint_Q |\mathscr{A}_\alpha^Q(\delta\nabla u_E)|^2 d\sigma \leq M, \tag{3.4.2}$$

then $\omega \in A_\infty^{\text{dyadic}}(\sigma, Q_0)$.

The lemma implies

Corollary 3.4.3 (Local $KCM \implies$ local A_∞)**.** *Let (Ω, m) be PDE friendly. Let L satisfy (3.1.1) and (3.1.2), and let $\omega := \{\omega^X\}_{X\in\Omega}$ be the associated elliptic measure.*

There exists $K > 0$ that depends only on the same parameters as α in Lemma 3.4.1 such that the following holds. Take Δ_0 to be a boundary ball. If for any Borel $E \subset \Delta_0$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies $\delta\nabla u_E \in KCM_{K\Delta_0}(\sigma, M)$ for a constant $M > 0$ and a doubling measure σ on $K\Delta_0$, then $\omega \in A_\infty(\sigma, \Delta_0)$.

Proof of Corollary 3.4.3 from Lemma 3.4.1. Let $\alpha \geq 2$ as in Lemma 3.4.1 and $K = 5(2+\alpha)$. We construct the collection $\{R_j\}_{j\in J}$ of dyadic cubes that covers Δ_0 as in the proof of Lemma 3.3.18, and the same reasoning as in the proof of Lemma 3.3.18 yields that

$$M_\alpha^{\text{dyadic}} := \sup_{E\subset\Delta_0} \sup_{j\in J} \sup_{R\in\mathbb{D}_{R_j}} \frac{1}{\sigma(R)} \int_R |\mathscr{A}_\alpha^R(\delta\nabla u_E)|^2 \, d\sigma \leq C'M < +\infty. \tag{3.4.4}$$

Lemma 3.4.1 gives then that $\omega \in A_\infty^{\text{dyadic}}(\sigma, R_j)$ for each $j \in J$ and Proposition 3.3.14 allows us to recover the non dyadic version $\omega \in A_\infty(\sigma, \Delta_0)$. $\square$

Proof of Theorem 3.1.13. If σ is a doubling measure on $\partial\Omega$, then Theorem 3.1.13 is a straightforward consequence of Corollary 3.4.3.

When σ is an elliptic measure, Theorem 3.1.13 is a consequence of Corollary 3.4.3, and the properties of the elliptic measure σ (doubling property (3.2.12), change of pole (3.2.13)). $\square$

The rest of the section is devoted to the proof of Lemma 3.4.1.

3.4.1 Step I: Construction of functions with large oscillations on small sets

The first order of business will be to construct the regions over which we will have large oscillations.

Definition 3.4.5 (Good ε_0 cover)**.** Fix $Q \in \mathbb{D}(\partial\Omega)$ and let ν be a regular Borel measure on Q. Given $\varepsilon_0 \in (0,1)$ and a Borel set $E \subset Q$, a *good* ε_0*−cover* of E with respect to ν, of length $k \in \mathbb{N}$, is a collection $\{\mathcal{O}_\ell\}_{\ell=1}^k$ of Borel subsets of Q, together with pairwise disjoint families $\mathcal{F}_\ell = \{S_i^\ell\} \subset \mathbb{D}_Q$, such that

(a) $E \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} \subset \cdots \subset \mathcal{O}_2 \subset \mathcal{O}_1 \subset \mathcal{O}_0 = Q$,

(b) $\mathcal{O}_\ell = \bigcup_i S_i^\ell, \qquad 0 \le \ell \le k$,

(c) $\nu(\mathcal{O}_\ell \cap S_i^{\ell-1}) \le \varepsilon_0 \nu(S_i^{\ell-1})$, for each $S_i^{\ell-1} \in \mathcal{F}_{\ell-1}$, $1 \le \ell \le k$.

(d) for each $S_i^{\ell-1} \in \mathcal{F}_{\ell-1}$, $1 \le \ell \le k$, the dyadic cube $S_i^{\ell-1}$ has at least two different children.

Remark 3.4.6. The *good* ε_0*−cover* has already been considered in multiple works, such as [KKPT16, DFM19a, CHMT20]. In all those works, the property (d) is not stated, but we can actually get this extra assumption for free, as explained in the following lines. First, we can always assume that S_i^ℓ intersects E, because otherwise we remove each S_i^ℓ that does not intersect E from the collections $\mathcal{F}_\ell$, and still get the same properties (a), (b), and (c). With this in hand, $\mathcal{O}_l \cap S_i^{\ell-1}$ will never be empty, and thus property (c) implies that $S_i^{\ell-1}$ cannot be an atom (that is, a set reduced to one point). At last, the cubes $\{S_i^\ell\}$ making up the good ε_0-cover are chosen *as sets*, meaning that the generation does not matter, and since $\{S_i^{\ell-1}\}$ are not atoms, we can always choose $S_i^{\ell-1}$ so that its child is not $S_i^{\ell-1}$, meaning that $S_i^{\ell-1}$ possesses at least two children.

As in [DFM19a], we write S_i^ℓ for the cubes making up $\mathcal{O}_\ell$ so as not to abuse the notation Q_i^ℓ, which is reserved for a dyadic cube of generation ℓ. Next, we have the fact that we may construct good ε_0-covers. Although the analogous statement in [CHMT20, Lemma 3.5] is formally only for the case of $n-$dimensional Ahlfors-David regular sets, a study of their proof reveals no dependence on the Ahlfors regularity *per se*, and their argument extends seamlessly to our setting. See also the remark that follows.

Lemma 3.4.7 (Existence of good ε_0-covers, [CHMT20, Lemma 3.5])**.** *Fix $Q \in \mathbb{D}(\partial\Omega)$. Let ν be a doubling measure on Q, with dyadic doubling constant C_ν^{dyadic}. For every $0 < \varepsilon_0 < e^{-1}$, if $E \subset Q$ is a Borel set with $\nu(E) \leq \zeta\nu(Q)$ and $0 < \zeta \leq \varepsilon_0^2/(\sqrt{2}C_\nu^{\text{dyadic}})^2$ then E has a good ε_0-cover with respect to ν of length $k_0 = k_0(\zeta, \varepsilon_0) \in \mathbb{N}$, $k_0 \geq 2$, which satisfies*

$$k_0 \gtrsim \frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})}.$$

In particular, if $\nu(E) = 0$, then E has a good ε_0-cover of arbitrary length.

Remark 3.4.8. The good ε_0-cover constructed in [CHMT20] does not specify the zeroth cover $\mathcal{O}_0$; however, it is an easy exercise to check that $\mathcal{O}_0 = Q$ with $\{S_i^0\} = \{Q\}$ can be appended to the cover $\{\mathcal{O}_\ell\}_{\ell=1}^k$ from [CHMT20, Lemma 3.5] to produce a good ε_0-cover in our sense of Definition 3.4.5.

We will eventually show that $\omega \in A_\infty^{\text{dyadic}}(\sigma, Q_0)$ (see Definition 3.3.11), but first we need to set the table. Fix $Q \in \mathbb{D}_{Q_0}$, let $X_0 \in \Omega \setminus B(x_{Q_0}, 2\ell(Q_0))$. Observe that ω^{X_0} is a regular Borel measure on $\partial\Omega$ which is dyadically doubling on Q_0 by (3.2.12). Henceforth we let $0 < \varepsilon_0 < e^{-1}$ and $0 < \zeta < \varepsilon_0^2/(2C_0^2)$ be sufficiently small to be chosen later, and we let $E \subset Q$ be a Borel set such that $\omega^{X_0}(E) \leq \zeta\omega^{X_0}(Q)$. We may apply Lemma 3.4.7 with $\nu = \omega^{X_0}$ to exhibit a good ε_0-cover for E of length $k \gtrsim \frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})}$ with $k \geq 2$. Thus let $\{\mathcal{O}_\ell\}_{\ell=0}^k$ and $\{S_i^\ell\}_{\mathcal{F}_\ell}$ be as described in Definition 3.4.5.

Owing to the property (d) of the ϵ_0-cover, for each S_i^ℓ, we let $\widehat{S}_i^\ell$ and $\widetilde{S}_i^\ell$ be two different children of S_i^ℓ. Following ideas of [KKPT16] and [DPP17], we set $\widehat{\mathcal{O}}_\ell := \bigcup_i \widehat{S}_i^\ell \subset \mathcal{O}_\ell$ for each $\ell = 0, \ldots, k$. Now, without loss of generality we may take k to be odd, and for each even ℓ with $0 \leq \ell \leq k-1$, we define

$$f_\ell := \mathbb{1}_{\widehat{\mathcal{O}}_\ell}, \qquad f_{\ell+1} := -f_\ell \mathbb{1}_{\mathcal{O}_{\ell+1}} = -\mathbb{1}_{\widehat{\mathcal{O}}_\ell \cap \mathcal{O}_{\ell+1}},$$

so that $f_\ell + f_{\ell+1} = \mathbb{1}_{\widehat{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1}}$ for ℓ even, and write

$$f := \sum_{\ell=0}^{k} f_\ell = \sum_{l=0}^{(k-1)/2} \mathbb{1}_{\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1}} = \mathbb{1}_{\bigcup_{l=0}^{(k-1)/2} \left(\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1}\right)}. \tag{3.4.9}$$

3.4.2 Step II: The solution with data f exhibits large oscillations on Whitney cubes

Let u solve $Lu = 0$ with data f on $\partial\Omega$, and according to (3.4.9), we have that $u(X) = \omega^X\big(\bigcup_{l=0}^{(k-1)/2}(\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1})\big)$. We shall present two balls, close to one another, over which u oscillates.

Take any $x \in E$, and $0 \leq \ell \leq k$, ℓ even. Let $S^\ell \in \{S_i^\ell\}$ be the unique cube that contains x, that possesses (at least) the two children $\widehat{S}^\ell$ and $\widetilde{S}^\ell$. We write r_ℓ for $\ell(S^\ell)$, we call $\widehat{x}_\ell$ and $\widetilde{x}_\ell$ the centers of $\widehat{S}^\ell$ and $\widetilde{S}^\ell$ respectively, and we set $\widehat{\Delta}_\ell := \Delta(\widehat{x}_\ell, a_0 r_\ell/2) \subset \widehat{S}^\ell$ and $\widetilde{\Delta}_\ell := \Delta(\widetilde{x}_\ell, a_0 r_\ell/2) \subset \widetilde{S}^\ell$.

By the Hölder continuity (3.2.14) of the elliptic measure at the boundary, we deduce that there exists $\rho > 0$ such that

$$\omega^X(\partial\Omega \setminus \widehat{\Delta}_\ell) \leq \frac{c_2}{8} \qquad \text{for } X \in B(\widehat{x}_\ell, \rho r_\ell) \cap \Omega, \tag{3.4.10}$$

where c_2 is the constant from the non-degeneracy of the elliptic measure (3.2.16)[4], and similarly

$$\omega^X(\partial\Omega \setminus \widetilde{\Delta}_\ell) \leq \frac{c_2}{8} \qquad \text{for } X \in B(\widetilde{x}_\ell, \rho r_\ell) \cap \Omega. \tag{3.4.11}$$

For the rest of the proof, $\widehat{X}_\ell$ and $\widetilde{X}_\ell$ are Corkscrew points associated to respectively $(\widehat{x}_\ell, \rho r_\ell)$ and $(\widetilde{x}_\ell, \rho r_\ell)$. That is, for a constant c that depends only on ρ, the constant c_1 in Definition 3.2.8, and c_2 from (3.2.16), we have

$$B(\widehat{X}_\ell, c r_\ell) \subset B(\widehat{x}_\ell, \rho r_\ell) \cap \Omega \text{ and } B(\widetilde{X}_\ell, c r_\ell) \subset B(\widetilde{x}_\ell, \rho r_\ell) \cap \Omega.$$

So if we set $\widehat{B}_\ell := B(\widehat{X}_\ell, c r_\ell/20)$ and $\widetilde{B}_\ell := B(\widetilde{X}_\ell, c r_\ell/20)$, the bounds (3.4.10),

[4] We use the estimate (3.2.16) to show that our argument is fundamentally local, and does not depend on the global properties of the elliptic measure; in particular, our argument does not directly use the fact that $\omega(\partial\Omega) = 1$.

(3.2.16), and (3.4.11) entail

$$\omega^X(\widehat{\Delta}_\ell) \geq \frac{7}{8}c_2 \;\text{ for } X \in \widehat{B}_\ell \qquad \text{and } \omega^X(\partial\Omega \setminus \widetilde{\Delta}_\ell) \leq \frac{c_2}{8} \;\text{ for } X \in \widetilde{B}_\ell. \tag{3.4.12}$$

We want to use the above bounds to estimate u on the balls $\widehat{B}_\ell$ and $\widetilde{B}_\ell$. For each $X \in \widehat{B}_\ell$, we have

$$u(X) \geq \omega^X\big(\widehat{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1}\big) \geq \omega^X\big(\widehat{\Delta}_\ell \backslash \mathcal{O}_{\ell+1}\big) = \omega^X(\widehat{\Delta}_\ell) - \omega^X(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1}). \tag{3.4.13}$$

and we want to show that the second term of the right-hand side above is small, smaller than $c_2/8$. Observe that

$$\omega^X(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1}) \lesssim \frac{\omega^{X_0}(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1})}{\omega^{X_0}(\widehat{\Delta}_\ell)} \leq \frac{\varepsilon_0 \omega^{X_0}(S^\ell)}{\omega^{X_0}(\widehat{\Delta}_\ell)} \lesssim \varepsilon_0 \frac{\omega^{X_0}(S^\ell)}{\omega^{X_0}(S^\ell)} = \varepsilon_0, \tag{3.4.14}$$

where we have used the change of pole (3.2.13), then property (c) of the good ϵ_0 cover, and at last the doubling property of ω^{X_0}. Therefore, there exists a constant M so that $\omega^{\widehat{X}_\ell}(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1}) \leq M\varepsilon_0$. If we ask that $\varepsilon_0 < c_2/(8M)$, then putting (3.4.13), (3.4.12), and (3.4.14) together we may conclude that

$$u(X) \geq \frac{3}{4}c_2, \qquad \text{for } X \in \widehat{B}_\ell. \tag{3.4.15}$$

Thus we have that u is large on a Whitney region associated to S^ℓ. Similarly, for $X \in \widetilde{B}_\ell$, we have

$$\begin{aligned}
&u(X) = \omega^X\big(\textstyle\bigcup_{l=0}^{(k-1)/2}(\widehat{\mathcal{O}}_{2l} \backslash \mathcal{O}_{2l+1})\big) \\
&\leq \omega^X\big(\partial\Omega \setminus \widetilde{\Delta}_\ell\big) + \sum_{l=0}^{(k-1)/2} \omega^X\big((\widehat{\mathcal{O}}_{2l} \backslash \mathcal{O}_{2l+1}) \cap \widetilde{\Delta}_\ell\big) \\
&\leq \omega^X\big(\partial\Omega \setminus \widetilde{\Delta}_\ell\big) + \omega^X\big(\widehat{\mathcal{O}}_\ell \cap \widetilde{\Delta}_\ell\big) + \sum_{2l+1<\ell} \omega^X\big(\widetilde{\Delta}_\ell \backslash \mathcal{O}_{2l+1}\big) + \sum_{2l>\ell} \omega^X\big(\widehat{\mathcal{O}}_{2l} \cap \widetilde{\Delta}_\ell\big).
\end{aligned}$$

By construction, $\widehat{\mathcal{O}}_\ell \cap \widetilde{\Delta}_\ell = \varnothing$. Notice also that $\widetilde{\Delta}_\ell \subset S^\ell \subset \mathcal{O}_{\ell-1}$, hence $\widetilde{\Delta}_\ell \backslash \mathcal{O}_{2l+1} = \varnothing$ when $2l+1 < \ell$. When $2l > \ell$, using the change of pole (3.2.13) and the property (c) of the good ϵ_0 cover like in (3.4.14), we obtain for $X \in \widetilde{B}_\ell$ that

$$\omega^X\big(\widehat{\mathcal{O}}_{2l} \cap \widetilde{\Delta}_\ell\big) \leq \omega^X\big(\widetilde{\Delta}_\ell \cap \mathcal{O}_{2l}\big) \lesssim \frac{\omega^{X_0}(\widetilde{\Delta}_\ell \cap \mathcal{O}_{2l})}{\omega^{X_0}(\widetilde{\Delta}_\ell)} \leq (\varepsilon_0)^{2l-\ell}\frac{\omega^{X_0}(S^\ell)}{\omega^{X_0}(\widehat{\Delta}_\ell)} \lesssim (\varepsilon_0)^{2l-\ell}.$$

Owing to (3.4.12) and the observations above, the bound on u when $X \in \widetilde{B}_\ell$ becomes $u(X) \leq \frac{c_2}{8} + M' \sum_{2l>\ell} (\varepsilon_0)^{2l-\ell}$ for some M' that is independent of all the important parameters. We choose ε_0 small enough so that $M' \sum_{2l>\ell}(\varepsilon_0)^{2l-\ell} < c_2/8$, and we conclude

$$u(X) \leq \frac{c_2}{4}, \qquad \text{for } X \in \widetilde{B}_\ell.$$

The last inequality together with (3.4.15) imply the desired large oscillation result. More precisely, if $B \subset \partial\Omega$ is a ball and we write $u_B := \frac{1}{m(B)} \iint_B u \, dm$, then we have that

$$|u_{\widehat{B}_\ell} - u_{\widetilde{B}_\ell}| \geq c_2/2. \tag{3.4.16}$$

3.4.3 Step III: Large oscillations on Whitney regions imply large square function

We now purport to pass from the large oscillation estimate (3.4.16) to a pointwise lower bound on the square function.

A Poincaré estimate

We ought to pass from the estimate on the difference over similarly sized balls to an estimate on the gradient, and this can be done via a delicate use of the Poincaré inequality. First of all, we recall that the radii of $\widehat{B}_\ell$ and $\widetilde{B}_\ell$ are equivalent to $r_\ell = \ell(S^\ell)$. Moreover, $\widehat{B}_\ell$, $\widetilde{B}_\ell$ are chosen so that both $20\widehat{B}_\ell$ and $20\widetilde{B}_\ell$ are subset of $\Omega \cap B(x_{S^\ell}, r_\ell)$. Therefore, we have that $\min\{\delta(\widehat{X}_\ell), \delta(\widetilde{X}_\ell)\} \geq r_\ell/M$ and $|\widehat{X}_\ell - \widetilde{X}_\ell| \leq 2r_\ell$. The Harnack chain condition from Definition 3.2.8 (and Remark 3.2.9) provides the existence of a Harnack Chain $\{B_j\}_{j=0}^N = \{B(X_j, \mathrm{rad}(B_j))\}_{j=0}^N$ of balls such that N is a uniformly bounded number (depending only on the allowable constants), $B_0 = \widehat{B}_\ell$, $B_N = \widetilde{B}_\ell$, $\delta(X_j) = 20\,\mathrm{rad}(B_j)$, and $B_j \cap B_{j+1} \neq \varnothing$ for each j (this last property can be ensured by adding in more balls of the same radius if necessary). Under this setup, (3.4.16) becomes

$$\frac{1}{2} \leq |u_{B_0} - u_{B_N}| \leq \sum_j |u_{B_j} - u_{B_{j+1}}| \lesssim \sum_j (|u_{B_j} - u_{3B_j}| + |u_{B_{j+1}} - u_{3B_j}|). \quad (3.4.17)$$

We now assume that $j = j(\ell)$ is the index at which the maximum in the right-hand side of (3.4.17) is taken. Since $B_j \cup B_{j+1} \subset 3B_j$, we may estimate

$$\begin{aligned} |u_{B_j} - u_{3B_j}| &= \fint\!\!\fint_{B_j} |u - u_{3B_j}|\, dm \leq \fint\!\!\fint_{3B_j} |u - u_{3B_j}|\, dm \\ &\lesssim \operatorname{rad}(3B_j)\Big(\fint\!\!\fint_{3B_j} |\nabla u(Y)|^2\, dm(Y)\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\iint_{3B_j} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)}\Big)^{\frac{1}{2}} \quad (3.4.18) \end{aligned}$$

where we have used the doubling property of m (3.2.2), the Poincaré inequality (3.2.3), and the fact that $\operatorname{rad}(3B_j) \approx \delta(X_j) \approx \delta(Y)$ for each $Y \in 3B_j$. A similar estimate holds for $|u_{B_{j+1}} - u_{3B_j}|$. The combination of (3.4.17) and (3.4.18) allows us to conclude

$$1 \lesssim \max_j \iint_{3B_j} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)}. \quad (3.4.19)$$

A strip decomposition of a wide cone

Recall that $x \in E$ and $S^\ell \in \mathcal{O}_\ell$ was chosen to contain x. The balls $\{B_{j(\ell)}\}$ are the Harnack chain between $\widehat{B}_\ell$ and $\widetilde{B}_\ell$ constructed in the beginning of the step. Let us show that there exist $K \geq 1$, $\alpha > 0$ and an even number $N_K \geq 2$ large enough so that for all even $\ell \geq N_K$,

$$3B_{j(\ell)} \subset \gamma_{\alpha,\ell}^{\ell(Q)} := \gamma_\alpha^{\ell(Q)}(x) \cap \big\{Y \in \Omega : \ell(S^\ell)/K \leq \delta(Y) \leq K\ell(S^\ell)\big\}. \quad (3.4.20)$$

Using the property (c) of the good ε_0-cover, and the fact that $S^\ell \cap S^m \supset \{x\}$ for each $0 \leq \ell \leq m$, it is easy to see that

$$\ell(S^m) \leq 2^{-(m-\ell)} \ell(S^\ell). \quad (3.4.21)$$

Now, by our constructions we have the chain

$$\delta(Y) \approx \operatorname{rad}(3B_j) \approx \delta(X_j) \approx \delta(\widetilde{X}_\ell) \approx r_\ell = \ell(S^\ell) \qquad \text{for } Y \in 3B_j, \tag{3.4.22}$$

and so in particular there exists $K \geq 1$ so that $r_\ell/K \leq \delta(Y) \leq K r_\ell$ for each $Y \in 3B_j$. We fix this K. Then, using (3.4.21), we have that $\delta(Y) \leq 2^{-\ell} K \ell(Q)$, and so we set N_K even and large enough such that $2^{-N_K} K \leq 1$. Hence for all even $\ell \geq N_K$, we have that $\delta(Y) \leq \ell(Q)$. It remains only to find α so that $|Y - x| \leq \alpha\delta(Y)$ for all $Y \in 3B_j$. However, for each $Y \in 3B_j$, armed with (3.4.22) we estimate

$$\begin{aligned} |Y - x| &\leq |Y - X_j| + |X_j - \widetilde{X}_\ell| + |\widetilde{X}_\ell - x_{\widetilde{S}^\ell}| + \operatorname{diam} S^\ell \\ &\lesssim \operatorname{rad}(3B_j) + \delta(\widetilde{X}_\ell) + \ell(\widetilde{S}^\ell) + \ell(S^\ell) \lesssim \delta(Y). \end{aligned}$$

In summary, $|Y - x| \leq \alpha\delta(Y)$ for some large α, as desired. With our choices of K, N_K, and α, (3.4.20) is proven for all even $\ell \geq N_K$. The combination of (3.4.20) and (3.4.19) yields that

$$1 \lesssim \iint_{\gamma^Q_{\alpha,\ell}} \delta(Y)^2 |\nabla u(Y)|^2 \, \frac{dm(Y)}{m(B_Y)}. \tag{3.4.23}$$

Conclusion of Step III

We are ready to estimate the square function. First, since $k \approx \frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})} \to \infty$ as $\zeta \to 0$, we consider only ζ small enough so that $k \geq 4N_K$. Owing to (3.4.21), the strips $\gamma^{\ell(Q)}_{\alpha,\ell}$ have uniformly bounded overlap. Reckon the bounds

$$\begin{aligned} \left|\mathscr{A}^Q_\alpha(\delta\nabla u)(x)\right|^2 &= \iint_{\gamma^{\ell(Q)}_\alpha} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)} \\ &\gtrsim \sum_{\ell = N_K,\, \ell \text{ even}}^{k} \iint_{\gamma^{\ell(Q)}_{\alpha,\ell}} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)} \gtrsim \sum_{\ell = N_K,\, \ell \text{ even}}^{k} 1 \gtrsim \frac{k - N_K}{2} \approx k, \end{aligned} \tag{3.4.24}$$

where in the second line we used the bounded overlap of the strips, the bound (3.4.23), and the fact that $k \gg N_K$.

3.4.4 Step IV: From large square function to A_∞

Integrate (3.4.24) over $x \in E$ with respect to σ to see that

$$\frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})}\sigma(E) \lesssim k\sigma(E) \lesssim \int_E |\mathscr{A}_\alpha^Q(\delta\nabla u)|^2\, d\sigma \le \int_Q |\mathscr{A}_\alpha^Q(\delta\nabla u)|^2\, d\sigma \lesssim_\beta M\sigma(Q),$$

where the last line is a consequence of the assumption (3.4.2). We deduce

$$\frac{\sigma(E)}{\sigma(Q)} \le C\frac{\log(\varepsilon_0^{-1})}{\log(\zeta^{-1})}, \qquad \text{for all Borel } E \subset Q \text{ with } \omega^{X_0}(E) \le \zeta\omega^{X_0}(Q). \tag{3.4.25}$$

Given $\xi > 0$ and $E \subset Q \in \mathbb{D}_{Q_0}$ such that $\omega^{X_0}(E) \le \zeta\omega^{X_0}(Q)$, we want to conclude that $\sigma(E) \le \xi\sigma(Q)$. It is clear that for $\zeta = \zeta(\xi)$ small enough, we achieve the desired result through the estimate (3.4.25). We have established that $\omega \in A_\infty^{\text{dyadic}}(\sigma, Q_0)$, as desired.□

3.5 Proof of Theorem 3.1.19

Lemma 3.5.1. *Let (Ω, m, μ) be PDE friendly. Let $L_0 = -\operatorname{div} w\mathcal{A}_0\nabla$ and $L_1 = -\operatorname{div} w\mathcal{A}_1\nabla$ be two elliptic operators satisfying (3.1.3) and (3.1.4), and construct the elliptic measure $\omega_0 := \{\omega_0^X\}_{X\in\Omega}$ and $\omega_1 := \{\omega_1^X\}_{X\in\Omega}$ as in (3.2.11).*

Assume that the weak solutions to $L_1 u = 0$ are the same as the ones of $\widehat{L}_1 = -\operatorname{div} w\widehat{\mathcal{A}}_1\nabla + w\widehat{\mathcal{B}}_1 \cdot \nabla$, and that $\widehat{\mathcal{A}}_1$ still satisfies (3.1.3)–(3.1.4). In addition, we require the existence of K such that $\mathcal{A}_0$, $\widehat{\mathcal{A}}_1$, and $\widehat{\mathcal{B}}_1$ satisfy

$$|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\sup}(\omega_0, K) \quad \text{and} \quad \delta|\widehat{\mathcal{B}}_1| \in KCM(\omega_0, K).$$

Then for any $x \in \partial\Omega$, any $r > 0$, any $X \in \Omega \setminus B(x, 1000r)$, and any weak solution u to $L_1 u = 0$, we have that

$$\int_{\Delta(x,r)} |\mathscr{A}^r(\delta\nabla u)|^2 d\omega_0^X \le C(1+K)\int_{\Delta(x,25r)} |N^{2r}(u)|^2 d\omega_0^X, \tag{3.5.2}$$

where the constants depends only on n, the elliptic constants of $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$, and the constants in (3.2.2), (3.2.7), (3.2.12), and (3.2.15).

Remark 3.5.3. The above lemma looks a bit technical, with the introduction of $\widehat{L}_1$. The key observation is that the cases in Theorem 3.1.19 (multiplicative Carleson perturbation and antisymmetric Carleson perturbation) can be reduced to drift perturbations via the identities (3.1.17)–(3.1.18), see the proof of Theorem 3.1.19 below.

Actually, Lemma 3.5.1 could be stated without any mention of L_1, because the constants in (3.5.2) depends on the properties of L_0 and $\widehat{L}_1$, and so only the latter operators matter. The only problem lies in the construction of the elliptic measure associated to the $\widehat{L}_1$. In Lemma 3.5.1, since $\widehat{L}_1$ has the same solutions as L_1, the elliptic measure associated to $\widehat{L}_1$ is the same as L_1, hence exists and has the desired properties.

If we had a definition and good properties (the ones presented in Section 3.2) of elliptic measure for (a class of) operators with drifts, for instance by deepening the theory in [DHM18], then we would not really need L_1. We could only consider two operators with drifts $\widehat{L}_i = -\operatorname{div} w\widehat{\mathcal{A}}_i\nabla + w\widehat{\mathcal{B}}_i \cdot \nabla$, $i \in \{0,1\}$, and their elliptic measures ω_i. And as long as $|\widehat{\mathcal{A}}_1 - \widehat{\mathcal{A}}_0| \in KCM_{\sup}(\omega_0)$ and $|\widehat{\mathcal{B}}_1 - \widehat{\mathcal{B}}_0| \in KCM(\omega_0)$, we would have $\omega_1 \in A_\infty(\omega_0)$.

Proof of Theorem 3.1.19. Since L_1 is a (generalized) Carleson perturbation of L_0, there exists a function b, a matrix $\mathcal{C}$, and an antisymmetric matrix $\mathcal{T}$ such that

$$|\mathcal{C}| \in KCM_{\sup}(\omega_0, K) \quad \text{and} \quad \frac{\delta|\nabla b|}{b} + \delta w^{-1}|\operatorname{div}(w\mathcal{T})| \in KCM(\omega_0, K) \tag{3.5.4}$$

for some $K > 0$, and

$$\mathcal{A}_1 = b(\mathcal{A}_0 + \mathcal{C} + \mathcal{T}).$$

We define

$$\begin{aligned}\widehat{L}_1 &:= -\operatorname{div}(w[\widehat{\mathcal{A}}_0 + \mathcal{C}]\nabla) - \Big[\operatorname{div}(w\mathcal{T}) + w\frac{\nabla b}{b}\Big] \cdot \nabla \\ &:= -\operatorname{div}(w\widehat{\mathcal{A}}_1\nabla) - w\widehat{\mathcal{B}}_1 \cdot \nabla.\end{aligned}$$

The identities (3.1.17)–(3.1.18) infer that the weak solutions of L_1 and $\widehat{L}_1$ are the same. Moreover, (3.5.4) implies that

$$|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\sup}(\omega_0, K) \quad \text{and} \quad \delta|\widehat{\mathcal{B}}_1| \in KCM(\omega_0, K).$$

So we can apply Lemma 3.5.1 to deduce the bound (3.5.2). We construct a finitely overlapping covering of $\Delta(x, r)$ by small boundary balls $\{\Delta(x_i, r')\}$ of radius $r' =$

$c_1r/10^6$, where c_1 in the constant in the Corkscrew point condition, so that our Corkscrew point X associated to (x,r) stays outside of every $B(x_i, 1000r')$. Then, by applying (3.5.2) to every small boundary ball $\Delta(x_i, r')$, we deduce that

$$\int_{\Delta(x,r)} |\mathscr{A}^{r'}(\delta\nabla u)|^2 d\omega_0^X \leq C \int_{\Delta(x,2r)} |N^{2r}(u)|^2 d\omega_0^X.$$

In order to change $\mathscr{A}^{r'}$ to $\mathscr{A}^r$ in the above estimate, and hence obtain (3.1.20), we need to bound the difference $T := |\mathscr{A}^r(\delta\nabla u)|^2 - |\mathscr{A}^{r'}(\delta\nabla u)|^2$. We have

$$T(y) = \iint_{W(y,r)} |\delta\nabla u|^2 \frac{dm}{m(B_Y)},$$

where $W(y,r) := \{Y \in \Omega,\, |Y-y| \leq 2\delta(X) \leq 2r,\, r' < \delta(Y)\}$. Notice that all the points $Y \in W(y,r)$ are Corkscrew points associated to (y,r). Therefore, for $Y \in W(y,r)$, we have $\delta(Y) \approx r$ and the doubling property of m infers that

$$m(B_Y) \approx m(B(y,r) \cap \Omega) \approx m(W(y,r)) \qquad \text{for } Y \in W(y,r).$$

We conclude that

$$T(y) \approx \frac{r^2}{m(W(y,r))} \iint_{W(y,r)} |\nabla u|^2\, dm.$$

Owning to Caccioppoli's inequality (see for instance Lemma 11.12 in [DFM]), one has that $T(y) \lesssim \frac{1}{m(W^*(y,r))} \iint_{W^*(y,r)} |u|^2 dm$, where $W^*(y,r)$ is a region slightly fatter than $W(y,r)$. From there, it is fairly easy to check that

$$\int_{\Delta(x,r)} T(y) d\omega_0^X(y) \lesssim \int_{\Delta(x,r)} \sup_{W^*(y,r)} |u|^2 d\omega_0^X(y) \lesssim \int_{\Delta(x,2r)} |N^{2r}(u)|^2 d\omega_0^X.$$

The theorem follows. $\square$

The rest of the section is devoted to the proof of Lemma 3.5.1.

3.5.1 Step 0: Carleson estimate

We shall need some preliminary results about the non-tangential maximal function N. Note that if one is not interested in the $S < N$ local L^2-estimate but only in establishing (3.1.14), then we could avoid these preliminary estimates and greatly simplify Step 5.

But we believe that the $S < N$ estimate is important on its own, and we decided to prove it.

We shall need the untruncated versions of $\mathscr{A}$ and N. We construct the infinite cone $\gamma_\alpha(x) = \{X \in \Omega,\ |X - x| \leq \alpha\delta(X)\}$, and we write $\gamma(x)$ for $\gamma_2(x)$. Then we define, for $f \in L^2_{\text{loc}}(\Omega, m)$ and $x \in \partial\Omega$,

$$\mathscr{A}(f)(x) := \Big(\iint_{\gamma(x)} |f(x)|^2 \frac{dm(X)}{m(B_X)} \Big)^{\frac{1}{2}} \qquad \text{and} \qquad N(f)(x) := \sup_{\gamma(x)} |f|.$$

We shall also need the variants

$$\widetilde{N}(f)(x) := \sup_{X \in \gamma(x)} \Big(⨌_{B_X} |f|^2 \, dm \Big)^{\frac{1}{2}} \qquad \text{and} \qquad N_{10}(f)(x) := \sup_{\gamma_{10}(x)} |f|.$$

Observe that $\widetilde{N}(f) \leq N_{10}(f)$, and if we take $2B_X$ instead of B_X in the definition of $\widetilde{N}$, the result would still hold. We also have

$$\|N_{10}(f)\|_{L_2(\sigma)} \leq \|N(f)\|_{L^2(\sigma)} \tag{3.5.5}$$

whenever σ is doubling on the support of $N_{10}(f)$. The L^1 nonlocal result in $\mathbb{R}^n$ can be found in Chapter II, § 2.5.1 from [Ste93a], but the proof goes through in our setting without difficulty. The area integral $\mathscr{A}$, the non-tangential maximal function N, and the Carleson measure condition are nicely related via the Carleson inequality. Indeed, if $v \in L^2_{\text{loc}}(\Omega, m)$, $f \in KCM(\sigma, M_f)$ and σ is doubling on a neighborhood of the support of $N(v)$, then

$$\int_{\partial\Omega} |\mathscr{A}(fv)|^2 d\sigma \leq C M_f \int_{\partial\Omega} |N(v)|^2 d\sigma, \tag{3.5.6}$$

where C depends only on the doubling constant of σ. If $f \in KCM_{\sup}(\sigma, M_f)$ instead, we can use (3.5.6) to $\widetilde{f}(X) = \sup_{B_X} f$ and $\widetilde{v} = (⨍_{B_X} |v^2| dm)^{1/2}$ and obtain the variant

$$\int_{\partial\Omega} |\mathscr{A}(fv)|^2 d\sigma \leq C M_f \int_{\partial\Omega} |\widetilde{N}(v)|^2 d\sigma. \tag{3.5.7}$$

The proof of (3.5.6) is classical, see for instance [Ste93a, Section II.2.2, Theorem 2] for the proof on the upper half plane, but which can easily adapted to our setting.

We fix now once for all the rest of this section $x \in \partial\Omega$ and $r > 0$.

3.5.2 Step 1: Construction of the cut-off function Ψ.

We choose then a function $\psi \in C_c^\infty(\mathbb{R})$ that satisfies $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $(-1,1)$, $\psi \equiv 0$ outside $(-2,2)$, and $|\psi'| \leq 2$. We construct $\Psi = \Psi_{x,r}$ on Ω as

$$\Psi(Y) = \psi\Big(\frac{\operatorname{dist}(Y,\Delta(x,r))}{4\delta(Y)}\Big)\psi\Big(\frac{\delta(Y)}{r}\Big)$$

and then

$$\Psi_\epsilon(Y) = \Psi(Y)\psi\Big(\frac{\epsilon}{\delta(Y)}\Big).$$

Observe that for any $y \in \Delta(x,r)$ and any $Y \in \gamma^r(y)$, we have $\Psi(Y) = 1$. That is, for any $X \in \Omega$, we have

$$\int_{\Delta(x,r)} |\mathscr{A}^r(\delta\nabla u)|^2 d\omega_0^X \leq \int_{\partial\Omega} |\mathscr{A}(\Psi^2\delta\nabla u)|^2 d\omega_0^X = \lim_{\epsilon\to 0}\int_{\partial\Omega} |\mathscr{A}(\Psi_\epsilon^2\delta\nabla u)|^2 d\omega_0^X.$$

Remark also that $\Psi(Y) \neq 0$ means that $\operatorname{dist}(Y,\Delta(x,r)) \leq 8\delta(Y) \leq 16r$, so if $y \in \partial\Omega$ is such that $Y \in \gamma(y)$ and $\Psi(Y) \neq 0$, we necessary have $|y-x| < 21r$. We conclude that

$$\sup_{\epsilon\to 0}\int_{\partial\Omega} |N(\Psi_\epsilon u)|^2 d\omega_0^X = \int_{\partial\Omega} |N(\Psi u)|^2 d\omega_0^X \leq \int_{\Delta(x,25r)} |N^{2r}(u)|^2 d\omega_0^X.$$

As a consequence, (3.5.2) will be proved once we establish that, for $\epsilon > 0$, we have

$$\int_{\partial\Omega} |\mathscr{A}(\Psi_\epsilon^2\delta\nabla u)|^2 d\omega_0^X \lesssim \int_{\partial\Omega} |N(\Psi_\epsilon u)|^2 d\omega_0^X. \tag{3.5.8}$$

3.5.3 Step 2: Properties of Ψ_ϵ.

In this step we show that $|\nabla\Psi_\epsilon| \in KCM_{\sup}(\omega_0)$. Notice that

$$|\nabla\Psi_\epsilon(Y)| \lesssim \frac{1}{\delta(Y)}\mathbb{1}_{E_1\cup E_2\cup E_3} \qquad \text{for } Y \in \Omega, \tag{3.5.9}$$

where

$$E_1 := \{Y \in \Omega, \operatorname{dist}(Y,\Delta(x,r))/8 \leq \delta(Y) \leq \operatorname{dist}(Y,\Delta)/4\},$$

$$E_2 := \{Y \in \Omega,\ r \leq \delta(Y) \leq 2r\}, \qquad \text{and} \qquad E_3 := \{Y \in \Omega,\ \epsilon/2 \leq \delta(Y) \leq \epsilon\}.$$

In addition, if $y \in \partial\Omega$, $Y \in \gamma(y)$, $Y' \in B_Y$, and $Y' \in E_1$, then $3\delta(Y)/4 \leq \delta(Y') \leq 5\delta(Y)/4$,

$$\begin{aligned}\operatorname{dist}(y, \Delta(x,r)) &\geq \operatorname{dist}(Y', \Delta(x,r)) - |Y' - Y| - |Y - y| \\ &\geq 4\delta(Y') - \frac{1}{4}\delta(Y) - 2\delta(Y) \geq \frac{3}{4}\delta(Y),\end{aligned}$$

and

$$\operatorname{dist}(y, \Delta(x,r)) \leq \operatorname{dist}(Y', \Delta(x,r)) + |Y' - Y| + |Y - y| \leq 13\delta(Y);$$

that is, for $Y \in \gamma(y)$ such that $B_Y \cap E_1 \neq \varnothing$,

$$\tfrac{1}{13}\operatorname{dist}(y, \Delta(x,r)) \leq \delta(Y) \leq \tfrac{4}{3}\operatorname{dist}(y, \Delta(x,r)). \tag{3.5.10}$$

We write $\widetilde{\mathbb{1}_{E_1}}$ for the function $Y \to \sup_{B_Y} \mathbb{1}_{E_1}$, the above estimates proves that $\delta(Y) \approx r_y := \operatorname{dist}(y, \Delta(x,r))$ whenever $Y \in \gamma(y) \cap \operatorname{supp} \widetilde{\mathbb{1}_{E_1}}$. As a consequence, for $y \in \partial\Omega$ and $s > 0$, we have that

$$|\mathscr{A}^s(\widetilde{\mathbb{1}_{E_1}})(y)|^2 \lesssim \iint_{Y \in \gamma(y),\, \delta(Y) \approx r_y} \frac{dm(Y)}{m(B_Y)} \lesssim 1$$

because (3.2.2) implies, for all $Y \in S_y := \{Y \in \gamma(y),\, \delta(Y) \approx r_y\}$, that $m(B_Y) \approx m(S_y) \approx m(B(y, r_y) \cap \Omega)$. The measure ω_0 does not matter to be able to conclude that $\mathbb{1}_{E_1} \in KCM_{\sup}(\omega_0, M)$, where M depends only on n and the constant in (3.2.2).

For $y \in \partial\Omega$, $Y \in \gamma(y)$, $B_Y \cap (E_2 \cup E_3) \neq \varnothing$, we easily deduce from the definition of E_2 and E_3 that $\delta(Y) \approx r$ or $\delta(Y) \approx \epsilon$. Those estimates are the analogue for E_2 and E_3 of the bounds (3.5.10). With the same arguments as the one used for E_1, we obtain that $\mathbb{1}_{E_2 \cup E_3} \in KCM_{\sup}(\omega_0, M)$, hence

$$\mathbb{1}_{E_1 \cup E_2 \cup E_3} \in KCM_{\sup}(\omega_0, M). \tag{3.5.11}$$

We combine (3.5.11) with (3.5.9) to conclude that

$$|\delta \nabla \Psi_\epsilon|^{1/2} + |\delta \nabla \Psi_\epsilon| \in KCM_{\sup}(\omega_0, M) \tag{3.5.12}$$

with a constant M that depends only on n and the constant in (3.2.2), as desired. Of course, we also have the weaker version

$$|\delta\nabla\Psi_\epsilon|^{1/2} + |\delta\nabla\Psi_\epsilon| \in KCM(\omega_0, M). \tag{3.5.13}$$

3.5.4 Step 3: Introduction of the Green function.

The pole X of the elliptic measure ω_0 is chosen in $\Omega\setminus B(x,1000r)$ as in the assumption of the lemma. As an intermediate tool, we shall call G^*_X the weak solution to $(L_0)^*u = 0$ in $B(x,500r)\cap\Omega$ that satisfies (3.2.15). More precisely, we have $\iint_\Omega \mathcal{A}_0\nabla\varphi\cdot\nabla G^*_X\, dm = 0$ for each $\varphi \in C^\infty_c(B(x,500r)\cap\Omega)$, and for $y \in \Delta(x,25r)$, $s\in(0,2r)$, and any Corkscrew point Y associated to (y,s), the bounds (3.2.15) show that

$$C^{-1}\omega_0^X(\Delta(y,s)) \leq \frac{m(B(y,s)\cap\Omega)}{s^2}G^*_X(Y) \leq C\omega_0^X(\Delta(y,s)). \tag{3.5.14}$$

The Green function will be used to replace the expression with the functional $\mathscr{A}$ by some integrals over Ω. We claim that, for any $v \in L^2_{\text{loc}}(\Omega)$, we have

$$\int_{\partial\Omega} |\mathscr{A}(\Psi v)|^2 d\omega_0^X \approx \iint_\Omega \Psi^2 v^2 \frac{G^*_X}{\delta^2}\, dm. \tag{3.5.15}$$

Observe that $Y \in \gamma(y)$ implies that $y \in 8\overline{B_Y}\cap\partial\Omega$. As a consequence, Fubini's lemma entails that

$$\int_{\partial\Omega} |\mathscr{A}(\Psi v)|^2 d\omega_0^X \approx \iint_\Omega \Psi^2(Y)v^2(Y)\frac{1}{m(B_Y)}\omega_0^X(8B_Y\cap\partial\Omega)dm(Y).$$

Take Y to be such that $\Psi(Y)\neq 0$, and then take $y\in\partial\Omega$ and $s>0$ be such that $s = |y-Y| = \delta(Y)$. The study in Step 1 showed that $y \in \Delta(x,25)$ and $s<2r$, so in particular $X \in \Omega\setminus B(y,2s)$. The doubling property of ω_0^X (3.2.12) shows that $\omega_0^X(8B_Y\cap\partial\Omega) \approx \omega(\Delta(y,s))$, and the doubling property of m, given by (3.2.2), entails that $m(B_Y)\approx m(B(y,s)\cap\Omega)$. Combined with (3.5.14),

$$\frac{1}{m(B_Y)}\omega_0^X(8B_Y\cap\partial\Omega) \approx \frac{G^*_X(Y)}{\delta(Y)^2}.$$

The claim (3.5.15) follows.

3.5.5 Step 4: Bound on the square function.

As explained in Step 1, we need to prove (3.5.8) for any $\epsilon > 0$. We define

$$I = I_\epsilon := \int_{\partial\Omega} |\mathscr{A}(\Psi_\epsilon^2 \delta \nabla u)|^2 d\omega_0^X,$$

which is the quantity that we want to bound. We also set

$$J = J_\epsilon := \int_{\partial\Omega} \Big| \widetilde{N}\Big(u\Psi_\epsilon^2 \frac{\delta \nabla G_X^*}{G_X^*}\Big)\Big|^2 d\omega_0^X + \int_{\partial\Omega} |N(u\Psi_\epsilon)|^2 d\omega_0^X.$$

If K is the constant in Theorem 3.1.19, we claim that,

$$I \lesssim (1+K)^{1/2} I^{1/2} J^{1/2} + J, \tag{3.5.16}$$

which self-improves, since I is finite, to $I \lesssim (1+K)J$, or

$$\int_{\partial\Omega} |\mathscr{A}(\Psi_\epsilon^2 \delta \nabla u)|^2 d\omega_0^X \lesssim (1+K) \int_{\partial\Omega} \Big| \widetilde{N}\Big(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*}\Big) + N(\Psi_\epsilon u)\Big|^2 d\omega_0^X. \tag{3.5.17}$$

Thanks to (3.5.15), we have

$$I \approx \iint_\Omega \Psi_\epsilon^4 |\nabla u|^2 G_X^* \, dm. \tag{3.5.18}$$

Using the ellipticity of $\widehat{\mathcal{A}}_1$, we have

$$\begin{aligned} I &\lesssim \iint_\Omega \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla u \Psi_\epsilon^4 G_X^* \, dm \\ &= \iint_\Omega \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla [u \Psi_\epsilon^4 G_X^*] \, dm - 4 \iint_\Omega \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla \Psi_\epsilon \, [u \Psi_\epsilon^3 G_X^*] \, dm \\ &\qquad - \iint_\Omega \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla G_X^* \, [u \Psi_\epsilon^4] \, dm =: I_1 + I_2 + I_3. \end{aligned}$$

We use the fact that u is a weak solution to L_1, and thus to $\widehat{L}_1$, to write that

$$I_1 = - \iint_\Omega \widehat{\mathcal{B}}_1 \cdot \nabla u \, [u \Psi_\epsilon^4 G_X^*] \, dm.$$

We use the Cauchy-Schwarz inequality, (3.5.18), and then (3.5.15) to obtain

$$
\begin{aligned}
I_1 &\le \Big(\iint_\Omega \Psi_\epsilon^4 |\nabla u|^2 G_X^* \, dm \Big)^{\frac12} \Big(\iint_\Omega |\widehat{\mathcal{B}}_1|^2 u^2 \Psi_\epsilon^4 G_X^* \, dm \Big)^{\frac12} \\
&\lesssim I^{1/2} \Big(\int_{\partial\Omega} |\mathscr{A}(\delta|\widehat{\mathcal{B}}_1| u \Psi_\epsilon^2)|^2 \, d\omega_0^X \Big)^{\frac12} \lesssim I^{1/2} K^{1/2} \Big(\int_{\partial\Omega} |N(u\Psi_\epsilon)|^2 \, d\omega_0^X \Big)^{\frac12} \\
&\lesssim I^{1/2} K^{1/2} J^{1/2},
\end{aligned}
$$

where the last line is due to the Carleson inequality (3.5.6) and the fact that $\delta|\widehat{\mathcal{B}}_1| \in KCM(\omega_0, K)$. For I_2, the argument is similar, but instead we use the fact that $\widehat{\mathcal{A}}_1$ is bounded, and then the fact that $\nabla\Psi \in KCM(\omega_0, M)$, proved previously in (3.5.13), to get

$$
\begin{aligned}
I_2 &\lesssim I^{1/2} \Big(\iint_\Omega |\nabla \Psi_\epsilon|^2 u^2 \Psi_\epsilon^2 G_X^* \, dm \Big)^{\frac12} \lesssim I^{1/2} \Big(\int_{\partial\Omega} |\mathscr{A}(\delta|\nabla\Psi_\epsilon| u \Psi_\epsilon)|^2 \, d\omega_0^X \Big)^{\frac12} \\
&\lesssim I^{1/2} \Big(\int_{\partial\Omega} |N(u\Psi_\epsilon)|^2 \, d\omega_0^X \Big)^{\frac12} \lesssim I^{1/2} J^{1/2}.
\end{aligned}
$$

For the term I_3, we replace $\widehat{\mathcal{A}}_1$ by $\mathcal{A}_0$:

$$
I_3 = -\iint_\Omega \mathcal{A}_0 \nabla u \cdot \nabla G_X^* \, [u\Psi_\epsilon^4] \, dm - \iint_\Omega (\widehat{\mathcal{A}}_1 - \mathcal{A}_0) \nabla u \cdot \nabla G_X^* \, [u\Psi_\epsilon^4] \, dm = I_{31} + I_{32}.
$$

We deal with I_{32} by invoking the assumption $|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\sup}(K)$. We have, using the Cauchy-Schwarz inequality, (3.5.18), and (3.5.15) as before, that

$$
\begin{aligned}
I_{32} &\lesssim I^{1/2} \Big(\int_{\partial\Omega} \Big| \mathscr{A}\Big(|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| u \Psi_\epsilon^2 \frac{\delta \nabla G_X^*}{G_X^*} \Big) \Big|^2 \, d\omega_0^X \Big)^{\frac12} \\
&\lesssim I^{1/2} K^{1/2} \Big(\int_{\partial\Omega} \Big| \widetilde{N}\Big(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*} \Big) \Big|^2 d\omega_0^X \Big)^{1/2} \lesssim I^{1/2} K^{1/2} J^{1/2}
\end{aligned}
$$

by (3.5.7), since $|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\sup}(K)$. It remains to bound I_{31}. We force everything into the first gradient, and we get that

$$
I_{31} = -\frac12 \iint_\Omega \widehat{\mathcal{A}}_0 \nabla [u^2 \Psi_\epsilon^4] \cdot \nabla G_X^* \, dm + 2 \iint_\Omega \widehat{\mathcal{A}}_0 \nabla \Psi_\epsilon \cdot \nabla G_X^* \, [u^2 \Psi_\epsilon^3] \, dm := I_{311} + I_{312}.
$$

The integral I_{311} is 0. Indeed G_X^* is a weak solution to $(L_0)^*$, and moreover $u^2\Psi_\epsilon^4$ is a valid test function because it is compactly supported in $\Omega \setminus \{X\}$ and $u^2\Psi^3 \in W^{1,2}(\Omega, m)$ [remember that u is a solution, so u is locally bounded]. As for I_{312}, we use the boundedness of $\mathcal{A}_0$ and the inequality $2ab \leq a^2 + b^2$ to infer that

$$\begin{aligned} I_{312} &\lesssim \iint_\Omega |\nabla \Psi_\epsilon| \Big(\Psi_\epsilon^2 + \Psi_\epsilon^4 \frac{\delta^2 |\nabla G_X^*|^2}{|G_X^*|^2} \Big) u^2 \frac{G_X^*}{\delta} \, dm \\ &\lesssim \int_{\partial\Omega} |\mathscr{A}(|\delta \nabla \Psi_\epsilon|^{1/2} u \Psi_\epsilon)|^2 \, d\omega_0^X + \int_{\partial\Omega} \Big| \mathscr{A}\Big(|\delta \nabla \Psi_\epsilon|^{1/2} u \Psi_\epsilon^2 \frac{\delta \nabla G_X^*}{G_X^*} \Big) \Big|^2 \, d\omega_0^X. \end{aligned}$$

By (3.5.12) and (3.5.13), that is, by the fact that $|\delta \nabla \Psi_\epsilon|^{1/2} \in KCM(\omega_0, M)$ and $|\delta \nabla \Psi_\epsilon|^{1/2} \in KCM_{\sup}(\omega_0, M)$, and the Carleson inequalities (3.5.6)–(3.5.7) we conclude that

$$I_{312} \lesssim \int_{\partial\Omega} \Big| \widetilde{N}\Big(u \Psi_\epsilon^2 \frac{\delta \nabla G_X^*}{G_X^*} \Big) \Big|^2 d\omega_0^X + \int_{\partial\Omega} |N(u\Psi_\epsilon)|^2 d\omega_0^X = J.$$

The claim (3.5.16) follows.

3.5.6 Step 5: A Caccioppoli inequality.

From (3.5.17) and (3.5.8), it remains to check that

$$\int_{\partial\Omega} \Big| \widetilde{N}\Big(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*} \Big) \Big|^2 d\omega_0^X \lesssim \int_{\partial\Omega} |N(\Psi_\epsilon u)|^2 d\omega_0^X. \tag{3.5.19}$$

We take $Y \in \Omega$ such that $2B_Y \cap \text{supp } \Psi \neq 0$, and we observe that $4B_Y$ does not contain $X \in \Omega \setminus B(x, 1000r)$. We want to prove the following variant of the Caccioppoli inequality:

$$⨌_{B_Y} \Psi_\epsilon^4 u^2 \frac{\delta^2 |\nabla G_X^*|^2}{|G_X^*|^2} \, dm \lesssim ⨌_{2B_Y} \Psi_\epsilon^2 u^2 \, dm. \tag{3.5.20}$$

Recall that G_X^* is positive (easy consequence of (3.5.14)) and a solution to $(L_0)^* u = 0$ on $4B_Y$. The Harnack inequality (Lemma 3.2.6) yields that

$$G_X^*(Z) \approx G_X^*(Y) \qquad \text{for } Z \in 2B_Y. \tag{3.5.21}$$

Using also the property that $\delta \approx \delta(Y)$ on B_Y, the claim (3.5.20) is equivalent to the estimate

$$\iint_{B_Y} \Psi_\epsilon^4 u^2 |\nabla G_X^*|^2\, dm \lesssim \Big(\frac{G_X^*(Y)}{\delta(Y)}\Big)^2 \iint_{2B_Y} \Psi_\epsilon^2 u^2\, dm.$$

We construct a cut-off function $\Phi = \Phi_Y$, using the smooth function ψ introduced in Step 1, by $\Phi(Z) = \psi\big(\frac{4|Z-Y|}{\delta(Y)}\big)$. Note that Φ is supported in $2B_Y$, $\Phi \equiv 1$ on B_Y, and $|\nabla\Phi| \leq \delta^{-1}(Y)$. So our claim (3.5.22) will be proven once we show that

$$T := \iint_\Omega \Psi_\epsilon^4 \Phi^2 u^2 |\nabla G_X^*|^2\, dm \lesssim U := \frac{|G_X^*(Y)|^2}{\delta(Y)^2} \iint_{2B_Y} \Psi_\epsilon^2 u^2\, dm. \tag{3.5.22}$$

We shall prove that $T \lesssim T^{1/2}U^{1/2}$, which self-improves to (3.5.22) because T is finite. Using the ellipticity of $\mathcal{A}_0$ given by (3.1.3), we have that

$$\begin{aligned}
T &\lesssim \iint_\Omega \mathcal{A}_0 \nabla G_X^* \cdot \nabla G_X^* \, [\Psi_\epsilon^4 \Phi^2 u^2]\, dm \\
&= \iint_\Omega \mathcal{A}_0 \nabla [G_X^* \Psi_\epsilon^4 \Phi^2 u^2] \cdot \nabla G_X^*\, dm - 4 \iint_\Omega \mathcal{A}_0 \nabla \Psi_\epsilon \cdot \nabla G_X^* \, [G_X^* \Psi_\epsilon^3 \Phi^2 u^2]\, dm \\
&\quad - 2 \iint_\Omega \mathcal{A}_0 \nabla \Phi \cdot \nabla G_X^* \, [G_X^* \Psi_\epsilon^4 \Phi u^2]\, dm - 2 \iint_\Omega \mathcal{A}_0 \nabla u \cdot \nabla G_X^* \, [G_X^* \Psi_\epsilon^4 \Phi^2 u]\, dm \\
&=: T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

The term T_1 equals 0, because G_X^* is a solution to $(L_0)^*$. Using the boundedness of $\mathcal{A}_0$ and the Cauchy-Schwarz inequality, the term T_3 is bounded as

$$T_3 \lesssim T^{1/2} \Big(\iint_\Omega |G_X^*|^2 \Psi_\epsilon^4 |\nabla\Phi|^2 u^2\, dm \Big)^{1/2} \lesssim T^{1/2} U^{1/2}$$

by (3.5.21) and $|\nabla\Phi| \lesssim \delta^{-1}(Y)$. With the same arguments, we treat T_2 as follows

$$T_2 \lesssim T^{1/2} \Big(\iint_\Omega |G_X^*(Y)|^2 \Psi_\epsilon^2 \Phi^2 |\nabla\Psi_\epsilon|^2 u^2\, dm \Big)^{1/2} \lesssim T^{1/2} U^{1/2}$$

by (3.5.21) and (3.5.9), i.e. the fact that $|\nabla\Psi_\epsilon| \lesssim \delta^{-1}(Y)$. As for T_4, we have

$$\begin{aligned}
T_4 &\lesssim T^{1/2} \Big(\iint_\Omega |G_X^*(Y)|^2 \Psi_\epsilon^4 \Phi^2 |\nabla u|^2\, dm \Big)^{1/2} \\
&\qquad\qquad \lesssim |G_X^*(Y)| T^{1/2} \Big(\iint_\Omega \Psi_\epsilon^4 \Phi^2 |\nabla u|^2\, dm \Big)^{1/2}
\end{aligned} \tag{3.5.23}$$

If we write $V = \iint_\Omega \Psi_\epsilon^4 \Phi^2 |\nabla u|^2 \, dm$, then the same argument as for T, using the fact that u is a weak solution to L_1 and $|\nabla \Psi_\epsilon| + |\nabla \Phi| \lesssim \delta(Y)$ on $2B_Y$, yields that

$$V \lesssim V^{1/2} \delta^{-1}(Y) \Big(\iint_{2B_Y} \Psi_\epsilon^2 u^2 \, dm \Big)^{\frac{1}{2}} = |G_X^*(Y)|^{-1} V^{1/2} U^{1/2},$$

which self-improves to $|G_X^*(Y)|^2 V \lesssim U$. Using the estimate in (3.5.23), we obtain that $T_4 \lesssim T^{1/2} U^{1/2}$. The claim (3.5.22) follows, and hence so does (3.5.20).

The inequality (3.5.20) entails the pointwise bound

$$\widetilde{N}\Big(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*} \Big) \lesssim N_{10}(\Psi_\epsilon u),$$

where $N_{10}(v)(y) := \sup_{\gamma_{10}(y)} |v|$, and $\gamma_{10}(y)$ is the cone with vertex at $y \in \partial\Omega$ with a bigger aperture than $\gamma(y)$ so that $\gamma_{10}(y) \supset \bigcup_{Y \in \gamma(y)} 2B_Y$. The estimate (3.5.19) comes then from classical fact that $\|N_{10}(v)\|_{L^2} \lesssim \|N(v)\|_{L^2}$, see (3.5.5). If we want to avoid this latter estimate, we can also define N using cones with bigger apertures than the ones of $\widetilde{N}$, and all our proofs are then identical. $\square$

Chapter 4

Critical Perturbations for Second Order Elliptic Operators. Part I: Square function bounds for layer potentials

The research in these chapters was done in collaboration with S. Bortz, S. Hofmann, J. L. Luna García, and S. Mayboroda. The results and proofs in these chapters will also appear in the doctoral thesis of J. L. Luna García.

4.1 Introduction

In this chapter, we lay the groundwork for the study of the L^2 Dirichlet, Neumann and Regularity problems for critical perturbations of second order divergence form equations by lower order terms, as discussed in Section 1.2.3. In particular, we produce the natural (L^2) square function estimates for (abstract) layer potential operators, which we will use in Chapter 5 to prove the well-posedness of Dirichlet, Neumann, and Regularity problems (with square-integrable data) for second-order elliptic operators with lower order terms. Relevant literature review lies in Section 1.3.2. Recall that we consider differential operators of the form (1.2.10) defined on $\mathbb{R}^n \times \mathbb{R} = \{(x,t)\}$, $n \geq 3$, where $A = A(x)$

is an $(n+1)\times(n+1)$ matrix of L^∞ complex coefficients, defined on $\mathbb{R}^n$ (independent of t) and satisfying the uniform ellipticity condition (1.1.4) for some $C_A>0$, and for all $\xi,\eta\in\mathbb{C}^{n+1}, x\in\mathbb{R}^n$. The first order complex coefficients $B_1=B_1(x), B_2=B_2(x)\in \big(L^n(\mathbb{R}^n)\big)^{n+1}$ (independent of t) and the complex potential $V=V(x)\in L^{\frac{n}{2}}(\mathbb{R}^n)$ (again independent of t) are such that

$$\max\left\{\|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\}\le\varepsilon_0$$

for some ε_0 depending on dimension and the ellipticity of A in order to ensure the accretivity of the form associated to the operator $\mathcal{L}$ on the space

$$Y^{1,2}(\mathbb{R}^{n+1}):=\left\{u\in L^{2^*}(\mathbb{R}^{n+1}):\nabla u\in L^2(\mathbb{R}^{n+1})\right\}$$

equipped with the norm

$$\|u\|_{Y^{1,2}(\mathbb{R}^{n+1})}:=\|u\|_{L^{2^*}(\mathbb{R}^{n+1})}+\|\nabla u\|_{L^2(\mathbb{R}^{n+1})},$$

where $p^*:=\frac{(n+1)p}{n+1-p}$. We interpret solutions of $\mathcal{L}u=0$ in the weak sense; that is, $u\in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$ is a solution of $\mathcal{L}u=0$ in $\Omega\subset\mathbb{R}^{n+1}$ if for every $\varphi\in C_c^\infty(\Omega)$ it holds that

$$\iint_{\mathbb{R}^{n+1}}\left((A\nabla u+B_1u)\cdot\overline{\nabla\varphi}+B_2\cdot\nabla u\overline{\varphi}\right)=0.$$

Examples of operators of the type defined above include the Schrödinger operator $-\Delta+V$ with $t-$independent electric potential $V\in L^{\frac{n}{2}}(\mathbb{R}^n)$ having a small $L^{\frac{n}{2}}$ norm, and the generalized magnetic Schrödinger operator $-(\nabla-i\mathbf{a})A(\nabla-i\mathbf{a})$, where A is a $t-$independent complex matrix satisfying (1.1.4), and the magnetic potential $\mathbf{a}\in L^n(\mathbb{R}^n)^{n+1}$ is $t-$independent and has small L^n norm. We treat the case $n\ge3$ because the Sobolev spaces we encounter are of the form $\dot{W}^{1,2}(\mathbb{R}^n)\cap L^s$ for some $s\ge1$, and in this case, these spaces embed continuously into Lebesgue spaces. This is not the situation when $n=2$, in which case the Sobolev spaces considered embed continuously into BMO. If one were to treat the case $n=2$, it would be natural to assume that $V=0$ and that B_i, $i=1,2$ are divergence-free. Under these additional hypotheses, one can use a compensated compactness argument [CLMS93] to obtain the boundedness and invertibility of the form associated to $\mathcal{L}$ (see [GHN16]).

However, there are several considerations in the case $n \geq 3$ that set it qualitatively apart from $n = 2$. For instance, when $n = 2$, all solutions are locally Hölder continuous and this is certainly not the case when $n \geq 3$. Indeed, let $u(x) = -\ln|x|$, $x \in \mathbb{R}^n$ and build $V(x)$ or $B_1(x)$ so that either $-\Delta u + Vu = 0$ or $-\Delta u + \operatorname{div} B_1 u = 0$ in the n-dimensional ball $B(0, 1/2)$. By extending u to be a function on $B(0, 1/2) \times \mathbb{R}$ by $u(x, t) = u(x)$, we may see that the analogous equations in $n + 1$ dimensions are satisfied by $u(x, t)$, and yet $u(x, t)$ fails to be locally bounded despite the fact that $V^2, B_1 \in L^n(\mathbb{R}^n)$. Moreover, by considering $u(x, t)$ on a smaller ball and replacing V or B_1 by $V_\epsilon = V\mathbb{1}_{B(0,\epsilon)}$ or $(B_1)_\epsilon = B_1\mathbb{1}_{B(0,\epsilon)}$ respectively, we may ensure that V_ϵ^2 or $(B_1)_\epsilon$ have arbitrarily small $L^n(\mathbb{R}^n)$ norm, provided that we choose $\epsilon > 0$ small enough. Therefore, solutions in our perturbative regime fail to be locally bounded and hence fail (miserably) to be locally Hölder continuous.

The lack of local Hölder continuity (or local boundedness) is one reason our results are not at all a straightforward adaptation of related works. For instance, in [AAA$^+$11] the authors are able to treat the fundamental solution as a Calderón-Zygmund-Littlewood-Paley kernel using pointwise estimates on the fundamental solution (and its t-derivatives) presented in [HK07]. Additionally, although establishing a Caccioppoli inequality (Proposition 4.3.1) is easy, constants are not necessarily null solutions to our operator and thus this Caccioppoli inequality does not yield the usual "reverse" Poincaré inequality for solutions. We remind the reader that if there are no lower order terms, the Caccioppoli inequality (becomes a "reverse" Poincaré inequality and) improves to an $L^p - L^2$ version; more precisely, we have that for each ball B_r and some $p > 2$, the estimate

$$\Big(\fint_{B_r} |\nabla u|^p \, dx\Big)^{1/p} \lesssim \frac{1}{r}\Big(\fint_{B_{2r}} |u|^2 \, dx\Big)^{1/2}$$

holds [Mey63, Geh73, Gia83]. We do not manage to obtain the above $L^p - L^2$ inequality, but rather a suitable $L^p - L^p$ version (Proposition 4.3.9). The unavailability of these desirable estimates makes it far less clear whether constructing the fundamental solution will be useful for us, and so we do not attempt it. We still endeavor to use the method of layer potentials, whence we appeal to (and adapt) the abstract constructions of Barton [Bar17], which avoid the use of fundamental solutions entirely. Fundamental solutions have been constructed in other situations with lower order terms in [DHM18] and [KS19], but they rely on sign conditions.

Our results in this series of chapters concern the unique solvability of the classical L^2 boundary value problems in the upper half space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \mathbb{R}_+$, $(\mathrm{D})_2$, $(\mathrm{N})_2$, and $(\mathrm{R})_2$, as stated in Section 1.2.3 (see (1.2.13), (1.2.14), (1.2.15)). The idea is to follow a (by now) familiar process for proving L^2 existence and uniqueness for these boundary value problems. This process has three steps, which can be (very) roughly summarized as:

1. Show square function (and/or non-tangential maximal function) bounds for a linear operator defined, perhaps by continuous extension, on L^2, where the operator necessarily produces weak solutions to the elliptic equation (for us, this operator is either the single or double layer potential).
2. Show the boundedness and invertibility of the appropriate boundary trace of the operator.
3. Show that any solution with square function (and/or non-tangential maximal function) bounds is, in fact, the solution produced by the linear operator with appropriate data.

The current chapter is concerned establishing the square function bounds for abstract layer potential operators, that is, step (1) in the process above. We prove the following.

Theorem 4.1.1 (Square function bound for the single layer potential)**.** *Suppose that $\mathcal{L}_0 = -\operatorname{div} A\nabla$ is a divergence form elliptic operator with t-independent coefficients, and that the matrix A is elliptic. Then, there exists $\varepsilon_0 > 0$, depending on n and C_A, such that if $M \in \mathcal{M}_{n+1}(\mathbb{R}^n, \mathbb{C})$, $V \in L^{n/2}(\mathbb{R}^n)$ and $B_i \in L^n(\mathbb{R}^n)$ are (all) t-independent with*

$$\|M\|_{L^\infty(\mathbb{R}^n)} + \|B_1\|_{L^n(\mathbb{R}^n)} + \|B_2\|_{L^n(\mathbb{R}^n)} + \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} < \varepsilon_0,$$

then for each $m \in \mathbb{N}$, we have the estimate

$$\iint_{\mathbb{R}^{n+1}_+} \left|t^m \partial_t^{m+1} \mathcal{S}_t^{\mathcal{L}} f(x)\right|^2 \frac{dx\,dt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)},$$

where C depends on m, n, and C_A, and

$$\mathcal{L} := -\operatorname{div}([A + M]\nabla + B_1) + B_2 \cdot \nabla + V.$$

Under the same hypothesis, the analogous bounds hold for $\mathcal{L}$ replaced by $\mathcal{L}^*$, and for $\mathbb{R}^{n+1}_+$ replaced by $\mathbb{R}^{n+1}_-$.

We point out that in the previous result, there is no restriction on the matrix A, other than that it be t-independent and satisfy the complex ellipticity condition (1.1.4). In the homogeneous, purely second order case (i.e., the case that B_1, B_2, and V are all zero), this result is due to Rosén [Ros13]; an alternative proof, with an extra hypothesis of De Giorgi/Nash/Moser regularity, appears in [GdlHH16].

We also obtain a uniform estimate on horizontal slices in terms of the square function.

Theorem 4.1.2 (Uniform control of $Y^{1,2}(\mathbb{R}^n)$ norm on each horizontal slice)**.** *Suppose that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ and $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense. Then for every $\tau > 0$,*

$$\|\operatorname{Tr}_\tau u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + \|\operatorname{Tr}_\tau \nabla u\|_{L^2(\mathbb{R}^n)} \lesssim \int_\tau^\infty \int_{\mathbb{R}^n} t|D^2_{n+1}u|^2\,dxdt \leq |||tD^2_{n+1}u|||, \tag{4.1.3}$$

where the traces exist in the sense of Lemma 4.2.3, and C depends on m, n, and C_A, provided that $\max\{\|B_1\|_n, \|B_2\|_n, \|V\|_{\frac{n}{2}}\}$ is sufficiently small depending on m, n, and C_A. Under the same hypothesis, the analogous bounds hold for $\mathcal{L}$ replaced by $\mathcal{L}^*$, and for $\mathbb{R}^{n+1}_+$ replaced by $\mathbb{R}^{n+1}_-$.

Our results in this series (Chapters 4 and 5) may be best thought of as extensions of the results in [AAA+11] to lower-order terms as well as complex matrices (and not only those arising from perturbations of real symmetric coefficients or constant coefficients), albeit with the important distiction that we do not require DeGiorgi-Nash-Moser [DeG57, Nas58, Mos61] estimates; this allows us to consider any complex elliptic matrix for A. Let us mention a few applications of our theorems. For the magnetic Schrödinger operator $-(\nabla - i\mathbf{a})^2$ when $\mathbf{a} \in L^n(\mathbb{R}^n)^{n+1}$ is $t-$independent and has small $L^n(\mathbb{R}^n)$ norm, we obtain in this chapter the first estimate for the square function and solvability of the modified problems $(\mathrm{D})'_2$, $(\mathrm{N})'_2$, $(\mathrm{R})'_2$ in the unbounded setting of the half-space. In fact, since our methods do not rely on an algebraic structure other than $t-$independence, we have similar novel conclusions for the generalized magnetic Schrödinger operators $-(\nabla - i\mathbf{a})A(\nabla - i\mathbf{a})$ where A is a real, symmetric, $t-$independent, elliptic, bounded matrix, and $\mathbf{a}$ is as above.

This chapter is organized as follows. In Section 4.3 we prove some elementary but essential PDE estimates, and in Section 4.4 we develop the notion of abstract layer potentials. Next, we show that for $\varepsilon_0 > 0$ small enough, the single and double layer potentials have square function estimates (Theorems 4.1.1 and 4.5.5, and Lemma 4.6.2), which, in turn, give us 'slice space' estimates for the single and double layer potentials (Theorems 4.6.12 and 4.6.17). In passing we remark that this analysis already allows us to establish the jump relations (as weak limits in $L^2(\mathbb{R}^n)$)

$$\mathcal{D}_{\pm} f \to (\mp\tfrac{1}{2} I + K) f$$

and

$$(\nabla S_t)|_{t=\pm s} f \to \mp\frac{1}{2}\frac{f(x)}{A_{n+1,n+1}} e_{n+1} + \mathcal{T} f$$

for f in L^2, where $\mathcal{D}$ and $\mathcal{S}$ are the double and single layer potentials.

4.2 Preliminaries

As stated above, our standing assumption will be that $n \geq 3$, and the ambient space will always be $\mathbb{R}^{n+1} = \{x, t : x \in \mathbb{R}^n,\ t \in \mathbb{R}\}$. We employ the following standard notation:

- We will use lower-case x, y, z to denote points in $\mathbb{R}^n$ and lower-case t, s, τ to denote real numbers. By convention, $x = (x_1, \ldots, x_n)$, and $x_{n+1} = t$. We will use capital X, Y, Z to denote points in $\mathbb{R}^{n+1}$. The symbols $e_1, \ldots, e_{n+1}$ are reserved for the standard basis vectors in $\mathbb{R}^{n+1}$.

- We will often be breaking up vectors into their parallel and perpendicular parts. For an $(n+1)$-dimensional vector $\vec{V} = (V_1, \ldots, V_n, V_{n+1})$, we define its 'horizontal' or 'parallel' component as
$$V_{\|} := (V_1, \ldots, V_n),$$
and its 'vertical' or 'transverse' component as $V_{\perp} = V_{n+1}$. Similarly, we label the horizontal component of the $(n+1)$-dimensional gradient operator as
$$\nabla_{\|} := \nabla_x := (\partial_{x_1}, \ldots, \partial_{x_n}),$$
and the 'vertical' component as D_{n+1} or $\nabla_{\perp}$.

- Given the $(n+1) \times (n+1)$ complex-valued matrix A, for each $i, j = 1, \ldots, n+1$, we denote by A_{ij} the $ij-$th entry of A. We denote by $\tilde{A}$ the $(n+1) \times n$ submatrix of A

consisting of the first n columns of A. We define $\vec{A}_{i,\cdot}$ as the $(n+1)-$dimensional row vector made up of the $i-$th row of A; similarly we let $\vec{A}_{\cdot,j}$ be the $(n+1)-$dimensional column vector made up of the $j-$th row of A.

- We set $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ and $\partial\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \{0\}$. We define $\mathbb{R}^{n+1}_-$ similarly and often we write $\mathbb{R}^n$ in place of $\partial\mathbb{R}^{n+1}_+$ when confusion may not arise. For $t \in \mathbb{R}$, we denote $\mathbb{R}^{n+1}_t = \mathbb{R}^{n+1}_{+,t} := \mathbb{R}^n \times (t, \infty)$, and $\mathbb{R}^{n+1}_{-,t} := \mathbb{R}^n \times (-\infty, t)$.

- The letter Q will always denote a cube in $\mathbb{R}^n$. By $\ell(Q)$ and x_Q we denote the side length and center of Q, respectively. We write $Q(x,r)$ to denote the cube with center x and sides of length r, parallel to the coordinate axes.

- Given a (closed) n-dimensional cube $Q = Q(x,r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa Q := Q(x, \kappa r)$. Similar dilations are defined for cubes in $\mathbb{R}^{n+1}$ as well as (open) balls in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$.

- For $a, b \in [-\infty, \infty]$, we set $\Sigma_a^b := \left\{X = (x,t) \in \mathbb{R}^{n+1} : t \in (a,b)\right\}$.

- Given a Borel set E and Borel measure μ, for any $\mu|_E$-measurable function f we define the μ-average of f over E as
$$\fint_E f\, d\mu := \frac{1}{\mu(E)} \int_E f\, d\mu.$$

- For a Borel set $E \subset \mathbb{R}^{n+1}$, we let $\mathbb{1}_E$ denote the usual indicator function of E; that is, $\mathbb{1}_E(x) = 1$ if $x \in E$, and $\mathbb{1}_E(x) = 0$ if $x \notin E$.

- For a Banach space X, we let $\mathcal{B}(X)$ denote the space of bounded linear operators on X. Simiarly, if X and Y are Banach spaces, we denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators $X \to Y$.

We will work with several function spaces; let us briefly describe them. For the rest of the chapter, we assume that the reader is familiar with the basics of the theory of distributions and Fourier Transform and the basics of the theory of Sobolev spaces (see [Leo17]). We delegate some of the basic definitions and results to these and other introductory texts.

Let Ω be an open set in $\mathbb{R}^k$ for some $k \in \mathbb{N}$. For any $m \in \mathbb{N}$ and any $p \in [1,\infty)$, the space $L^p(\Omega)^m = L^p(\Omega, \mathbb{C}^m)$ consists of the complex-valued $p-$th integrable $m-$dimensional vector functions over Ω. We equip $L^p(\Omega, \mathbb{C}^m)$ with the norm

$$\|\vec{f}\|_{L^p(\Omega,\mathbb{C}^m)} = \Big(\sum_{i=1}^m \int_\Omega |f_i|^p\Big)^{\frac{1}{p}}, \qquad \vec{f} = (f_1, \ldots, f_m).$$

For simplicity of notation, we often write $\|\vec{f}\|_p = \|\vec{f}\|_{L^p(\Omega)} = \|\vec{f}\|_{L^p(\Omega,\mathbb{C}^m)}$ when the domain Ω and the dimension of the vector function $\vec{f}$ are clear from the context (most often, when Ω is the ambient space, which for us means either $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^{n+1}$).

The space $C_c^\infty(\Omega)$ consists of all compactly supported smooth complex-valued functions in Ω. As usual, we denote $\mathscr{D} = C_c^\infty(\mathbb{R}^{n+1})$, and we let $\mathscr{D}' = \mathscr{D}^*$ be the space of distributions on $\mathbb{R}^{n+1}$. The space $\mathscr{S}$ consists of the Schwartz functions on $\mathbb{R}^{n+1}$, and $\mathscr{S}' = \mathscr{S}^*$ is the space of tempered distributions on $\mathbb{R}^{n+1}$.

For $p \in [1,\infty)$, we denote by $W^{1,p}(\Omega)$ the usual Sobolev space of functions in $L^p(\Omega)$ whose weak gradients exist and lie in $(L^p(\Omega))^{n+1}$. We endow this space with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

We define $W_0^{1,p}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ in the above norm. We shall have occasion to discuss the homogeneous Sobolev spaces as well: by $\dot{W}^{1,p}(\Omega)$ we denote the space of functions in $L^1_{\mathrm{loc}}(\Omega)$ whose weak gradients exist and lie in $L^p(\Omega)$. We equip this space with the seminorm

$$|u|_{\dot{W}^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)},$$

and point out that $\dot{W}^{1,p}(\Omega)$ coincides with the completion of the quotient space $C^\infty(\Omega)/\mathbb{C}$ in the $|\cdot|_{\dot{W}^{1,p}(\Omega)}$ (quotient) norm. For $p \in (1, n+1)$ and $\Omega \subset \mathbb{R}^{n+1}$ an open set, we define the space $Y^{1,p}(\Omega)$ as

$$Y^{1,p}(\Omega) := \Big\{u \in L^{\frac{(n+1)p}{n+1-p}}(\Omega) : \nabla u \in L^p(\Omega)\Big\}.$$

Write $p^* := \frac{(n+1)p}{n+1-p}$. We equip this space with the norm

$$\|u\|_{Y^{1,p}(\Omega)} := \|u\|_{L^{p^*}(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

We define $Y_0^{1,p}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ in this norm. By virtue of the Sobolev embedding, when $p \in (1, n+1)$ we have that $Y_0^{1,p}(\Omega)$ coincides with the completion of $C_c^\infty(\Omega)$ in the $\dot{W}^{1,p}(\Omega)$ seminorm. Moreover, we have that $Y_0^{1,p}(\mathbb{R}^{n+1}) = Y^{1,p}(\mathbb{R}^{n+1})$.

The $Y^{1,2}$ spaces exhibit the following useful property.

Lemma 4.2.1 (Integrability up to a constant of a function with square integrable gradient on a half-space)**.** *Suppose that $u \in L^1_{\mathrm{loc}}(\Sigma_a^b)$ for some $a < b$, $a, b \in [-\infty, \infty]$, either $a = -\infty$ or $b = +\infty$, and that the distributional gradient satisfies $\nabla u \in L^2(\Sigma_a^b)$. Then there exists $c \in \mathbb{C}$ such that $u - c \in Y^{1,2}(\Sigma_a^b)$.*

The proof is very similar to that of Theorem 1.78 in [MZ97], thus we omit it.

In this chapter, whenever we write $u(t)$ for $t \in \mathbb{R}$, we mean

$$u(t) = u(\cdot, t); \tag{4.2.2}$$

thus $u(t)$ is a measurable function on $\mathbb{R}^n$. Let us present a fact regarding the regularity of functions in $Y^{1,2}(\mathbb{R}^{n+1})$ when seen as single-variable vector-valued maps. The proof is omitted as it is straightforward.

Lemma 4.2.3 (Local Hölder continuity in the transversal direction)**.** *Suppose that $u \in Y^{1,2}(\Sigma_a^b)$ for some $a < b$. Then it holds that $u \in C^\alpha_{\mathrm{loc}}((a,b); L^{2^*}(\mathbb{R}^n))$ for some exponent $\alpha > 0$ (see (4.2.2)). Moreover, if $\partial_t \nabla u \in L^2(\Sigma_a^b)$, then we also have that $\nabla u \in C^\beta_{\mathrm{loc}}((a,b); L^2(\mathbb{R}^n))$ for some $\beta > 0$.*

Remark 4.2.4. Note that the functions above are representatives of $u(t)$ and $\nabla u(t)$, but that these retain the same properties as their smooth counterparts when acting on functions defined on the slice $\{x_{n+1} = t\}$. More precisely, for any $\vec{\varphi} \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^n)$ and any $t \in (a,b)$, we have the identity

$$\int_{\mathbb{R}^n} u(x,t) \operatorname{div}_\parallel \vec{\varphi}(x)\, dx = -\int_{\mathbb{R}^n} \nabla_\parallel u(x,t) \cdot \vec{\varphi}(x)\, dx.$$

The above identity is already true for a.e. $t \in (a,b)$, and is seen to be true for arbitrary $t \in (a,b)$ by the continuity of u and ∇u.

Analogously, we introduce $Y^{1,2}(\mathbb{R}^n)$ as

$$Y^{1,2}(\mathbb{R}^n) := \Big\{ u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n) \Big\}, \tag{4.2.5}$$

and equip it with the norm

$$\|u\|_{Y^{1,2}(\mathbb{R}^n)} := \|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)}.$$

Note carefully that in our convention, $2^* = \frac{2(n+1)}{n-1} \neq \frac{2n}{n-2}$.

Some fractional Sobolev spaces will be useful for us when discussing trace operators. Let $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the Fourier transform. Throughout this chapter, we shall also denote $\hat{u} := \mathcal{F}u$. We write

$$H^{\frac{1}{2}}(\mathbb{R}^n) = \Big\{u \in L^2(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} \big(1 + |\xi|\big)|\hat{u}(\xi)|^2\, d\xi < +\infty\Big\}.$$

The space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$ consists of those tempered distributions $u \in \mathscr{S}'$ whose Fourier transform $\hat{u} \in \mathscr{S}'$ is a measurable function verifying that $\int_{\mathbb{R}^n} |\xi||\hat{u}(\xi)|^2\, d\xi < +\infty$. Naturally, this space comes equipped with the seminorm $|u|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi||\hat{u}(\xi)|^2\, d\xi$. We define the space $H_0^{\frac{1}{2}}(\mathbb{R}^n) = \dot{H}_0^{\frac{1}{2}}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$ seminorm. We write $H^{-\frac{1}{2}}(\mathbb{R}^n) := (H_0^{\frac{1}{2}}(\mathbb{R}^n))^*$, and emphasize that we are departing from notation used elsewhere in the literature. Since $H_0^{\frac{1}{2}}(\mathbb{R}^n) \supsetneq H^{\frac{1}{2}}(\mathbb{R}^n)$, it follows that $H^{-\frac{1}{2}}(\mathbb{R}^n)$ is contained in the dual space of $H^{\frac{1}{2}}(\mathbb{R}^n)$, which is the usual (inhomogeneous) fractional Sobolev space of order $-1/2$ that coincides with the space

$$\Big\{u \in \mathscr{S}'(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-\frac{1}{2}}|\hat{u}(\xi)|^2\, d\xi < +\infty\Big\}.$$

For a survey on the properties of fractional Sobolev spaces, see [DPV12]. We state without proof two easy results which are nevertheless useful.

Proposition 4.2.6 (Sobolev embeddings of the fractional Sobolev spaces)**.** *Let* $p_+ := \frac{2n}{n-1}$ *and* $p_- := \frac{2n}{n+1}$*. Then we have the continuous embeddings* $H_0^{\frac{1}{2}}(\mathbb{R}^n) \hookrightarrow L^{p_+}(\mathbb{R}^n)$*,* $L^{p_-}(\mathbb{R}^n) \hookrightarrow H^{-\frac{1}{2}}(\mathbb{R}^n)$*.*

Proposition 4.2.7. *The map* $\nabla : H_0^{\frac{1}{2}}(\mathbb{R}^n) \to H^{-\frac{1}{2}}(\mathbb{R}^n)$ *is bounded.*

For fixed $t \in \mathbb{R}$ and any open set $\Omega \subset \mathbb{R}^{n+1}$ with nice enough (but possibly unbounded) boundary such that $\mathbb{R}^n \times \{\tau = t\} \subset \Omega$, we define the *trace operator*

$$\mathrm{Tr}_t : C_c^\infty(\overline{\Omega}) \to C_c^\infty(\mathbb{R}^n), \qquad \mathrm{Tr}_t\, u = u(\cdot, t). \tag{4.2.8}$$

The relevance of the fractional Sobolev spaces to our theory comes from the following trace result; we cite a paper with the proof for traces of functions in $\dot{W}^{1,2}(\mathbb{R}^2)$, but the result is straightforwardly extended to our situation.

Lemma 4.2.9 (Traces of $Y^{1,2}$ functions; [Str16])**.** *Fix $t > 0$. Let Ω be either $\mathbb{R}^{n+1}$, $\mathbb{R}^{n+1}_t$, or $\mathbb{R}^{n+1}_{-,t}$. Then, for each $s \in \mathbb{R}$ such that there exists $x \in \mathbb{R}^n$ with $(x, s) \in \Omega$, the trace operator Tr_s (see (4.2.8)) extends uniquely to a bounded linear operator $Y^{1,2}(\Omega) \to H_0^{\frac{1}{2}}(\mathbb{R}^n)$.*

Definition 4.2.10 (Local weak solutions)**.** Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set with Lipschitz (but possibly unbounded) boundary, and fix $f \in L^1_{\mathrm{loc}}(\Omega)$, $F \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^{n+1})$, and $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. We say that u solves the equation $\mathcal{L}u = f - \operatorname{div} F$ in Ω *in the weak sense* if, for every $\varphi \in C_c^\infty(\Omega)$, the identity

$$\iint_{\mathbb{R}^{n+1}} \Big((A\nabla u + B_1 u) \cdot \overline{\nabla \varphi} + B_2 \cdot \nabla u \overline{\varphi} \Big) = \iint_{\mathbb{R}^{n+1}} \left(f\overline{\varphi} + F \cdot \overline{\nabla \varphi} \right) \tag{4.2.11}$$

holds.

Remark 4.2.12. Suppose that Ω is as in Lemma 4.2.9. By a standard density argument, if $u \in Y^{1,2}(\Omega)$ solves $\mathcal{L}u = f + \operatorname{div} F$ in Ω in the weak sense and either

- $F \in L^2(\Omega)$ and $f \in L^{(2n+2)/(n+3)}(\Omega)$, or
- $\Omega = D \times I$, where D is a domain with nice enough (but possibly unbounded) boundary and I is an interval, and

$$F \in L^2(\Omega), \quad f \in L^2(I; L^{(2n)/(n+2)}(D)) + L^{(2n+2)/(n+3)}(\Omega), \tag{4.2.13}$$

then (4.2.11) holds for all $\varphi \in Y_0^{1,2}(\Omega)$. A similar observation to the second item can be made if Ω is a ball in $\mathbb{R}^{n+1}$.

For an infinite interval $I \subset \mathbb{R}$ and a Banach space X, let $C_0^k(I; X)$ be the space of functions $f : I \to X$ such that all their first k derivatives $f^{(l)} : I \to X$, $0 \leq l \leq k$, exist, are continuous on I, and satisfy that $\lim_{t\to\infty} \|f^{(l)}(t)\|_X = 0$ for all $0 \leq l \leq k$. When $k = 0$, we will omit the superscript and simply write $C^0 = C$.

We also state, without proof, the following criterion for the existence of weak derivatives in $L^2(I; X)$. See [CH98] for further results and definitions.

Theorem 4.2.14 (Vector-valued weak derivatives; [CH98] Theorem 1.4.40)**.** *Suppose that X is a reflexive Banach space and let $I \subset \mathbb{R}$ be a (not necessarily bounded) interval. Let $f \in L^2(I; X)$. Then $f \in W^{1,2}(I; X)$ if and only if there exists $\varphi \in L^2(I; \mathbb{R})$ such that for any $t, s \in I$, the estimate*

$$\|f(t) - f(s)\|_X \leq \left| \int_s^t \varphi(r)\, dr \right|$$

holds. Moreover, for a.e. $t \in I$, the difference quotients

$$\Delta^h f(t) := \frac{f(t+h) - f(t)}{h}, \qquad h \in \mathbb{R}, |h| \ll 1,$$

converge weakly in X to $f'(t)$ as $h \to 0$.

Remark 4.2.15. We will see that if $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+) \cap S^2_+$ and $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$, then by Caccioppoli's inequality (on slices) we have that

$$\begin{aligned} \|u\|_{S^2_+} &\approx \sup_{t>0} \|u(t)\|_{Y^{1,2}(\mathbb{R}^n)} + \sup_{t>0} \|u'(t)\|_{L^2(\mathbb{R}^n)} \\ &\approx \sup_{t>0} \|\nabla_\| \operatorname{Tr}_t u\|_{L^2(\mathbb{R}^n)} + \sup_{t>0} \|\operatorname{Tr}_t(D_{n+1}u)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

We now state a Trace Theorem in cubes. We set

$$I_R^\pm := (-R, R)^n \times (0, \pm R), \quad I_R := (-R, R)^{n+1}, \quad \Delta_R := (-R, R)^n \times \{0\}.$$

Proposition 4.2.16 (Trace operator on a cube)**.** *Let $H^{\frac{1}{2}}(\Delta_R)$ be the space consisting of pointwise restrictions of functions in $H^{\frac{1}{2}}(\mathbb{R}^n)$ to Δ_R. There exists a bounded linear operator $\operatorname{Tr}_0{}^\pm : W^{1,2}(I_R^\pm) \to H^{\frac{1}{2}}(\Delta_R)$ (called* the trace operator associated to *$I_R^\pm$) with the following properties.*

(i) For each $u \in C^\infty(\overline{I_R^\pm})$, $\operatorname{Tr}_0{}^\pm u(\cdot) = u(\cdot, 0)$.

(ii) For each $\Phi \in C_c^\infty(I_R)$, the identity

$$\int_{\Delta_R} (\operatorname{Tr}_0{}^\pm u)\overline{\phi} = \mp \iint_{I_R^\pm} (u\overline{D_{n+1}\Phi} + D_{n+1}u\overline{\Phi})$$

holds, where $\phi(\cdot) = \Phi(\cdot, 0)$.

In particular, the traces are consistent in the sense that for every $R' < R$, the restriction *to $I^{\pm}_{R'}$ of the trace operator associated to $I^{\pm}_{R}$, agrees with the trace in $I^{\pm}_{R'}$.*

Proof. The result follows from the usual Trace Theorem on Lipchitz domains (see, for instance, [Leo17] Theorem 15.23 and the results which follow this theorem) and the fact that I^{+}_{R} is an extension domain for $W^{1,2}$ (see [Leo17] Theorem 12.15). $\square$

We now remark that the zeroth-order term V in our differential equation can be absorbed into the first order terms.

Lemma 4.2.17 (Zeroth order term absorbed by first order terms)**.** *Let $\mathcal{L}$ be as in* (1.2.10) *with*

$$\max\left\{\|B_1\|_n,\ \|B_2\|_n,\ \|V\|_{\frac{n}{2}}\right\} \leq \varepsilon_0.$$

Then

$$\mathcal{L} = -\operatorname{div}(A\nabla + \tilde{B}_1) + \tilde{B}_2 \cdot \nabla,$$

where

$$\max\left\{\|\tilde{B}_1\|_n,\ \|\tilde{B}_2\|_n,\right\} \leq C_n\varepsilon_0.$$

Proof. We write

$$V(x) = -\operatorname{div}_x \nabla_{\|} I_2 V(x) = c_n \operatorname{div}_x \vec{R} I_1 V(x),$$

where I_α is the α-order Riesz potential

$$(I_\alpha f)(x) = \frac{1}{c_\alpha}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\alpha}\,dy,$$

and $\vec{R}$ is the Riesz transform on $\mathbb{R}^n$. For definitions and properties, see [Ste70b]. To conclude the lemma, we note that $I_1 : L^{n/2}(\mathbb{R}^n) \to L^n(\mathbb{R}^n)$ and $\vec{R}$ is a bounded operator $L^n(\mathbb{R}^n) \to [L^n(\mathbb{R}^n)]^n$. $\square$

Observe that it suffices that $V \in \dot{L}^n_{-1} = \{V \in \mathscr{D}' : I_1 V \in L^n\}$, with small norm. Thus, our results hold under this slightly more general assumption on V.

Accordingly, from now on we drop the term V from our operator. We obtain invertibility of the operator $\mathcal{L}$ on the Hilbert space $Y^{1,2}(\mathbb{R}^{n+1})$ when the size of the lower order terms is small enough.

Definition 4.2.18 (Sesquilinear form and associated operator)**.** Define the sesquilinear form $B_{\mathcal{L}} : C_c^\infty(\mathbb{R}^{n+1}) \times C_c^\infty(\mathbb{R}^{n+1}) \to \mathbb{C}$ via

$$B_{\mathcal{L}}[u,v] := \iint_{\mathbb{R}^{n+1}} \Big[A\nabla u \cdot \overline{\nabla v} + uB_1 \cdot \overline{\nabla v} + \overline{v}B_2 \cdot \nabla u\Big], \qquad u,v \in C_c^\infty(\mathbb{R}^{n+1}).$$

Define the operator $\mathcal{L} : \mathscr{D} \to \mathscr{D}'$ via the identity

$$\langle \mathcal{L}u, v\rangle = B_{\mathcal{L}}[u,v], \qquad u,v \in C_c^\infty(\mathbb{R}^{n+1}).$$

It is clear that $\mathcal{L}$ is linear.

In fact, the form $B_{\mathcal{L}}$ extends to a bounded, coercive form on the product space $Y^{1,2}(\mathbb{R}^{n+1}) \times Y^{1,2}(\mathbb{R}^{n+1})$, and the operator $\mathcal{L}$ extends to an isomorphism $Y^{1,2}(\mathbb{R}^{n+1}) \to (Y^{1,2}(\mathbb{R}^{n+1}))^*$. This is precisely the content of the following result.

Proposition 4.2.19 (Extension of operator to $Y^{1,2}$)**.** *The form $B_{\mathcal{L}}$ extends to a bounded form on $Y^{1,2}(\mathbb{R}^{n+1})$; that is,*

$$|B_{\mathcal{L}}[u,v]| \lesssim \|\nabla u\|_2 \|\nabla v\|_2, \quad \textit{for all } u,v \in C_c^\infty(\mathbb{R}^{n+1}),$$

with the implicit constant depending on n, C_A and $\max\big\{\|B_1\|_n, \|B_2\|_n\big\}$. Hence $\mathcal{L}$ extends to a bounded operator $Y^{1,2}(\mathbb{R}^{n+1}) \to (Y^{1,2}(\mathbb{R}^{n+1}))^*$.

Moreover, there exists a constant $\varepsilon_0 = \varepsilon_0(n, C_A) > 0$ such that if $\max\big\{\|B_1\|_n, \|B_2\|_n\big\} < \varepsilon_0$, then $B_{\mathcal{L}}$ is also coercive in $Y^{1,2}(\mathbb{R}^{n+1})$ with lower bound $1/(2C_A)$; that is,

$$\frac{1}{2C_A}\|\nabla u\|_2^2 \lesssim \Re e\, B_{\mathcal{L}}[u,u], \quad \textit{for all } u \in C_c^\infty(\mathbb{R}^{n+1}).$$

In particular, if $\max\big\{\|B_1\|_n, \|B_2\|_n\big\} < \varepsilon_0$, then by the Lax-Milgram Theorem the operator $\mathcal{L}^{-1} : (Y^{1,2}(\mathbb{R}^{n+1}))^* \to Y^{1,2}(\mathbb{R}^{n+1})$ exists as a bounded linear operator.

Proof. The proof is straightforward, thus omitted. □

Remark 4.2.20. We will always assume that $\max\{\|B_1\|_n, \|B_2\|_n\} < \varepsilon_0$, as above. The value of ε_0 may be made smaller, but it will always depend only on n and C_A, and we will explicitly state when we impose further smallness.

Definition 4.2.21 (Dual operator)**.** Associated to $\mathcal{L}$ we also have the dual operator, denoted $\mathcal{L}^* : Y^{1,2}(\mathbb{R}^{n+1}) \to (Y^{1,2}(\mathbb{R}^{n+1}))^*$, defined by the relation

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle.$$

It is a matter of algebra to check that

$$\mathcal{L}^* v = -\operatorname{div}(A^*\nabla v + \overline{B}_2 v) + \overline{B}_1 \cdot \nabla v$$

holds in the weak sense.

In particular, $\mathcal{L}^*$ is an operator of the same type as $\mathcal{L}$ and if $\max\{\|B_1\|_n, \|B_2\|_n\} < \varepsilon_0$ so that $\mathcal{L}^{-1}$ is defined, then $(\mathcal{L}^*)^{-1}$ is well defined, bounded, and satisfies $(\mathcal{L}^*)^{-1} = (\mathcal{L}^{-1})^*$.

4.2.1 Generalized Littlewood-Paley Theory

In this subsection, we review some of the known results from the generalized Littlewood-Paley theory. Here, the generalization is that one replaces the classical smoothness assumption by a so-called *quasi-orthogonality* condition, and one replaces the classical pointwise decay condition by off-diagonal decay in an L^2 sense.

First, we introduce the *square function norm* $|||\cdot|||$. We define

$$|||F|||_{\pm} := \Big(\iint_{\mathbb{R}^{n+1}_{\pm}} |F(x,t)|^2 \, \tfrac{dx\,dt}{t}\Big)^{1/2}, \; |||F|||_{all} := \Big(\iint_{\mathbb{R}^{n+1}} |F(x,t)|^2 \, \tfrac{dx\,dt}{t}\Big)^{1/2}.$$

For a family of linear operators on $L^2(\mathbb{R}^n)$, $\{\theta_t\}_{t>0}$, we define

$$|||\theta_t|||_{+,op} := \sup_{\|f\|_2=1} |||\theta_t f|||_+,$$

and similarly define $|||\theta_t|||_{-,op}$ and $|||\theta_t|||_{all,op}$. We will often drop the sign in the subscript when in context it is understood that we work in the upper half space.

Recall that a Borel measure μ on $\mathbb{R}^{n+1}_+$ is called *Carleson* if there exists a constant C such that $\mu(R_Q) \leq C|Q|$ for all cubes $Q \subset \mathbb{R}^n$, where $R_Q = Q \times (0, \ell(Q))$ is the

Carleson box above Q. Given a measurable function Υ on $\mathbb{R}^{n+1}_+$, we define

$$\|\Upsilon\|_{\mathcal{C}} := \sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Upsilon(x,t)|^2 \, \frac{dx\,dt}{t},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. In other words, $\|\Upsilon\|_{\mathcal{C}} < \infty$ if and only if $|\Upsilon(x,t)|^2 \frac{dx\,dt}{t}$ is a Carleson measure; in this case, we say that $\Upsilon \in \mathcal{C}$. There is a deep connection between Carleson measures and square function estimates, as seen in the $T1$ theorem for square functions of Christ and Journé [CJ87]. In this chapter, we use a generalized version of their result [GdlHH16, Theorem 4.3].

We record several results from [AAA$^+$11], which will be crucial in establishing square function estimates for solutions.

Definition 4.2.22 (Good off-diagonal decay)**.** We say that a family of linear operators on $L^2(\mathbb{R}^n)$, $\{\theta_t\}_{t>0}$, has *good off-diagonal decay* if there exist $M \geq 0$ and $C > 0$ such that for all $f \in L^2(\mathbb{R}^n)$, the estimate

$$\|\theta_t(f\mathbb{1}_{2^{k+1}Q\setminus 2^kQ})\|^2_{L^2(Q)} \lesssim_M 2^{-nk}\Big(\frac{t}{2^k\ell(Q)}\Big)^{2M+2}\|f\|^2_{L^2(2^{k+1}Q\setminus 2^kQ)}$$

holds for every cube $Q \subset \mathbb{R}^n$, every $k \geq 2$ and all $0 < t \leq C\ell(Q)$. Here, the implicit constants may depend only on dimension, M, and on the family of operators.

If $b \in L^\infty(\mathbb{R}^n)$, then for any cube Q in $\mathbb{R}^n$ and any $t \in (0, C\ell(Q))$, it can be shown via the good off-diagonal decay that $\theta_t(b\mathbb{1}_{\mathbb{R}^n\setminus Q}) \in L^2(Q)$. This allows us to define $\theta_t b := \theta(b\mathbb{1}_Q) + \theta_t(b\mathbb{1}_{\mathbb{R}^n\setminus Q}) \in L^2(Q)$ for any $t > 0$ and Q with $\ell(Q) \geq t/C$ (the independence of $\theta_t b$ over Q is given by the linearity). Thus, for $b \in L^\infty(\mathbb{R}^n)$, $\theta_t b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for each $t > 0$. We omit further details.

Lemma 4.2.23 (Consequences of off-diagonal decay; [FS72], [AAA$^+$11, Lemma 3.2])**.** *Suppose that $\{\theta_t\}_{t>0}$ is a family of linear operators on $L^2(\mathbb{R}^n)$ with good off-diagonal decay which verifies that $|||\theta_t|||_{op} \leq C$. Then, for every $b \in L^\infty(\mathbb{R}^n)$ (see the above remarks), the family $\{\theta_t\}_{t>0}$ satisfies the estimate*

$$\|\theta_t b\|_{\mathcal{C}} \lesssim (1 + |||\theta_t|||^2_{op})\|b\|^2_\infty.$$

Moreover, if $\|\theta_t\|_{L^2\to L^2} \lesssim 1$ *and* $\theta_t 1 = 0$ *for all* $t > 0$*, then for every* $b \in BMO(\mathbb{R}^n)$*,*

$$\|\theta_t b\|_{\mathcal{C}} \lesssim (1 + \||\theta_t\||_{op}^2)\|b\|_{BMO}^2.$$

Lemma 4.2.24 ([AAA⁺11, Lemma 3.11])**.** *Suppose that* $\{\theta_t\}_{t>0}$ *is a family of linear operators on* $L^2(\mathbb{R}^n)$ *with good off-diagonal decay and which satisfies* $\|\theta_t\|_{L^2\to L^2} \lesssim 1$ *for all* $t > 0$*. For each* $t > 0$*, let* $\mathcal{A}_t$ *denote a self-adjoint averaging operator on* $L^2(\mathbb{R}^n)$*, given as* $\mathcal{A}_t f = \int_{\mathbb{R}^n} f(y)\varphi_t(\cdot, y)\, dy$*, whose kernel satisfies*

$$0 \leq \varphi_t(x,y) \lesssim t^{-n}\mathbb{1}_{|x-y|\leq Ct}, \quad \textit{and} \quad \int_{\mathbb{R}^n} \varphi_t(x,y)\, dy = 1.$$

Then for each $t > 0$ *and any* $b \in L^\infty(\mathbb{R}^n)$*, the function* $\theta_t b$ *is well defined as an element of* $L^2_{\mathrm{loc}}(\mathbb{R}^n)$*, and we have that*

$$\sup_{t>0} \|(\theta_t b)\mathcal{A}_t f\|_{L^2(\mathbb{R}^n)} \lesssim \|b\|_\infty \|f\|_2.$$

Lemma 4.2.25 ([AAA⁺11, Lemma 3.5])**.** *Suppose that* $\{R_t\}_{t>0}$ *is a family of operators on* $L^2(\mathbb{R}^n)$ *with good off-diagonal decay, and suppose further that* $\|R_t\|_{L^2\to L^2} \lesssim 1$ *and* $R_t 1 = 0$ *for all* $t > 0$ *(note that by Lemma 4.2.24,* $R_t 1$ *is defined as an element of* $L^2_{\mathrm{loc}}(\mathbb{R}^n)$*). Then for each* $h \in \dot{W}^{1,2}(\mathbb{R}^n)$*, we have that*

$$\int_{\mathbb{R}^n} |R_t h|^2 \lesssim t^2 \int_{\mathbb{R}^n} |\nabla_x h|^2.$$

If, in addition, $\|R_t \operatorname{div}_x\|_{L^2\to L^2} \lesssim \frac{1}{t}$*, then we also have for each* $f \in L^2(\mathbb{R}^n)$ *that*

$$\int_{\mathbb{R}^{n+1}_+} |R_t f(x)|^2 \frac{dx\, dt}{t} \lesssim \|f\|_2^2.$$

The following definition is important in establishing quasi-orthogonality estimates (compare to the notion of an ϵ-family in [CJ87]).

Definition 4.2.26 (CLP Family)**.** We say that a family of convolution operators on $L^2(\mathbb{R}^n)$, $\{\mathcal{Q}_s\}_{s>0}$, is a *CLP family* ("Calderón-Littlewood-Paley" family), if there exist $\sigma > 0$ and $\psi \in L^1(\mathbb{R}^n)$ satisfying

$$|\psi(x)| \lesssim (1+|x|)^{-n-\sigma}, \qquad \text{and} \qquad |\hat{\psi}(\xi)| \lesssim \min(|\xi|^\sigma, |\xi|^{-\sigma}),$$

such that the following four statements hold.

i) The representation $\mathcal{Q}_s f = s^{-n}\psi(\cdot/s) * f$ holds for each $f \in L^2(\mathbb{R}^n)$.

ii) For each $f \in L^2(\mathbb{R}^n)$, we have control of the following L^2 norms uniformly in s:

$$\sup_{s>0} (\|\mathcal{Q}_s f\|_2 + \|s\nabla \mathcal{Q}_s f\|_2) \lesssim \|f\|_2.$$

iii) For each $f \in L^2(\mathbb{R}^n)$, we have the square function estimate

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathcal{Q}_s f(x)|^2 \frac{dx\, ds}{s} \lesssim \|f\|_2^2.$$

iv) Let $I : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the identity operator. The equation

$$\int_0^\infty \mathcal{Q}_s^2 \frac{ds}{s} = I$$

holds in the sense that the Bochner integrals $\int_\delta^R \mathcal{Q}_s^2 \frac{ds}{s}$ converge to I in the strong operator topology on $\mathcal{B}(L^2(\mathbb{R}^n))$ as $\delta \to 0$ and $R \to \infty$.

Proposition 4.2.27 (Qualitative mappings)**.** *Let $f \in Y^{1,2}(\mathbb{R}^n)$ and $\{\mathcal{Q}_s\}_{s>0}$ be either*

a) A standard Littlewood-Paley family as in Definition 4.2.26, with kernel ψ, with the additional condition that there exists $\sigma > 1$ such that $|\hat{\psi}(\xi)| \lesssim \min(|\xi|^\sigma, |\xi|^{-\sigma})$.

b) $\mathcal{Q}_s = I - P_s$, where P_s is a nice approximate identity.

Then for all $s > 0$, we have that $\mathcal{Q}_s f \in W^{1,2}(\mathbb{R}^n)$.

Proof. In either case, via Plancherel's Theorem, it will suffice to estimate the L^2 norm of $\widehat{\mathcal{Q}_s f}$. In case a), by basic properties of the Fourier Transform, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{\mathcal{Q}_s f}(\xi)|^2\, d\xi &= \int_{\mathbb{R}^n} |\hat{\psi}(s\xi)|^2 |\hat{f}(\xi)|^2\, d\xi \\ &\lesssim \int_{\mathbb{R}^n} \min(|s\xi|^{\sigma-1}, |s\xi|^{-\sigma-1})^2 |\xi|^2 |\hat{f}(\xi)|^2\, d\xi,\end{aligned}$$

whence the desired conclusion follows in this case. For case b), we similarly compute, using Plancherel's Theorem and the Fundamental Theorem of Calculus, that if φ is the radial kernel of the nice approximate identity P_s,

$$\int_{\mathbb{R}^n} |\widehat{\mathcal{Q}_s f}(\xi)|^2\,d\xi = \int_{\mathbb{R}^n} |1-\hat{\varphi}(s|\xi|)|^2|\hat{f}(\xi)|^2\,d\xi = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \Big|\int_0^{s|\xi|} \hat{\varphi}'(\tau)\,d\tau\Big|^2\,d\xi$$
$$\leq \int_{\mathbb{R}^n} s^2|\xi|^2|\hat{f}(\xi)|^2 \fint_0^{s|\xi|} |\hat{\varphi}'(\tau)|^2\,d\tau\,d\xi \leq s^2\|\hat{\varphi}'\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\xi|^2|\hat{f}(\xi)|^2\,d\xi.$$

□

4.3 Elliptic theory estimates

In this section, we establish several estimates for the operators under consideration, which are 'standard' in the elliptic theory. We begin with Caccioppoli-type estimates.

4.3.1 Caccioppoli-type inequalities

Let us first show

Proposition 4.3.1 (Caccioppoli inequality, [DHM18]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $f \in L^2_{\mathrm{loc}}(\Omega)$, $\vec{F} \in L^2_{\mathrm{loc}}(\Omega)^{n+1}$, and that $\mathcal{L}u = f - \operatorname{div}\vec{F}$ in Ω in the weak sense. Then, for every ball $B \subset 2B \subset \Omega$, the estimate*

$$\iint_B |\nabla u|^2 \lesssim \iint_{2B} \Big(\frac{1}{r(B)^2}|u|^2 + |\vec{F}|^2 + r(B)^2|f|^2\Big),$$

holds, with the implicit constant depending only on n and C_A.

The above estimate is a particular case of a Caccioppoli inequality obtained in a very general setting of elliptic systems in [DHM18]. Since our techniques will be exploited in several calculations later, we present here a self-contained proof.

Proof. Consider $\eta \in C_c^\infty(2B)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in B and $|\nabla\eta| \lesssim r(B)^{-1}$. Note that $u\eta^2$ is a valid testing function in (4.2.11), and therefore we obtain that

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}} C_A^{-1}|\nabla u|^2\eta^2 &\leq \iint_{\mathbb{R}^{n+1}} A\nabla u\cdot\overline{\nabla u}\eta^2 \\ &= \iint_{\mathbb{R}^{n+1}} \Big(-2(A\nabla u\cdot\nabla\eta)\eta\bar{u} + B_1 u\cdot\overline{\nabla(u\eta^2)} - B_2\cdot\nabla u\overline{u\eta^2}\Big) \\ &\qquad + \iint_{\mathbb{R}^{n+1}} \Big(\vec{F}\cdot\overline{\nabla(u\eta^2)} + f\overline{u\eta^2}\Big)\end{aligned}$$

$$=: I + II + III + IV + V.$$

To handle the term I, we use Cauchy's inequality with $\varepsilon > 0$ and the boundedness of A to obtain that

$$|I| \le 2C_A \iint_{\mathbb{R}^{n+1}} |\nabla u|\eta|\nabla\eta||u| \le C_A\varepsilon \iint_{\mathbb{R}^{n+1}} |\nabla u|^2\eta^2 + \frac{C_A}{\varepsilon} \iint_{\mathbb{R}^{n+1}} |u|^2|\nabla\eta|^2.$$

with ε small enough (depending only on C_A) that we can hide the first term. The second term is seen to be of a desired form after using the bound on $|\nabla\eta|$.

To handle the term III, we use the Hölder and Sobolev inequalities in $\mathbb{R}^n$ coupled with the $t-$independence of B_2, as follows:

$$\begin{aligned} |III| \le \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |B_2|(|\nabla u|\eta)|u|\eta\, dxdt \\ \le \int_{-\infty}^{\infty} \|B_2\|_{L^n(\mathbb{R}^n)}\|\eta\nabla u\|_{L^2(\mathbb{R}^n)}\|u\eta\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}\, dt \\ \lesssim \|B_2\|_{L^n(\mathbb{R}^n)} \int_{-\infty}^{\infty} \|\eta\nabla u\|_{L^2(\mathbb{R}^n)}\|\nabla_{\|}(u\eta)\|_{L^2(\mathbb{R}^n)}\, dt \\ \le \|B_2\|_{L^n(\mathbb{R}^n)} \int_{-\infty}^{\infty} \Big(\|\eta\nabla u\|^2_{L^2(\mathbb{R}^n)} + \|\eta\nabla u\|_{L^2(\mathbb{R}^n)}\|u\nabla\eta\|_{L^2(\mathbb{R}^n)}\Big)\, dt. \end{aligned}$$

Using the Cauchy inequality on the second term, we arrive at the estimate

$$|III| \lesssim \|B_2\|_n \iint_{\mathbb{R}^{n+1}} \left(|\nabla u|^2\eta^2 + |u|^2|\nabla\eta|^2\right).$$

If we choose $\|B_2\|_n < \varepsilon_0$ (see Proposition 4.2.19) with ε_0 small enough (depending only on n and C_A), we can hide the first term, while the second term is of a desired form.

To handle the term II, notice that the product rule allows us to write the estimate

$$|II| \le \iint_{\mathbb{R}^{n+1}} \left(|B_1||u||\nabla u|\eta^2 + 2|B_1||u|^2\eta|\nabla\eta|\right) =: II_1 + II_2.$$

The first term is handled similarly as III. As for II_2, we appeal again to the Hölder and Sobolev inequalities, together with the t-independence of B_1, to see that

$$|II| \lesssim \int_{-\infty}^{\infty} \|B_1\|_{L^n(\mathbb{R}^n)}\|u\nabla\eta\|_{L^2(\mathbb{R}^n)}\|u\eta\|_{L^{2n/n-2}(\mathbb{R}^n)}\, dt$$

$$\lesssim \|B_1\|_{L^n(\mathbb{R}^n)} \int_{-\infty}^{\infty} \|u\nabla\eta\|_{L^2(\mathbb{R}^n)} \|\nabla_\|(u\eta)\|_{L^2(\mathbb{R}^n)}\, dt,$$

and this last expression may be handled in the same way as in II.

For the term IV, we use the product rule to obtain that

$$|IV| \leq \iint_{\mathbb{R}^{n+1}} \big(|\vec{F}|\eta|\nabla u|\eta + 2|\vec{F}|\eta|u||\nabla\eta|\big) =: IV_1 + IV_2.$$

The first term may be estimated with the Cauchy's inequality with ε:

$$IV_1 \leq \iint_{2B} \Big(\frac{1}{\varepsilon}|\vec{F}|^2 + \varepsilon|\nabla u|^2\eta^2\Big),$$

and we can hide the second term. The term IV_2, since after using the Cauchy inequality, both terms are of a desired form:

$$IV_2 \leq \iint_{2B} \big(|\vec{F}|^2 + |u|^2|\nabla\eta|^2\big).$$

Combining these estimates gives

$$\iint_B |\nabla u|^2 \leq \iint_{\mathbb{R}^{n+1}} |\nabla u|^2\eta^2 \lesssim \frac{1}{r(B)^2} \iint_{2B} \Big(|u|^2 + |\vec{F}|^2\Big) + |V|.$$

To handle the term V, we use the Cauchy inequality to obtain that

$$|V| \leq \iint_{\mathbb{R}^{n+1}} |f||u|\eta^2 \leq \iint_{2B} \Big(r(B)^2|f|^2 + \frac{1}{r(B)^2}|u|^2\Big).$$

This completes the proof. □

Remark 4.3.2 ($Y^{1,p}$ form a complex interpolation scale). In the case of purely second order operators (that is, $B_1 = B_2 = 0$), we may exploit the fact that constants are always null-solutions. Applying the Poincaré inequality, we obtain a weak reverse Hölder inequality for ∇u, which in particular implies L^p integrability for the gradient, for some $p > 2$. We do not obtain the analogous estimate here, but rather a suitable substitute. More precisely, we shall muster an L^p version of the Caccioppoli inequality. In order to prove this result, we remark that the spaces $Y^{1,p}(\mathbb{R}^{n+1})$ and their dual spaces, $(Y^{1,p})^*$, form a complex interpolation scale, with

$$[Y^{1,p_1}, Y^{1,p_2}]_\theta = Y^{1,p_\theta}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$

for $\theta \in (0,1)$ and $1 < p_1 < p_2 < n$. We may show this fact by gathering the following two ingredients. First, the homogeneous spaces $\dot{W}^{1,p}$ form a complex interpolation scale (see [Tri95]). Next, one uses that the map that sends an element in $\dot{W}^{1,p}$ to its unique representative in $Y^{1,p}$ is a 'retract' (see [KMM07, Lemma 7.11] and the discussion preceding it). Thus, we employ [KMM07, Lemma 7.11] and conclude that the spaces $Y^{1,p}$ form a complex interpolation scale. The fact that $(Y^{1,p})^*$ form a complex interpolation scale is a general consequence of the interpolation scale for $Y^{1,p}$; see, for instance, [BL76, Theorem 4.5.1].

The L^p Caccioppoli inequality will also make use of the well-known lemma of Šneĭberg [Šne74]. The (explicitly) quantitative version stated here appears in [ABES19].

Theorem 4.3.3 (Šneĭberg's Lemma [ABES19, Theorem A.1],[Šne74]). *Let $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ be interpolation couples of Banach spaces, and $T \in \mathcal{B}(\overline{X}, \overline{Y})$. Suppose that for some $\theta^* \in (0,1)$ and some $\kappa > 0$, the lower bound $\|Tx\|_{Y_{\theta^*}} \geq \kappa \|x\|_{X_{\theta^*}}$ holds for all $x \in X_{\theta^*}$. Then the following statements are true.*

(i) Given $0 < \varepsilon < 1/4$, the lower bound $\|Tx\|_{Y_\theta} \geq \varepsilon\kappa\|x\|_{X_\theta}$ holds for all $x \in X_\theta$, provided that $|\theta - \theta^*| \leq \frac{\kappa(1-4\varepsilon)\min\{\theta^*, 1-\theta^*\}}{3\kappa + 6M}$, where $M = \max_{j=0,1} \|T\|_{X_j \to Y_j}$.

(ii) If $T : X_{\theta^*} \to Y_{\theta^*}$ is invertible, then the same is true for $T : X_\theta \to Y_\theta$ if θ is as in (i). The inverse mappings agree on $X_\theta \cap X_{\theta^*}$ and their norms are bounded by $\frac{1}{\varepsilon\kappa}$.

Using the above result, we can easily obtain

Lemma 4.3.4 (Invertibility of $\mathcal{L}$ in a window around 2). *Let $p \in (1, n)$ be such that $p' < n$, where p' is the Hölder conjugate of p. The operator $\mathcal{L}$ extends to a bounded operator $Y^{1,p}(\mathbb{R}^{n+1}) \to (Y^{1,p'}(\mathbb{R}^{n+1}))^*$. Moreover, the operator is invertible if $|p-2|$ is small enough depending on n and C_A.*

Remark 4.3.5. Here and throughout, we assume that the range of p near 2 in Lemma 4.3.4 is such that $p_* = \frac{(n+1)p}{n+1+p} < 2$.

The following lemma details the modification to the operator output upon multiplying a solution by a cut-off function.

Lemma 4.3.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Suppose that $u \in W^{1,2}_{\text{loc}}(\Omega)$ satisfies $\mathcal{L}u = 0$ in Ω in the weak sense. Then for any $\chi \in C_c^\infty(\Omega, \mathbb{R})$, we have that*

$$\mathcal{L}(\chi u) = \operatorname{div} \vec{F} + f \tag{4.3.7}$$

in $\mathbb{R}^{n+1}$ in the weak sense, where $\vec{F} = A(\nabla\chi)u$, and $f = -A\nabla u \cdot \nabla\chi - B_1 u\nabla\chi + B_2 u\nabla\chi$.

Proof. We apply the operator $\mathcal{L}$ to $u\chi$ and test against $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ with the goal in mind of extracting a term of the form $\langle \mathcal{L}u, \varphi\chi\rangle = 0$. Observe that

$$\int_{\mathbb{R}^{n+1}} A\nabla(u\chi)\cdot\overline{\nabla\varphi} = \int_{\mathbb{R}^{n+1}} A\nabla u\cdot\overline{\nabla(\chi\varphi)} + \int_{\mathbb{R}^{n+1}} uA\nabla\chi\cdot\overline{\nabla\varphi} - \int_{\mathbb{R}^{n+1}} [A\nabla u\cdot\nabla\chi]\overline{\varphi},$$
$$\int_{\mathbb{R}^{n+1}} (B_1 u\chi)\cdot\overline{\nabla\varphi} = + \int_{\mathbb{R}^{n+1}} B_1 u\overline{\nabla(\chi\varphi)} - \int_{\mathbb{R}^{n+1}} [B_1 u\nabla\chi]\overline{\varphi},$$
$$\int_{\mathbb{R}^{n+1}} B_2\nabla(u\chi)\overline{\varphi} = \int_{\mathbb{R}^{n+1}} B_2\nabla u\overline{\chi\varphi} + \int_{\mathbb{R}^{n+1}} [B_2 u\nabla\chi]\overline{\varphi},$$

where we use that χ is real-valued. Collecting the first terms in each inequality and noting that $\varphi\chi \in C_c^\infty(\Omega)$, we realize that the contribution of these terms is $\langle \mathcal{L}u, \varphi\chi\rangle = 0$. Then we have that $\langle \mathcal{L}(\chi u), \varphi\rangle = \langle \operatorname{div} \vec{F} + f, \varphi\rangle$, as desired. □

We are now ready to combine the past few results and obtain the local high integrability of the gradient.

Lemma 4.3.8 (Local high integrability of the gradient of a solution)**.** *Let Ω be an open set. Suppose that $u \in W^{1,2}_{\text{loc}}(\Omega)$ solves $\mathcal{L}u = 0$ in Ω in the weak sense. Then $u \in W^{1,p}_{\text{loc}}(\Omega)$, where p is close to 2 and depends only on n, C_A and ε_0. Moreover, for any $\chi \in C_c^\infty(\Omega, \mathbb{R})$ we have the estimate*

$$\|\chi u\|_{Y^{1,p}(\mathbb{R}^{n+1})} \leq \|\mathcal{L}^{-1}(\operatorname{div} \vec{F} + f)\|_{Y^{1,p}(\mathbb{R}^{n+1})} \lesssim \|\vec{F}\|_p + \|f\|_{p_*},$$

where $\vec{F}$ and f are as in Lemma 4.3.6.

Proof. Let $\vec{F}$ and f be as in the previous lemma. One may verify, using the Sobolev embedding and the fact that χ is smooth and compactly supported, that $\vec{F} \in L^1(\mathbb{R}^{n+1}) \cap L^{2^*}(\mathbb{R}^{n+1})$ and that $f \in L^1(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$. Choosing $p > 2$ with $|p-2|$ sufficiently small, we may apply Lemma 4.3.4 to show that the operator $\mathcal{L}$ extends to a bounded and invertible operator $Y^{1,p}(\mathbb{R}^{n+1}) \to (Y^{1,p'}(\mathbb{R}^{n+1}))^*$. Hence $\mathcal{L}^{-1}$ is bounded. Applying $\mathcal{L}^{-1}$ to each side of (4.3.7), we obtain that

$$\|\chi u\|_{Y^{1,p}} \leq \|\mathcal{L}^{-1}(\operatorname{div} \vec{F} + f)\|_{Y^{1,p}} \lesssim \|\vec{F}\|_p + \|f\|_{p_*}.$$

Here, we note that L^{p_*} embeds continuously into $(Y^{1,p'})^*$, and $\operatorname{div} \vec{F} \in (Y^{1,p'})^*$ since $\vec{F} \in L^p$. This observation uses the identity $[(p')^*]' = p_*$ and the continuous embedding $Y^{1,p'}(\mathbb{R}^{n+1}) \hookrightarrow L^{(p')^*}(\mathbb{R}^{n+1})$. $\square$

Finally, we provide a more precise version of the above Lemma, namely the L^p-Caccioppoli inequality.

Proposition 4.3.9 (L^p-Caccioppoli inequality)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $u \in W^{1,2}_{\text{loc}}(\Omega)$ solve $\mathcal{L}u = 0$ in Ω in the weak sense. Suppose that B is a ball such that $\kappa B \subset \Omega$ for some $\kappa > 1$. Then, for every $p > 0$ such that $|p-2|$ is small enough that the conditions of Lemma 4.3.8 are satisfied, the estimate*

$$\|\nabla u\|_{L^p(B)} \lesssim \frac{1}{r(B)} \|u\|_{L^p(\kappa B)} \tag{4.3.10}$$

holds, where the implicit constants depend on κ, p, n, C_A, and ε_0.

Proof. Set $r := r(B)$ and let $\chi = \eta^2$ with $\eta \in C_c^\infty(\frac{1+\kappa}{2}B, \mathbb{R})$, $0 \leq \eta \leq 1$, $|\nabla \eta| \lesssim \frac{1}{r}$. Note that χ has the same properties as η. The estimate (4.3.10) will follow immediately from the estimate

$$\|u\chi\|_{Y^{1,p}(\mathbb{R}^{n+1})} \lesssim \frac{1}{r} \|u\|_{L^p(\kappa B)}, \tag{4.3.11}$$

since $\|\nabla u\|_{L^p(B)} \lesssim \|(\nabla u)\chi\|_p$ and (the reverse triangle inequality yields)

$$\|(\nabla u)\chi\|_p - \|(\nabla \chi) u\|_p \lesssim \|\nabla(u\chi)\|_p \leq \|u\chi\|_{Y^{1,p}(\mathbb{R}^{n+1})}.$$

We immediately note that we have already established (4.3.11) in the case $p = 2$; this is the classical Caccioppoli inequality. Applying Lemma 4.3.8, we have that

$$\|\chi u\|_{Y^{1,p}(\mathbb{R}^{n+1})} \lesssim \|\vec{F}\|_p + \|f\|_{p_*}, \tag{4.3.12}$$

where $\vec{F}$ and f are as in Lemma 4.3.6. The bound

$$\|\vec{F}\|_p = \|A\nabla\chi u\|_p \lesssim \frac{1}{r}\|u\|_{L^p(\kappa B)} \tag{4.3.13}$$

is trivial from the properties of A and χ and desirable from the standpoint of (4.3.11). It remains to find appropriate bounds for the terms appearing in the expression for f. To this end, we have by Minkowski's inequality that

$$\|f\|_{p_*} \leq \|A\nabla u\cdot\nabla\chi\|_{p_*} + \|B_1 u\nabla\chi\|_{p_*} + \|B_2 u\nabla\chi\|_{p_*} = I + II + III.$$

Before continuing, we remark that the relation $\frac{n+1}{p_*} = \frac{n+1}{(n+1)p}[(n+1)+p] = \frac{n+1}{p}+1$

holds. Using the L^2 Caccioppoli inequality, Jensen's inequality and the fact that $p > 2$, we have that

$$\begin{aligned} I = \|A\nabla u\cdot\nabla\chi\|_{p_*} &\lesssim r^{\frac{n+1}{p}}\left(\fint_{\frac{1+\kappa}{2}B}|\nabla u|^2\right)^{\frac{1}{2}} \lesssim \frac{1}{r}r^{\frac{n+1}{p}}\left(\fint_{\kappa B}|u|^2\right)^{\frac{1}{2}} \\ &\lesssim \frac{1}{r}r^{\frac{n+1}{p}}\left(\fint_{\kappa B}|u|^p\right)^{\frac{1}{p}} \lesssim \frac{1}{r}\Big(\int_{\kappa B}|u|^p\Big)^{\frac{1}{p}}. \end{aligned} \tag{4.3.14}$$

Next we bound II and III. The Sobolev embedding on $\mathbb{R}^n$ and the Caccioppoli inequality [1] yield for $i = 1, 2$ the estimate

$$\begin{aligned} \|B_i u(\nabla\chi)\|_{p_*} &\lesssim \frac{1}{r}\|B_i(u\eta)\|_{p_*} \lesssim \frac{1}{r}r^{\frac{n+1}{p_*}}\left(\fint_{\frac{1+\kappa}{2}B}|B_i(u\eta)|^{p_*}\right)^{\frac{1}{p_*}} \\ &\lesssim \frac{1}{r}r^{\frac{n+1}{p_*}}r^{-\frac{n+1}{2}}\Big(\int_{\frac{1+\kappa}{2}B}|B_i(u\eta)|^2\Big)^{\frac{1}{2}} \lesssim \frac{1}{r}r^{\frac{n+1}{p_*}}r^{-\frac{n+1}{2}}\Big(\int_{\mathbb{R}^{n+1}}|\nabla(u\eta)|^2\Big)^{\frac{1}{2}} \\ &\lesssim \frac{1}{r}r^{\frac{n+1}{p}}\left(\fint_{\kappa B}|u|^2\right)^{\frac{1}{2}} \lesssim \frac{1}{r}\Big(\int_{\kappa B}|u|^p\Big)^{\frac{1}{p}}. \end{aligned} \tag{4.3.15}$$

Combining (4.3.13), (4.3.14) and (4.3.15) with (4.3.12) and the definitions of $\vec{F}$ and f, we obtain (4.3.11). As we had reduced the proof of the statement of the Proposition to (4.3.11), we have thus shown our claim. $\square$

[1] More precisely, we use (4.3.11) with $p = 2$.

4.3.2 Properties of solutions and their gradients on slices

Our next goal is to study the t-regularity of our solutions as well as their properties on 'slices', which are sets of the form $\{(x,t) : t = t_0\}$. Let us first note that $t-$derivatives of solutions are solutions.

Proposition 4.3.16 (The t-derivatives of solutions are solutions). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, let $f, \vec{F} \in L^2_{\mathrm{loc}}(\Omega)$, and suppose that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies $\mathcal{L}u = f - \operatorname{div} \vec{F}$ in Ω in the weak sense. Assume further that $f_t := \partial_t f \in L^2_{\mathrm{loc}}(\Omega)$ and $\vec{F}_t := \partial_t \vec{F} \in L^2_{\mathrm{loc}}(\Omega)$. Then the function $v = \partial_t u$ lies in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies $\mathcal{L}v = f_t - \operatorname{div} F_t$ in Ω in the weak sense.*

Proof. Fix a ball $B \subset 2B \subset \Omega$ and consider the difference quotients

$$u_h := \frac{u(\cdot + he_{n+1}) - u(\cdot)}{|h|}, \qquad |h| < \operatorname{dist}(B, \partial\Omega).$$

We define f_h and $\vec{F}_h$ similarly. By t-independence of the coefficients, we have that $\mathcal{L}u_h = f_h - \operatorname{div} \vec{F}_h$ in B for any such h. By the Caccioppoli inequality (Proposition 4.3.1), we obtain that for any h as above,

$$\iint_B |\nabla u_h|^2 \lesssim \iint_{2B} \Big(\frac{1}{r(B)^2}|u_h|^2 + |\vec{F}_h|^2 + r(B)^2|f_h|^2\Big)$$
$$\lesssim \iint_{2B} \Big(\frac{1}{r(B)^2}|\partial_t u|^2 + |\vec{F}_t|^2 + r(B)^2|f_t|^2\Big).$$

In particular, the difference quotients of ∇u are bounded, which implies that $\partial_t u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Consequently, we must have that the difference quotients u_h converge weakly (in $W^{1,2}_{\mathrm{loc}}(\Omega)$) to $v = \partial_t u$ (and similarly for f_h and $\vec{F}_h$). From (4.2.11) and the fact that $\mathcal{L}u_h = f_h - \operatorname{div} \vec{F}_h$, we conclude that $\mathcal{L}v = f_t - \operatorname{div} \vec{F}_t$, as desired. $\square$

We now check that $t-$derivatives of solutions are well-behaved on horizontal strips.

Lemma 4.3.17 (Good integrability of the t-derivative of a solution on a strip). *Denote $\Sigma_a^b := \big\{(x,t) \in \mathbb{R}^{n+1} : a < t < b\big\}$. Suppose that u and $v := \partial_t u$ are as in Proposition 4.3.16 with $\Omega = \Sigma_a^b$, and suppose further that $v \in L^2(\Sigma_a^b)$. Then $\nabla v \in L^2(\Sigma_{a'}^{b'})$ for each $a < a' < b' < b$.*

Proof. Let $\chi_R = \phi(x)\psi(t)$ be a product of infinitely smooth cut-off functions with $0 \le \phi_R, \psi \le 1$, $\psi \equiv 1$ on (a', b'), $\psi \in C_c^\infty(a, b)$, and $\phi_R \equiv 1$ on B_R, $\phi_R \in C_c^\infty(B_{2R})$. Then, for all $R \gg \min\{a' - a, b - b'\}$, we claim that

$$\begin{aligned}\int_{a'}^{b'} \int_{B_R} |\nabla v|^2 \, dx \, dt &\lesssim \iint_{\mathbb{R}^{n+1}} \chi_R^2 |\nabla v|^2 \\ &\lesssim \iint_{\mathbb{R}^{n+1}} \Big(|v|^2 + |\vec{F}_t|^2 + |f_t|^2 \Big) \big(|\nabla \chi_R|^2 + 1 \big) \\ &\lesssim \frac{1}{(\min\{a' - a, b - b', 1\})^2} \int_a^b \int_{\mathbb{R}^n} \Big(|v|^2 + |\vec{F}_t|^2 + |f_t|^2 \Big).\end{aligned}$$

We provide the details of the second line in a moment; note that in the third line we used that the dominant contribution for the gradient of χ_R is its t component when R is large. Sending $R \to \infty$ finishes the proof modulo the aforementioned line.

To see the computation above, denote $\chi := \chi_R$ and observe that

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}} \chi^2 |\nabla v|^2 &\lesssim \iint_{\mathbb{R}^{n+1}} \chi^2 \Re e \big(A \nabla v \overline{\nabla v} \big) \\ &\le \Re e \Big[\iint_{\mathbb{R}^{n+1}} A \nabla v \overline{\nabla (v \chi^2)} - 2 \iint_{\mathbb{R}^{n+1}} \chi \overline{v} A \nabla v \nabla \chi \Big] =: \Re e [I + II].\end{aligned}$$

Clearly,

$$|II| \lesssim \epsilon \iint_{\mathbb{R}^{n+1}} |\chi \nabla v|^2 + \frac{1}{\epsilon} \iint_{\mathbb{R}^{n+1}} |\nabla \chi v|^2,$$

and the first term can be absorbed to the left-hand side. It remains to handle I. We use the equation $\mathcal{L} v = f_t - \operatorname{div} \vec{F}_t$ to write $I = I_1 + I_2 + I_3 + I_4$, where each I_j is a term of the equation and each will be given explicitly below. First, note that

$$|I_4| := \Big| \iint_{\mathbb{R}^{n+1}} f_t \overline{v} \chi^2 \Big| \lesssim \iint_{\mathbb{R}^{n+1}} |v \chi|^2 + \iint_{\mathbb{R}^{n+1}} |f_t \chi|^2,$$

which handles this term. Next, we have that

$$|I_3| := \Big| \iint_{\mathbb{R}^{n+1}} \vec{F}_t \overline{\nabla (v \chi^2)} \Big| \lesssim \iint_{\mathbb{R}^{n+1}} |\vec{F}_t \nabla v \chi|^2 + \iint_{\mathbb{R}^{n+1}} |\vec{F}_t \chi \nabla \chi v|.$$

We handle the first term as in II, and we handle the second term as I_4. Moving on, we see that

$$|I_1| := \Big| \iint_{\mathbb{R}^{n+1}} B_1 v \overline{\nabla(v\chi^2)} \Big| \lesssim \iint_{\mathbb{R}^{n+1}} |(B_1 v\chi)\nabla v\chi| + \iint_{\mathbb{R}^{n+1}} |(B_1 v\chi)\nabla \chi v|.$$

Both of the terms above are handled by using the smallness of B_1 as in the proof of the Caccioppoli inequality. Now, for the last term, we have that

$$|I_2| := \Big| \iint_{\mathbb{R}^{n+1}} B_2 \nabla v \chi^2 \overline{v} \Big| \lesssim \iint_{\mathbb{R}^{n+1}} \big|(B_2 \chi v)\nabla v \chi\big|,$$

so that we may handle this term exactly as we did I_1. □

Remark 4.3.18. We may bring the above lemma and Lemma 4.2.3 together to conclude that if u solves $\mathcal{L}u = 0$ in Σ_a^b, then automatically we have the transversal Hölder continuity of its gradient, and $u \in C^{\alpha'}_{\text{loc}}\Big((a,b), L^{\frac{2n}{n-2}}(\mathbb{R}^n)\Big)$ for some $\alpha > 0$.

Next, we present a formula for our equation on a slice. Recall that $\tilde{A}$ denotes the $(n+1) \times n$ submatrix of A consisting of the first n columns of A.

Proposition 4.3.19 (Integration by parts on slices for $\mathcal{L}$)**.** *Let $u \in Y^{1,2}(\Sigma_a^b)$ and suppose that $\mathcal{L}u = g$ in Σ_a^b for some $g \in C_c^\infty(\mathbb{R}^{n+1})$. Then, for every $t \in (a,b)$ and $\varphi \in W^{1,2}(\mathbb{R}^n)$, the identity*

$$\int_{\mathbb{R}^n} \Big((A(x)\nabla u(x,t))_\| + (B_1)_\| u(x,t)\Big) \cdot \overline{\nabla_\| \varphi(x)}\, dx + \int_{\mathbb{R}^n} B_2(x) \cdot \nabla u(x,t) \overline{\varphi(x)}\, dx$$
$$= \int_{\mathbb{R}^n} \Big(\vec{A}_{n+1,\cdot}(x) \cdot \partial_t \nabla u(x,t) + (B_1(x))_\perp \partial_t u(x,t)\Big) \overline{\varphi(x)}\, dx + \int_{\mathbb{R}^n} g(x,t) \overline{\varphi(x)}\, dx$$

holds. If $v, \partial_t v \in Y^{1,2}(\Sigma_a^b)$, and $\mathcal{L}^* v = 0$ in Σ_a^b for some $g \in C_c^\infty(\mathbb{R}^n)$, then for every $t \in (a,b)$ and $\varphi \in W^{1,2}(\mathbb{R}^n)$, the identity

$$\int_{\mathbb{R}^n} \Big[\nabla_\| \varphi \cdot \overline{((\overline{B}_2)_\| v(t))} + \tilde{A} \nabla_\| \varphi \cdot \overline{\nabla v(t)} + B_1 \varphi \cdot \overline{\nabla v(t)}\Big]$$
$$= \int_{\mathbb{R}^n} \Big[\varphi \overline{(\overline{B}_2)_\perp D_{n+1} v(t)} + \varphi \vec{A}_{\cdot, n+1} \overline{\nabla D_{n+1} v(t)}\Big]$$

holds. Finally, for v and φ as above, we also have the identity

$$\int_{\mathbb{R}^n} \nabla_\| \varphi \cdot \overline{(A^* \nabla v(t))_\|} = \int_{\mathbb{R}^n} \varphi \cdot \overline{\vec{A}^*_{n+1,\cdot} D_{n+1} \nabla v(t)} - \int_{\mathbb{R}^n} \nabla_\| \varphi \cdot \overline{(\overline{B}_2)_\| v(t)}$$

$$+\int_{\mathbb{R}^n} \varphi\overline{(\overline{B}_2)_\perp v(t)} - \int_{\mathbb{R}^n} \varphi\overline{\overline{B}_1 \cdot \nabla v(t)}.$$

Proof. Fix $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $t \in (a,b)$. Let $\varphi_\varepsilon(x,s) := \varphi(x)\eta_\varepsilon(t-s)$ with $\varepsilon < \min\{b-t, t-a\}$, and where $\eta_\varepsilon(\cdot) = \varepsilon^{-1}\eta(\cdot/\varepsilon)$, $\eta \in C_c^\infty(-1,1)$, $\int_{\mathbb{R}} \eta = 1$. In particular, $\varphi_\varepsilon \in C_c^\infty(\Sigma_a^b)$ is an admissible test function in the definition of the weak solution. Thus, from the definition of $\mathcal{L}u = g$, we have that

$$\begin{aligned}
&\iint_{\mathbb{R}^{n+1}} \Big\{ \Big(\big(A(x)\nabla u(x,s)\big)_\| + (B_1)_\| u(x,s) \Big) \cdot \overline{\nabla_\| \varphi_\varepsilon(x,s)} \\
&\qquad\qquad\qquad + B_2(x) \cdot \nabla u(x,s) \overline{\varphi_\varepsilon(x,s)} \Big\}\, dxds \\
&= \iint_{\mathbb{R}^{n+1}} \Big(\vec{A}_{n+1,\cdot}(x)\partial_s \nabla u(x,s) + (B_1(x))_\perp \partial_s u(x,s) + g(x,s) \Big) \overline{\varphi_\varepsilon(x,s)}\, dxds.
\end{aligned}$$

Notice, for instance, that the map

$$t \mapsto \int_{\mathbb{R}^n} \Big(\big(A(x)\nabla u(x,t)\big)_\| + (B_1)_\|(x) u(x,t) \Big) \cdot \overline{\nabla_\| \varphi(x)}\, dx$$

is continuous in (a,b), owing to Lemma 4.2.3 and the continuity of the duality pairings in each of its entries. A similar statement holds for all the other integrals. The desired conclusion now follows from the fact that for any continuous function $h : (a,b) \to \mathbb{C}$, we have that $\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \eta_\varepsilon(t-\cdot)h = h(t)$, for each $t \in (a,b)$. ☐

As in [AAA+11], but now employing Lemma 4.3.9, the $t-$independence of our coefficients allows us to obtain L^p estimates on cubes lying in horizontal slices.

Lemma 4.3.20 (L^p estimates on slices; [AAA+11, Proposition 2.1])**.** *Let $t \in \mathbb{R}$, $Q \subset \mathbb{R}^n$ be a cube, and I_Q be the box $I_Q = 4Q \times (t - \ell(Q), t + \ell(Q))$. Let $p \geq 2$ with $|p-2|$ small enough that the conclusion of Lemma 4.3.4 holds. Suppose that $u \in W^{1,2}(I_Q)$ satisfies $\mathcal{L}u = 0$ in I_Q. Then the estimates*

$$\Big(\frac{1}{|Q|} \int_Q |\nabla u(x,t)|^p \Big)^{1/p} \lesssim \Big(\frac{1}{|Q'|} \iint_{Q'} |\nabla u(x,t)|^p \Big)^{1/p}, \tag{4.3.21}$$

and

$$\Big(\frac{1}{|Q|} \int_Q |\nabla u(x,t)|^p \Big)^{1/p} \lesssim_p \frac{1}{\ell(Q)} \Big(\frac{1}{|Q''|} \iint_{Q''} |u(x,t)|^p \Big)^{1/p} \tag{4.3.22}$$

hold, where $Q' := 2Q \times (t - \ell(Q)/4, t + \ell(Q)/4)$ is an $(n+1)-$dimensional rectangle, *and* $Q'' := 3Q \times (t - \ell(Q)/2, t + \ell(Q)/2)$ *is a slight dilation of* Q'.

In [AAA+11], the analogue of the preceding lemma is proved in the purely second order case. However, the argument there extends almost verbatim to the present situation, given Lemma 4.3.9. We omit the details.

Let us consider how the shift operator acts on $\mathcal{L}^{-1}$. For each $\tau \in \mathbb{R}$, denote by $\mathscr{T}^\tau$ the (positive) *shift by* τ in the $t-$direction: If $u \in C_c^\infty(\mathbb{R}^{n+1})$, then $(\mathscr{T}^\tau u) = u(\cdot, \cdot + \tau)$. More generally, if $f \in \mathscr{D}'$ is a distribution, we define the distribution $\mathscr{T}^\tau f$ by $\langle \mathscr{T}^\tau f, \varphi \rangle = \langle f, \mathscr{T}^{-\tau} \varphi \rangle$, for each $\varphi \in \mathscr{D}$.

Proposition 4.3.23. *Suppose that* $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ *solves* $\mathcal{L}u = 0$ *in* $\mathbb{R}^{n+1}_+$. *Then*

(i) Let $f \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ *and fix* $s \in \mathbb{R}$. *Then* $\mathscr{T}^s \mathcal{L}^{-1} f \in Y^{1,2}(\mathbb{R}^{n+1})$ *and satisfies* $\mathscr{T}^s \mathcal{L}^{-1} f = \mathcal{L}^{-1} \mathscr{T}^s f$.

(ii) Let $s > 0$. *Then* $\mathscr{T}^s u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ *and* $\mathcal{L}\mathscr{T}^s u = 0$ *in* $\mathbb{R}^{n+1}_+$.

(iii) We have that $D_{n+1}u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ *and* $\mathcal{L}D_{n+1}u = 0$ *in* $\mathbb{R}^{n+1}_+$.

(iv) For any $s > 0$, *we have that* $D_{n+1}\mathscr{T}^s u \in Y^{1,2}(\mathbb{R}^{n+1}_+) \cap L^2(\mathbb{R}^{n+1}_+) = W^{1,2}(\mathbb{R}^{n+1}_+)$. *In particular, for any* $t > 0$, *the trace* $\mathrm{Tr}_t\, D_{n+1}u$ *is an element of* $H^{\frac{1}{2}}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap H_0^{\frac{1}{2}}(\mathbb{R}^n)$. *Moreover, for each* $t > 0$, *the estimate*

$$\|t\, \mathrm{Tr}_t\, \nabla \partial_t u\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{Y^{1,2}(\mathbb{R}^{n+1}_{t/2})} \tag{4.3.24}$$

holds. In particular, for each $s > 0$ *we have that*

$$\sup_{t \geq 0} \|(t+s)\, \mathrm{Tr}_t\, \nabla \partial_t \mathscr{T}^s u\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)}. \tag{4.3.25}$$

Finally, for each $t > 0$ *and* $\zeta \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, *we have the identity*

$$(\mathrm{Tr}_t\, D_{n+1}u, \zeta) = \frac{d}{dt}(\mathrm{Tr}_t\, u, \zeta). \tag{4.3.26}$$

Proof. The proofs of (i)), (ii)), and (iii)) are very similar to the proof of Proposition 4.3.16, and are thus omitted. We prove (iv)), and to this end fix $s > 0$. By assumption, it is clear that $\mathscr{T}^s u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$, and by (ii)), we have that $\mathcal{L}\mathscr{T}^s u = 0$ in $\mathbb{R}^{n+1}_+$. Hence, by (iii)), we have that $D_{n+1}\mathscr{T}^s u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and $\mathcal{L}D_{n+1}\mathscr{T}^s u = 0$ in $\mathbb{R}^{n+1}_+$. Let $\mathbb{G}(s/2)$ be a grid of pairwise disjoint cubes $R \subset \mathbb{R}^{n+1}_s$ such that $\mathbb{R}^{n+1}_s = \cup_{R \in \mathbb{G}(s/2)} R$

and $\ell(R) = \frac{s}{2}$. Consider the estimate

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}_+} |\nabla D_{n+1} \mathscr{T}^s u|^2 &= \iint_{\mathbb{R}^{n+1}_s} |\nabla D_{n+1} u|^2 = \sum_{R \in \mathbb{G}(s/2)} \iint_R |\nabla D_{n+1} u|^2 \\ &\lesssim \sum_{R \in \mathbb{G}(s/2)} \frac{1}{s^2} \iint_{\tilde{R}} |D_{n+1} u|^2 \lesssim \frac{1}{s^2} \|D_{n+1} u\|^2_{L^2(\mathbb{R}^{n+1}_{s/2})} \leq \frac{1}{s^2} \|u\|^2_{Y^{1,2}(\mathbb{R}^{n+1}_+)},\end{aligned}$$

which proves that $\nabla D_{n+1} \mathscr{T}^s u \in L^2(\mathbb{R}^{n+1}_+)$. Since $D_{n+1} \mathscr{T}^s u \in L^2(\mathbb{R}^{n+1}_+)$ by the assumption that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$, it is proven that $D_{n+1} \mathscr{T}^s u \in W^{1,2}(\mathbb{R}^{n+1}_+)$. Hence, for each $t \geq 0$, $\operatorname{Tr}_t D_{n+1} \mathscr{T}^s u \in H^{\frac{1}{2}}(\mathbb{R}^n)$. But $\operatorname{Tr}_t D_{n+1} \mathscr{T}^s u = \operatorname{Tr}_{t+s} D_{n+1} u$. The estimate (4.3.24) is true by Caccioppoli on slices (Proposition 4.3.20), as follows: break $\mathbb{R}^n$ into a grid $\mathbb{G}_n(t/2)$ of cubes $Q \subset \mathbb{R}^n$, $\ell(Q) = t/2$, and use Caccioppoli on slices in each cube.

It remains to check the identity (4.3.26), so fix $t > 0$. We have seen that $\operatorname{Tr}_\tau D_{n+1} u \in H^{\frac{1}{2}}_0(\mathbb{R}^n)$ for each $\tau > 0$. Fix $\zeta \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, and define $g(\tau) := (\operatorname{Tr}_\tau u, \zeta)$ for each $\tau > 0$. We will show that g is differentiable at t, and compute its derivative. To this end, note that

$$\tfrac{g(t+h) - g(t)}{h} = \tfrac{(\operatorname{Tr}_{t+h} u, \zeta) - (\operatorname{Tr}_t u, \zeta)}{h} = \big(\operatorname{Tr}_t \tfrac{\mathscr{T}^h u - u}{h}, \zeta \big) = \big(\operatorname{Tr}_0 \tfrac{\mathscr{T}^h \mathscr{T}^t u - \mathscr{T}^t u}{h}, \zeta \big).$$

By our previous computations, we have that $\frac{\mathscr{T}^h \mathscr{T}^t u - \mathscr{T}^t u}{h} \longrightarrow D_{n+1} \mathscr{T}^t u$ in $Y^{1,2}(\mathbb{R}^{n+1}_+)$ as $h \to 0$, which implies that $\operatorname{Tr}_0 \left(\frac{\mathscr{T}^h \mathscr{T}^t u - \mathscr{T}^t u}{h} \right) \longrightarrow \operatorname{Tr}_0 D_{n+1} \mathscr{T}^t u$ in $H^{\frac{1}{2}}_0(\mathbb{R}^n)$ as $h \to 0$, and hence we have that $\frac{g(t+h) - g(t)}{h} \longrightarrow (\operatorname{Tr}_0 D_{n+1} \mathscr{T}^t u, \zeta) = (\operatorname{Tr}_t D_{n+1} u, \zeta)$ as $h \to 0$. This finishes the proof. $\square$

4.4 Abstract Layer Potential Theory

In this section, we develop the abstract layer potential theory. Our methods often closely follow the constructions of Ariel Barton [Bar17]; but see also [Ros13].

Definition 4.4.1 (Single layer potential)**.** Define the *single layer potential of* $\mathcal{L}$ as the operator $\mathcal{S}^{\mathcal{L}} : H^{-\frac{1}{2}}(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^{n+1})$ given by

$$\mathcal{S}^{\mathcal{L}} := \big(\operatorname{Tr}_0 \circ (\mathcal{L}^{-1})^* \big)^*,$$

which is well defined by virtue of Lemma 4.2.9 and Proposition 4.2.19. For $t \in \mathbb{R}$, we denote $\mathcal{S}_t^{\mathcal{L}} := \mathrm{Tr}_t \circ \mathcal{S}^{\mathcal{L}}$. When the operator under consideration is clear from the context, we will sometimes drop the superscript, so that we write $\mathcal{S} = \mathcal{S}^{\mathcal{L}}$. For each $t \in \mathbb{R}$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^{n+1}$ and $\vec{f} : \mathbb{R}^n \to \mathbb{C}^n$, define $(\mathcal{S}_t^{\mathcal{L}} \nabla_{\|})\vec{f} := -\mathcal{S}_t^{\mathcal{L}}(\mathrm{div} \vec{f})$, $\mathcal{S}_t^{\mathcal{L}} D_{n+1} := -\partial_t \mathcal{S}_t^{\mathcal{L}}$, and $(\mathcal{S}_t^{\mathcal{L}} \nabla)\mathbf{f} = (\mathcal{S}_t^{\mathcal{L}} \nabla_{\|})\mathbf{f}_{\|} + \mathcal{S}_t^{\mathcal{L}} D_{n+1} \mathbf{f}_{n+1}$.

Let us elucidate a few properties of this "abstract" single layer potential.

Proposition 4.4.2 (Properties of the single layer potential)**.** *Fix $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$. The following statements hold.*

(i) The function $\mathcal{S}^{\mathcal{L}}\gamma \in Y^{1,2}(\mathbb{R}^{n+1})$ is the unique element in $Y^{1,2}(\mathbb{R}^{n+1})$ such that

$$B_{\mathcal{L}}[\mathcal{S}^{\mathcal{L}}\gamma, \Phi] = \langle \gamma, \mathrm{Tr}_0 \Phi \rangle, \qquad \text{for all } \Phi \in Y^{1,2}(\mathbb{R}^{n+1}). \tag{4.4.3}$$

Accordingly, $\mathcal{S}^{L} : H^{-\frac{1}{2}}(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^{n+1})$ is a bounded linear operator.

(ii) The function $\mathcal{S}^{\mathcal{L}}\gamma$ satisfies $\mathcal{L}\mathcal{S}^{\mathcal{L}}\gamma = 0$ in Ω, where $\Omega = \mathbb{R}^{n+1}_+, \mathbb{R}^{n+1}_-$.

(iii) Suppose that γ has compact support. Then $\mathcal{L}\mathcal{S}^{\mathcal{L}}\gamma = 0$ in $\mathbb{R}^{n+1} \setminus \operatorname{supp} \gamma$.

(iv) Define p_-, p_+ as in Proposition 4.2.6 and suppose that $\gamma \in L^{p_-}(\mathbb{R}^n)$. Then the bound $\|\mathrm{Tr}_t \mathcal{S}^{\mathcal{L}}\gamma\|_{L^{p_+}(\mathbb{R}^n)} \lesssim \|\gamma\|_{L^{p_-}(\mathbb{R}^n)}$ holds for each $t \in \mathbb{R}$.

(v) For each $t \in \mathbb{R}$, the operators $\mathcal{S}_t^{\mathcal{L}}$ and $\mathcal{S}_{-t}^{\mathcal{L}^*}$ are adjoint to one another. That is, for each $\gamma, \psi \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, the identity $\langle \mathcal{S}_t^{\mathcal{L}}\gamma, \psi \rangle = \langle \gamma, \mathcal{S}_{-t}^{\mathcal{L}^*}\psi \rangle$ holds.

(vi) For each $t \in \mathbb{R}$, we have the characterization

$$\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}}\gamma = \big(\mathrm{Tr}_t \circ (\mathcal{L}^{-1})^*\big)^*. \tag{4.4.4}$$

(vii) For each $t \in \mathbb{R}\setminus\{0\}$, we have that $\mathrm{Tr}_t D_{n+1}\mathcal{S}^{\mathcal{L}}\gamma \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$. Moreover, for each $t \in \mathbb{R}\setminus\{0\}$ and each $\zeta \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, we have that $\langle \mathrm{Tr}_t D_{n+1}\mathcal{S}^{\mathcal{L}}\gamma, \zeta \rangle = \frac{d}{dt}\langle \mathcal{S}_t^{\mathcal{L}}\gamma, \zeta \rangle = -\langle \gamma, \mathrm{Tr}_{-t} D_{n+1}\mathcal{S}^{\mathcal{L}^*}\zeta \rangle$.

(viii) Let $t \in \mathbb{R}\setminus\{0\}$. Let $\mathbf{g} = (\vec{g_{\|}}, g_{\perp}) : \mathbb{R}^n \to \mathbb{C}^{n+1}$ be such that $g_{\|}, g_{\perp} \in C_c^{\infty}(\mathbb{R}^n)$. In the sense of distributions, we have the adjoint relation

$$\langle \nabla \mathcal{S}_t^{\mathcal{L}}\gamma, \mathbf{g} \rangle_{\mathscr{D}', \mathscr{D}} = \langle \gamma, (\mathcal{S}_{-t}^{\mathcal{L}^*}\nabla)\mathbf{g} \rangle_{H^{-\frac{1}{2}}(\mathbb{R}^n), H_0^{\frac{1}{2}}(\mathbb{R}^n)}. \tag{4.4.5}$$

Proof. Fix $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$. Proof of (i). Since $\mathrm{Tr}_0 : Y^{1,2}(\mathbb{R}^{n+1}) \to H_0^{\frac{1}{2}}(\mathbb{R}^n)$ is a bounded linear operator, then $T_\gamma := \langle \gamma, \mathrm{Tr}_0 \cdot \rangle$ is a bounded linear functional on

$Y^{1,2}(\mathbb{R}^{n+1})$. By the Lax-Milgram theorem, there exists a unique $u_\gamma \in Y^{1,2}(\mathbb{R}^{n+1})$ such that $B_{\mathcal{L}}[u_\gamma, \Phi] = \langle T_\gamma, \Phi\rangle = \langle \gamma, \operatorname{Tr}_0 \Phi\rangle$, for all $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$. Now let $\Psi \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ be arbitrary, and observe that

$$\begin{aligned}\langle \Psi, \mathcal{S}^{\mathcal{L}}\gamma\rangle = \big\langle \Psi, \big(\operatorname{Tr}_0 \circ (\mathcal{L}^{-1})^*\big)^* \gamma\big\rangle = \big\langle \operatorname{Tr}_0 \circ (\mathcal{L}^{-1})^* \Psi, \gamma\big\rangle = \overline{\langle T_\gamma, (\mathcal{L}^*)^{-1}\Psi\rangle} \\ = \overline{B_{\mathcal{L}}[u_\gamma, (\mathcal{L}^*)^{-1}\Psi]} = \overline{\langle \mathcal{L}u_\gamma, (\mathcal{L}^*)^{-1}\Psi\rangle} = \overline{\langle u_\gamma, \Psi\rangle} = \langle \Psi, u_\gamma\rangle.\end{aligned}$$

Proof of (ii). Let $\Phi \in C_c^\infty(\mathbb{R}^{n+1}_+)$, and let $\tilde{\Phi}$ be an extension of Φ to $C_c^\infty(\mathbb{R}^{n+1})$ with $\tilde{\Phi} \equiv 0$ on $\mathbb{R}^{n+1} \setminus \operatorname{supp} \Phi$. In particular, $\operatorname{Tr}_0 \tilde{\Phi} \equiv 0$. Then (4.4.3) gives that $B_{\mathcal{L}}[\mathcal{S}^{\mathcal{L}}\gamma, \Phi] = B_{\mathcal{L}}[\mathcal{S}^{\mathcal{L}}\gamma, \tilde{\Phi}] = 0$. Since Φ was arbitrary, the claim follows.

Proof of (iii). Let $\Omega := \mathbb{R}^{n+1} \setminus \operatorname{supp} \gamma$, and let $\Phi \in C_c^\infty(\Omega)$. Let $\tilde{\Phi}$ be an extension of Φ to $C_c^\infty(\mathbb{R}^{n+1})$ with $\tilde{\Phi} \equiv 0$ on $\mathbb{R}^{n+1} \setminus \operatorname{supp} \gamma$. In particular, the supports of $\tilde{\Phi}$ and γ are disjoint. It follows that $\langle \gamma, \operatorname{Tr}_0 \tilde{\Phi}\rangle = 0$. Using (4.4.3) now yields the result.

Proof of (iv). By the boundedness of $\mathcal{S}^{\mathcal{L}}$ and the Sobolev embeddings, we have that

$$\|\mathcal{S}_t^{\mathcal{L}} g\|_{L^{p_+}(\mathbb{R}^n)} \lesssim \|\mathcal{S}_t^{\mathcal{L}} g\|_{H_0^{\frac{1}{2}}(\mathbb{R}^n)} \lesssim \|\mathcal{S}^{\mathcal{L}} g\|_{Y^{1,2}(\mathbb{R}^{n+1})} \lesssim \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p_-}(\mathbb{R}^n)}.$$

Proof of (v). Fix $t \in \mathbb{R}$ and $\gamma, \zeta \in H^{-\frac{1}{2}}(\mathbb{R}^n)$. By the Lax-Milgram theorem, there exists a unique $v^{\zeta,t} \in Y^{1,2}(\mathbb{R}^{n+1})$ such that $B_{\mathcal{L}^*}[v^{\zeta,t}, \Phi] = \langle \zeta, \operatorname{Tr}_t \Phi\rangle$, for all $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$. Observe that

$$\langle \operatorname{Tr}_t \mathcal{S}^{\mathcal{L}}\gamma, \zeta\rangle = \overline{\langle \zeta, \operatorname{Tr}_t \mathcal{S}^{\mathcal{L}}\gamma\rangle} = \overline{B_{\mathcal{L}^*}[v^{\zeta,t}, \mathcal{S}^{\mathcal{L}}\gamma]} = B_{\mathcal{L}}[\mathcal{S}^{\mathcal{L}}\gamma, v^{\zeta,t}] = \langle \gamma, \operatorname{Tr}_0 v^{\zeta,t}\rangle.$$

Thus it suffices to show that $\operatorname{Tr}_0 v^{\zeta,t}$ and $\mathcal{S}_{-t}^{\mathcal{L}^*}\zeta$ coincide as elements in $H_0^{\frac{1}{2}}(\mathbb{R}^n)$. In turn, this will follow if we prove that $\mathcal{S}^{\mathcal{L}^*}\zeta = \mathscr{T}^t v^{\zeta,t} = v^{\zeta,t}(\cdot, \cdot + t)$, in $Y^{1,2}(\mathbb{R}^{n+1})$. Let $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$ be arbitrary. Note then that $\mathscr{T}^t\Phi$ also lies in $Y^{1,2}(\mathbb{R}^{n+1})$. By the $t-$independence of the coefficients of $\mathcal{L}$ and a change of variables we have that

$$B_{\mathcal{L}^*}[\mathscr{T}^t v^{\zeta,t}, \mathscr{T}^t\Phi] = B_{\mathcal{L}^*}[v^{\zeta,t}, \Phi] = \langle \gamma, \operatorname{Tr}_t \Phi\rangle = \langle \gamma, \operatorname{Tr}_0 \mathscr{T}^t\Phi\rangle.$$

By (4.4.3) with $\mathcal{L}$ replaced by $\mathcal{L}^*$ throughout, $\mathcal{S}^{\mathcal{L}^*}\zeta$ is the unique element of $Y^{1,2}(\mathbb{R}^{n+1})$ for which the above identity can hold for all $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$, as desired.

Proof of (vi). In (v)), we proved that for each $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, $\mathcal{S}^{\mathcal{L}}\gamma = \mathscr{T}^t\mathcal{L}^{-1}(T_\gamma^t)$, where $T^t_\gamma \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ is given by $\langle T^t_\gamma, \Phi\rangle = \langle \gamma, \operatorname{Tr}_t \Phi\rangle$ for $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$. Hence $\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}}\gamma = \mathcal{L}^{-1}(T^t_\gamma)$. Reproduce the proof of (i)) in reverse to obtain the claim.

Proof of (vii). Let $t > 0$ (the case $t < 0$ is analogous). By (ii)) we have that $\mathcal{L}\mathcal{S}^{\mathcal{L}}\gamma = 0$ in $\mathbb{R}^{n+1}_+$. Therefore, using Proposition 4.3.23 (iv)) we have that $\operatorname{Tr}_\tau D_{n+1}\mathcal{S}^{\mathcal{L}}\gamma \in H^{\frac{1}{2}}_0(\mathbb{R}^n)$ for each $\tau > 0$. Using (4.3.26) and (v)), we calculate that

$$\begin{aligned}
\tfrac{d}{d\tau}\langle \operatorname{Tr}_\tau \mathcal{S}^{\mathcal{L}}\gamma, \zeta\rangle\big|_{\tau=t} &= \overline{\tfrac{d}{d\tau}\langle \zeta, \operatorname{Tr}_\tau \mathcal{S}^{\mathcal{L}}\gamma\rangle}\big|_{\tau=t} = \overline{\tfrac{d}{d\tau}\langle \operatorname{Tr}_{-\tau} \mathcal{S}^{\mathcal{L}^*}\zeta, \gamma\rangle}\big|_{\tau=t} \\
&= -\overline{\tfrac{d}{d(-\tau)}\langle \operatorname{Tr}_{-\tau} \mathcal{S}^{\mathcal{L}^*}\zeta, \gamma\rangle}\big|_{-\tau=-t} = -\overline{\langle \operatorname{Tr}_{-t} D_{n+1}\mathcal{S}^{\mathcal{L}^*}\zeta, \gamma\rangle} \\
&= -\langle \gamma, \operatorname{Tr}_{-t} D_{n+1}\mathcal{S}^{\mathcal{L}^*}\zeta\rangle.
\end{aligned}$$

Proof of (viii). It is clear by an easy induction procedure that ((vii)) holds for higher $t-$derivatives in the expected manner. Note that

$$\begin{aligned}
\langle \nabla\mathcal{S}^{\mathcal{L}}_t\gamma, \mathbf{g}\rangle_{\mathscr{D}',\mathscr{D}} &= \langle \nabla_\parallel\mathcal{S}^{\mathcal{L}}_t\gamma, \vec{g_\parallel}\rangle_{\mathscr{D}',\mathscr{D}} + \langle \operatorname{Tr}_t D_{n+1}\mathcal{S}^{\mathcal{L}}\gamma, g_\perp\rangle_{\mathscr{D}',\mathscr{D}} \\
&= -\langle \mathcal{S}^{\mathcal{L}}_t\gamma, \operatorname{div}\vec{g_\parallel}\rangle_{\mathscr{D}',\mathscr{D}} - \langle \gamma, \operatorname{Tr}_{-t} D_{n+1}\mathcal{S}^{\mathcal{L}^*} g_\perp\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^n), H^{\frac{1}{2}}_0(\mathbb{R}^n)} \\
&= -\langle \gamma, \mathcal{S}^{\mathcal{L}^*}_{-t}\operatorname{div}\vec{g_\parallel}\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^n), H^{\frac{1}{2}}_0(\mathbb{R}^n)} + \langle \gamma, (\mathcal{S}^{\mathcal{L}^*}_{-t} D_{n+1})g_\perp\rangle_{H^{-\frac{1}{2}}(\mathbb{R}^n), H^{\frac{1}{2}}_0(\mathbb{R}^n)} \\
&= \langle \gamma, (\mathcal{S}^{\mathcal{L}^*}_{-t}\nabla)\mathbf{g}\rangle.
\end{aligned}$$

□

In preparation for defining the double layer potential, let us make the following remark.

Remark. Given $\varphi \in H^{\frac{1}{2}}_0(\mathbb{R}^n)$, there exists $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$ with $\operatorname{Tr}_0 \Phi = \varphi$ and $\|\Phi\|_{Y^{1,2}(\mathbb{R}^{n+1})} \lesssim \|\varphi\|_{H^{\frac{1}{2}}_0(\mathbb{R}^n)}$.

For a fixed $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$, let $\mathscr{F}^+_u$ be the functional on $Y^{1,2}(\mathbb{R}^{n+1})$ defined by

$$\langle \mathscr{F}^+_u, v\rangle := B_{\mathcal{L},\mathbb{R}^{n+1}_+}[u, v] = \iint_{\mathbb{R}^{n+1}_+} \Big[A\nabla u \cdot \overline{\nabla v} + B_1 u \cdot \overline{\nabla v} + B_2 \cdot \nabla u\overline{v}\Big],$$

for each $v \in Y^{1,2}(\mathbb{R}^{n+1})$. Then $\mathscr{F}^+_u$ is clearly bounded on $Y^{1,2}(\mathbb{R}^{n+1})$. We define $B_{\mathcal{L},\mathbb{R}^{n+1}_-}$ and $\mathscr{F}^-_u$ in a similar way (using $\mathbb{R}^{n+1}_-$ instead of $\mathbb{R}^{n+1}_+$), and we note that if $u \in Y^{1,2}(\mathbb{R}^{n+1})$, then $\mathcal{L}u = \mathscr{F}^+_u + \mathscr{F}^-_u$.

Definition 4.4.6 (Double layer potential)**.** Given $\varphi \in H^{\frac{1}{2}}_0(\mathbb{R}^n)$, let $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$ be any extension of φ to $\mathbb{R}^{n+1}$. Define $\mathcal{D}^{\mathcal{L},+}(\varphi) := -\Phi\big|_{\mathbb{R}^{n+1}_+} + \mathcal{L}^{-1}(\mathscr{F}^+_\Phi)\big|_{\mathbb{R}^{n+1}_+}$ (see below for a proof that this is well defined). We call the operator $\mathcal{D}^{\mathcal{L},+} : H_0^{\frac{1}{2}}(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^{n+1}_+)$ the *double layer potential* associated to operator $\mathcal{L}$ on the upper half-space. Analogously, we define $\mathcal{D}^{\mathcal{L},-}$, the double-layer potential associated to operator $\mathcal{L}$ on the lower half-space, by extending φ to $\mathbb{R}^{n+1}_-$. We define $\mathcal{D}^{\mathcal{L}^*,\pm}$ similarly, by replacing $\mathcal{L}$ with $\mathcal{L}^*$.

Proposition 4.4.7 (Properties of the double layer potential)**.** *Fix $\varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$ and let Φ be any $Y^{1,2}(\mathbb{R}^{n+1})-$extension of φ to $\mathbb{R}^{n+1}$ with $\operatorname{Tr}_0 \Phi = \varphi$. The following statements hold.*

(i) The double layer potential $\mathcal{D}^{\mathcal{L},+}$ is well defined.

(ii) We have the characterizations

$$\mathcal{D}^{\mathcal{L},+}\varphi = -\mathcal{L}^{-1}(\mathscr{F}^-_\Phi)\big|_{\mathbb{R}^{n+1}_+}, \qquad \mathcal{D}^{\mathcal{L},-}\varphi = -\mathcal{L}^{-1}(\mathscr{F}^+_\Phi)\big|_{\mathbb{R}^{n+1}_-}. \tag{4.4.8}$$

(iii) The bound $\|\mathcal{D}^{\mathcal{L},+}\varphi\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)} \lesssim \|\varphi\|_{H_0^{\frac{1}{2}}(\mathbb{R}^n)}$ holds.

(iv) The function $\mathcal{D}^{\mathcal{L},+}\varphi$ satisfies $\mathcal{L}\mathcal{D}^{\mathcal{L},+}\varphi = 0$ in the weak sense in $\mathbb{R}^{n+1}_+$.

Proof. Proof of (i)). Let $\Phi, \Phi' \in Y^{1,2}(\mathbb{R}^{n+1})$ be any two extensions of φ to $\mathbb{R}^{n+1}$. Then $(\Phi - \Phi')(\cdot, 0) = 0$. If w is defined as $w|_{\mathbb{R}^{n+1}_+} = \Phi - \Phi'$ with $w|_{\mathbb{R}^{n+1}_-} \equiv 0$, then $w \in Y^{1,2}(\mathbb{R}^{n+1})$. Thus observe that $\langle \mathcal{L}w, \Psi\rangle = B_{\mathcal{L}}[w, \Psi] = \langle \mathscr{F}^+_{\Phi-\Phi'}, \Psi\rangle$, for all $\Psi \in Y^{1,2}(\mathbb{R}^{n+1})$, whence we conclude that $w = \mathcal{L}^{-1}(\mathscr{F}^+_{\Phi-\Phi'})$. Hence

$$\begin{aligned}\big[-\Phi \,+\, \mathcal{L}^{-1}(\mathscr{F}^+_\Phi)\big]_{\mathbb{R}^{n+1}_+} - \big[-\Phi' \,+\, \mathcal{L}^{-1}(\mathscr{F}^+_{\Phi'})\big]_{\mathbb{R}^{n+1}_+} \\ = \big[\Phi' - \Phi + \mathcal{L}^{-1}(\mathscr{F}^+_\Phi - \mathscr{F}^+_{\Phi'})\big]_{\mathbb{R}^{n+1}_+} = \big[\Phi' - \Phi + \mathcal{L}^{-1}(\mathscr{F}^+_{\Phi-\Phi'})\big]_{\mathbb{R}^{n+1}_+} \equiv 0.\end{aligned}$$

Proof of (ii)). Simply note that

$$\mathcal{D}^{\mathcal{L},+}\varphi = \big[-\Phi + \mathcal{L}^{-1}(\mathscr{F}^+_\Phi)\big]_{\mathbb{R}^{n+1}_+} = \big[\mathcal{L}^{-1}(-\mathcal{L}\Phi + \mathscr{F}^+_\Phi)\big]_{\mathbb{R}^{n+1}_+} = \big[\mathcal{L}^{-1}(-\mathscr{F}^-_\Phi)\big]_{\mathbb{R}^{n+1}_+}.$$

Proof of (iii)). Owing to (4.4.8) we write

$$\|\mathcal{D}^{\mathcal{L},+}\varphi\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)} = \|\mathcal{L}^{-1}(\mathscr{F}^-_\Phi)\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)} \lesssim \|\mathscr{F}^-_\Phi\|_{(Y^{1,2}(\mathbb{R}^{n+1}))^*}.$$

Let $0 \neq \Psi \in Y^{1,2}(\mathbb{R}^{n+1})$. We have

$$|(\mathscr{F}_\Phi^-, \Psi)| = |B_{\mathcal{L},\mathbb{R}_-^{n+1}}[\Phi, \Psi]| \lesssim \|\Phi\|_{Y^{1,2}(\mathbb{R}_-^{n+1})} \|\Psi\|_{Y^{1,2}(\mathbb{R}^{n+1})},$$

whence we deduce that $\|\mathscr{F}_\Phi^-\|_{(Y^{1,2}(\mathbb{R}^{n+1}))^*} \lesssim \|\Phi\|_{Y^{1,2}(\mathbb{R}_-^{n+1})} \lesssim \|\varphi\|_{H_0^{\frac{1}{2}}(\mathbb{R}^n)}$. Putting these estimates together we obtain the desired result.

Proof of (iv)). Let $\Psi \in C_c^\infty(\mathbb{R}_+^{n+1})$ and extend it as a function in $\Psi \in C_c^\infty(\mathbb{R}^{n+1})$ so that $\Psi \equiv 0$ in $\mathbb{R}_-^{n+1}$. Observe that

$$\begin{aligned} B_{\mathcal{L},\mathbb{R}_+^{n+1}}[\mathcal{D}^{\mathcal{L},+}\varphi, \Psi] = B_{\mathcal{L},\mathbb{R}_+^{n+1}}[-\mathcal{L}^{-1}(\mathscr{F}_\Phi^-), \Psi] &= B_{\mathcal{L}}[-\mathcal{L}^{-1}(\mathscr{F}_\Phi^-), \Psi] \\ &= -\langle \mathscr{F}_\Phi^-, \Psi \rangle = -B_{\mathcal{L},\mathbb{R}_-^{n+1}}[\Phi, \Psi] \equiv 0. \end{aligned}$$

□

We may now introduce the definition of the conormal derivative. First let us make the quick observation that since $Y_0^{1,2}(\mathbb{R}_+^{n+1}) \hookrightarrow Y^{1,2}(\mathbb{R}_+^{n+1})$, then we have a surjection $(Y^{1,2}(\mathbb{R}_+^{n+1}))^* \to (Y_0^{1,2}(\mathbb{R}_+^{n+1}))^*$ given by restriction of the test space for the functional. In particular, if $f \in (Y^{1,2}(\mathbb{R}_+^{n+1}))^*$, then we can also think of $f \in (Y_0^{1,2}(\mathbb{R}_+^{n+1}))^*$.

Definition 4.4.9 (Conormal derivative)**.** Say that $u \in Y^{1,2}(\mathbb{R}_+^{n+1})$, $f \in (Y^{1,2}(\mathbb{R}_+^{n+1}))^*$ (note carefully that this space is not $(Y_0^{1,2}(\mathbb{R}_+^{n+1}))^*$), and that $\mathcal{L}u = f$ in $\mathbb{R}_+^{n+1}$ in the sense that for each $\Phi \in C_c^\infty(\mathbb{R}_+^{n+1})$, the identity

$$B_{\mathcal{L},\mathbb{R}_+^{n+1}}[u, \Phi] = \langle f, \Phi \rangle_{(Y_0^{1,2}(\mathbb{R}_+^{n+1}))^*, Y_0^{1,2}(\mathbb{R}_+^{n+1})} \tag{4.4.10}$$

holds. Define the *conormal derivative* of u associated to $\mathcal{L}$ with respect to $\mathbb{R}_+^{n+1}$, $\partial_\nu^{\mathcal{L},+}u \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, by

$$\langle \partial_\nu^{\mathcal{L},+}u, \varphi \rangle = B_{\mathcal{L},\mathbb{R}_+^{n+1}}[u, \Phi] - \langle f, \Phi \rangle_{(Y^{1,2}(\mathbb{R}_+^{n+1}))^*, Y^{1,2}(\mathbb{R}_+^{n+1})}, \qquad \varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n),$$

where $\Phi \in Y^{1,2}(\mathbb{R}_+^{n+1})$ is any bounded extension of φ to $\mathbb{R}_+^{n+1}$. Note that we also define the objects $\partial_\nu^{\mathcal{L}^*,+}u, \partial_\nu^{\mathcal{L},-}u, \partial_\nu^{\mathcal{L}^*,-}u$ similarly.

When $f = \tilde{f} - \operatorname{div} \tilde{F}$ and $\tilde{f}, |\tilde{F}|$ verify the assumptions in (4.2.13) (with $\Omega = \mathbb{R}_+^{n+1}$, $D = \mathbb{R}^n$, and $I = (0, \infty)$), then the sense (4.4.10) of weak solutions coincides with the one previously given in Definition 4.2.10 (see Remark 4.2.12). In particular, if $f \equiv 0$, the two senses (4.2.11), (4.4.10) of weak solutions coincide, and there is no ambiguity.

Let us show that $\partial_\nu^{\mathcal{L},+}u$ is well defined. Let Φ, Φ' be $Y^{1,2}(\mathbb{R}^{n+1}_+)-$extensions of φ with $\mathrm{Tr}_0\,\Phi = \mathrm{Tr}_0\,\Phi' = \varphi$. Then $\Phi - \Phi' \in Y^{1,2}_0(\mathbb{R}^{n+1}_+)$, and so

$$B_{\mathcal{L},\mathbb{R}^{n+1}_+}[u,\Phi] - B_{\mathcal{L},\mathbb{R}^{n+1}_+}[u,\Phi'] = B_{\mathcal{L},\mathbb{R}^{n+1}_+}[u,\Phi-\Phi'] \\ = \langle f, \Phi - \Phi'\rangle_{(Y^{1,2}_0(\mathbb{R}^{n+1}_+))^*, Y^{1,2}_0(\mathbb{R}^{n+1}_+)}$$

since u solves $\mathcal{L}u = f$ in $\mathbb{R}^{n+1}_+$ in the sense (4.4.10). Finally, observe that

$$\langle f, \Phi\rangle_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*, Y^{1,2}(\mathbb{R}^{n+1}_+)} - \langle f, \Phi'\rangle_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*, Y^{1,2}(\mathbb{R}^{n+1}_+)} \\ = \langle f, \Phi - \Phi'\rangle_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*, Y^{1,2}(\mathbb{R}^{n+1}_+)} = \langle f, \Phi - \Phi'\rangle_{(Y^{1,2}_0(\mathbb{R}^{n+1}_+))^*, Y^{1,2}_0(\mathbb{R}^{n+1}_+)},$$

so that, upon subtracting these two identities, we see that $\partial_\nu^{\mathcal{L},+}u$ does not depend on the particular extension Φ taken. It remains to show that $\partial_\nu^{\mathcal{L},+}u \in H^{-\frac{1}{2}}(\mathbb{R}^n)$. Observe that

$$|\langle \partial_\nu^{\mathcal{L},+}u, \varphi\rangle| \le |B_{\mathcal{L},\mathbb{R}^{n+1}_+}[u,\Phi]| + |\langle f, \Phi\rangle_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*, Y^{1,2}(\mathbb{R}^{n+1}_+)}| \\ \lesssim \big(\|u\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)} + \|f\|_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*}\big)\|\Phi\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)} \\ \lesssim \big(\|u\|_{Y^{1,2}(\mathbb{R}^{n+1}_+)} + \|f\|_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*}\big)\|\varphi\|_{H^{\frac{1}{2}}_0(\mathbb{R}^n)}.$$

It will also be helpful to consider conormal derivatives on slices other than $t = 0$, denoted $\partial_{\nu,t}^{\mathcal{L},\pm}$. The definition is entirely analogous.

The following identities tie these definitions of the conormal derivatives together.

Lemma 4.4.11. *Let $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$. The following statements are true.*

(i) Suppose that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ solves $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense. Then for any $t > 0$, $\partial_\nu^{\mathcal{L},+}\mathscr{T}^t u = \partial_{\nu,t}^{\mathcal{L},+}u$. Moreover, for any $t > 0$, $\partial_{\nu,t}^{\mathcal{L},+}u \in L^2(\mathbb{R}^n)$, and we have the identity

$$\partial_{\nu,t}^{\mathcal{L},+}u = -e_{n+1}\cdot \mathrm{Tr}_t[A\nabla u + B_1 u] \qquad \text{in } L^2(\mathbb{R}^n). \tag{4.4.12}$$

(ii) Suppose that $u \in Y^{1,2}(\mathbb{R}^{n+1}_-)$ solves $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_-$. Then for any $t > 0$, $\partial_\nu^{\mathcal{L},-}\mathscr{T}^{-t}u = \partial_{\nu,-t}^{\mathcal{L},-}u$.

(iii) Let $t > 0$. Then for each $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, the identity $-\partial_{\nu,-t}^{\mathcal{L},-}\mathcal{S}^{\mathcal{L}}\gamma = \partial_{\nu,-t}^{\mathcal{L},+}\mathcal{S}^{\mathcal{L}}\gamma$

holds in the space $H^{-\frac{1}{2}}(\mathbb{R}^n)$.

Proof. Proof of (i)) and (ii)). Let $\varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$, and $\Phi \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ is any extension of φ. Thenm

$$\langle \partial_\nu^{\mathcal{L},+} \mathscr{T}^t u, \varphi\rangle = B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\mathscr{T}^t u, \Phi] = B_{\mathcal{L},\mathbb{R}^{n+1}_t}[u, \mathscr{T}^{-t}\Phi]$$
$$= \langle \partial_{\nu,t}^{\mathcal{L},+} u, \operatorname{Tr}_t \mathscr{T}^{-t}\Phi\rangle = \langle \partial_{\nu,t}^{\mathcal{L},+} u, \varphi\rangle.$$

We turn our attention now to (4.4.12). By Remark 4.3.18, we have that $F(x,t) = -e_{n+1}\cdot \operatorname{Tr}_t[A\nabla u + B_1 u]$ is continuous in t taking values in $L^2(\mathbb{R}^n)$. In order to prove the lemma we will regularize our coefficients and solution simultaneously.

Let P_ε be an $(n+1)-$dimensional approximate identity; that is, $P_\varepsilon(f) = \eta_\varepsilon * f$, where $\eta_\varepsilon(X) = \frac{1}{\varepsilon^{n+1}}\eta(X/\varepsilon)$ $(X \in \mathbb{R}^{n+1})$, $\eta \in C_c^\infty(B(0,1))$, η non-negative and radially decreasing with $\int_{\mathbb{R}^{n+1}} \eta = 1$. We claim that

$$-e_{n+1}\cdot P_\varepsilon(A\nabla u + B_1 u)(x,t_0) \longrightarrow -e_{n+1}\cdot (A\nabla u + B_1 u)(x,t_0) \tag{4.4.13}$$

strongly in $L^2(\mathbb{R}^n)$. Assume (4.4.13) for a moment. Then to show (i)) and (ii)) in the lemma, it is enough to show that for every $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$ with $\Phi(x,t_0) = \varphi(x)$, we have that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} -e_{n+1}\cdot P_\varepsilon(A\nabla u + B_1 u)(x,t_0)\overline{\varphi(x)}\,dx$$
$$= \iint_{\mathbb{R}^{n+1}_{t_0}} A\nabla u\overline{\nabla\Phi} + B_1 u\overline{\nabla\Phi} + B_2\cdot\nabla u\overline{\Phi}.$$

To prove the above equality, first define for any cube $Q\subset\mathbb{R}^n$, $R_Q^{t_0} := Q\times[t_0, t_0+\ell(Q)]$. Now choose any cube $Q \subset \mathbb{R}^n$ such that $\operatorname{supp}\Phi \cap \{t\geq t_0\} \subset \overline{R^{t_0}_{\frac{1}{2}Q}}$. Integrating by parts, we have for $0<\varepsilon \ll \min\{\ell(Q), t_0\}$ the identity

$$\int_{\mathbb{R}^n} -e_{n+1}\cdot P_\varepsilon(A\nabla u + B_1 u)(x,t_0)\overline{\varphi(x)}\,dx = \iint_{R_Q^{t_0}} \operatorname{div}[P_\varepsilon(A\nabla u + B_1 u)\overline{\Phi}]$$
$$= \iint_{R_Q^{t_0}} \operatorname{div}[P_\varepsilon(A\nabla u + B_1 u)]\overline{\Phi} + \iint_{R_Q^{t_0}} P_\varepsilon(A\nabla u + B_1 u)\cdot\overline{\nabla\Phi}. \tag{4.4.14}$$

Now let $X = (x,t)\in \operatorname{supp}\Phi\cap\{t\geq t_0\}$, and $\varepsilon < \frac{t_0}{2}$. Then, since $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$, we have that

$$\begin{aligned}\operatorname{div}_X[P_\varepsilon(A\nabla u+B_1u)](X)&=\operatorname{div}_X\iint_{\mathbb{R}^{n+1}}\eta_\varepsilon(X-Y)(A\nabla_Y u+B_1u)(Y)\,dY\\&=-\iint_{\mathbb{R}^{n+1}}\nabla_Y\eta_\varepsilon(X-Y)(A\nabla_Y u+B_1u)(Y)\,dY\\&=\iint_{\mathbb{R}^{n+1}_+}\eta_\varepsilon(X-Y)B_2\nabla_Y u(Y)\,dY=P_\varepsilon(B_2\nabla u)(X),\end{aligned}$$

and therefore the identity

$$\iint_{R_Q^{t_0}}\operatorname{div}[P_\varepsilon(A\nabla u+B_1u)]\overline{\Phi}=\iint_{R_Q^{t_0}}P_\varepsilon(B_2\nabla u)\overline{\Phi}\tag{4.4.15}$$

holds. Finally, we want to pass in the limit as $\varepsilon\to0$ the identity (4.4.14) while using (4.4.15), so we use the Lebesgue dominated convergence theorem. Observe that for some $p>1$, $|A\nabla u|+|B_1u|+|B_2\nabla u|\in L^p(U_Q^{t_0})$, where $U_Q^{t_0}:=R_Q^{t_0}+B(0,\frac{t_0}{4})$ (the $\frac{t_0}{4}$-neighborhood of $R_Q^{t_0}$). It follows that for $\varepsilon\in(0,\frac{t_0}{4})$,

$$P_\varepsilon(A\nabla u+B_1u)(x,t)+P_\varepsilon(B_2\nabla u)(x,t)\le\mathscr{M}\big(\big[|A\nabla u|+|B_1u|+|B_2\nabla u|\big]\mathbb{1}_{U_Q^{t_0}}\big)(x,t)$$

for all $(x,t)\in R_Q^{t_0}$, where $\mathscr{M}$ is the usual Hardy-Littlewood maximal operator in $\mathbb{R}^{n+1}$. Hence we have that

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}-e_{n+1}\cdot P_\varepsilon(A\nabla u+B_1u)(x,t_0)\overline{\varphi(x)}\,dx\\&\qquad=\lim_{\varepsilon\to0}\iint_{R_Q^{t_0}}P_\varepsilon(A\nabla u+B_1u)\overline{\nabla\Phi}+P_\varepsilon(B_2\nabla u)\overline{\Phi}\\&\qquad=\iint_{R_Q^{t_0}}A\nabla u+B_1u\overline{\nabla\Phi}+B_2\nabla u\overline{\Phi}.\end{aligned}$$

Thus it remains to prove (4.4.13). Set $F_\varepsilon(x,t):=-e_{n+1}\cdot P_\varepsilon(A\nabla u+B_1u)(x,t)$, $F(x,t):=-e_{n+1}\cdot(A\nabla u+B_1u)(x,t)$. For $\varepsilon<\frac{t_0}{2}$, we have that

$$\begin{aligned}&\limsup_{\varepsilon\to0}\|F_\varepsilon(\cdot,t_0)-F_0(\cdot,t_0)\|_2\\&=\limsup_{\varepsilon\to0}\Big(\int_{\mathbb{R}^n}\Big|\iint_{\mathbb{R}^{n+1}}\big[F_0(x-\varepsilon y,t_0-\varepsilon s)-F_0(x,t_0)\big]\eta_\varepsilon(y,s)\,dy\,ds\Big|^2dx\Big)^{\frac12}\end{aligned}$$

$$\leq \limsup_{\varepsilon\to 0} \iint_{\mathbb{R}^{n+1}} \eta(y,s)\Big[\int_{\mathbb{R}^n} |F_0(x-\varepsilon y, t_0-\varepsilon s) - F_0(x,t_0)|^2\,dx\Big]^{\frac{1}{2}}\,dy\,ds$$

$$\leq \limsup_{\varepsilon\to 0} \sup_{|y|,|s|<1} \|F_0(\cdot-\varepsilon y, t_0-\varepsilon s) - F_0(\cdot,t_0)\|_2$$

$$\leq \limsup_{\varepsilon\to 0} \sup_{|\hat{y}|,|\hat{s}|<\varepsilon} \|F_0(\cdot-\hat{y}, t_0-\hat{s}) - F_0(\cdot-\hat{y},t_0)\|_2 + \|F_0(\cdot-\hat{y},t_0) - F_0(\cdot,t_0)\|_2,$$

which drops to 0 as $\varepsilon \to 0$, finishing the proof.

Proof of (iii)). Let $\varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$ and let $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$ be any extension of φ. Note that $\mathcal{L}\mathcal{S}^{\mathcal{L}}\gamma = 0$ in $\mathbb{R}^{n+1}_{-,-t}$, while $\mathcal{L}\mathcal{S}^{\mathcal{L}}\gamma = T_\gamma$ in $\mathbb{R}^{n+1}_{+,-t}$ in the sense (4.4.10), where $T_\gamma \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ is the distribution given by $\langle T_\gamma, \Psi\rangle = \langle \gamma, \mathrm{Tr}_0\,\Psi\rangle$, for $\Psi \in Y^{1,2}(\mathbb{R}^{n+1})$. Then,

$$\begin{aligned}
\langle \partial^{\mathcal{L},+}_{\nu,-t}\mathcal{S}^{\mathcal{L}}\gamma, \varphi\rangle &= B_{\mathcal{L},\mathbb{R}^{n+1}_{+,-t}}[\mathcal{S}^{\mathcal{L}}\gamma, \Phi] - \langle \gamma, \mathrm{Tr}_0\,\Phi\rangle \\
&= -B_{\mathcal{L},\mathbb{R}^{n+1}_{-,-t}}[\mathcal{S}^{\mathcal{L}}\gamma, \Phi] + B_{\mathcal{L}}[\mathcal{S}^{\mathcal{L}}\gamma, \Phi] - \langle \gamma, \mathrm{Tr}_0\,\Phi\rangle \\
&= -B_{\mathcal{L},\mathbb{R}^{n+1}_{-,-t}}[\mathcal{S}^{\mathcal{L}}\gamma, \Phi] + \langle \gamma, \mathrm{Tr}_0\,\Phi\rangle - \langle \gamma, \mathrm{Tr}_0\,\Phi\rangle = -\langle \partial^{\mathcal{L},-}_{\nu,-t}\mathcal{S}^{\mathcal{L}}\gamma, \varphi\rangle.
\end{aligned}$$

□

4.4.1 Green's formula and jump relations

Let us remark that the functional $\mathscr{F}^+_u$ makes sense even if we only have that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ and $u \notin Y^{1,2}(\mathbb{R}^{n+1})$. Also, if $\Omega \subset \mathbb{R}^{n+1}$ is an open set with Lipschitz boundary, and $f \in (Y^{1,2}(\Omega))^*$, define the functional $\mathbb{1}_\Omega f \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ by $\langle \mathbb{1}_\Omega f, \Psi\rangle := \langle f, \mathbb{1}_\Omega \Psi\rangle$ for each $\Psi \in Y^{1,2}(\mathbb{R}^{n+1})$.

Theorem 4.4.16 (Green's formula)**.** *Suppose that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ solves $\mathcal{L}u = f$ in $\mathbb{R}^{n+1}_+$ for some $f \in (Y^{1,2}(\mathbb{R}^{n+1}_+))^*$ in the sense (4.4.10). Then the following statements hold.*

(i) We have the identity

$$\mathcal{S}^{\mathcal{L}}(\partial^{\mathcal{L},+}_\nu u) = \mathcal{L}^{-1}(\mathscr{F}^+_u) - \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+} f) \qquad \text{in } Y^{1,2}(\mathbb{R}^{n+1}). \tag{4.4.17}$$

(ii) The identity $u = -\mathcal{D}^{\mathcal{L},+}(\mathrm{Tr}_0\,u) + \mathcal{S}^{\mathcal{L}}(\partial^{\mathcal{L},+}_\nu u)|_{\mathbb{R}^{n+1}_+} + \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+} f)|_{\mathbb{R}^{n+1}_+}$ holds in $Y^{1,2}(\mathbb{R}^{n+1}_+)$.

(iii) We have that

$$-\mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f)|_{\mathbb{R}^{n+1}_-} = \mathcal{D}^{\mathcal{L},-}(\operatorname{Tr}_0 u) + \mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},+}u)|_{\mathbb{R}^{n+1}_-}$$

in $Y^{1,2}(\mathbb{R}^{n+1}_-)$.

(iv) Suppose that $\mathcal{L}u = 0$ *in* $\mathbb{R}^{n+1}_-$. *Then* $\mathcal{D}^{\mathcal{L},+}(\operatorname{Tr}_0 u) = -\mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},-}u)$ *holds in* $\mathbb{R}^{n+1}_+$.

Proof. Proof of (i)). Let $\Psi \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$. Then

$$\begin{aligned}
\langle \Psi, \mathcal{S}^{\mathcal{L}}\partial_\nu^{\mathcal{L},+}u\rangle &= \langle \operatorname{Tr}_0(\mathcal{L}^*)^{-1}\Psi, \partial_\nu^{\mathcal{L},+}u\rangle \\
&= \overline{B_{\mathcal{L},\mathbb{R}^{n+1}_+}[u,(\mathcal{L}^*)^{-1}\Psi]} - \overline{\langle f,(\mathcal{L}^*)^{-1}\Psi\rangle}_{(Y^{1,2}(\mathbb{R}^{n+1}_+))^*,Y^{1,2}(\mathbb{R}^{n+1}_+)} \\
= \overline{\langle \mathscr{F}_u^+, (\mathcal{L}^{-1})^*\Psi\rangle} &- \overline{\langle \mathbb{1}_{\mathbb{R}^{n+1}_+}f, (\mathcal{L}^{-1})^*\Psi\rangle} = \overline{\langle \mathcal{L}^{-1}(\mathscr{F}_u^+),\Psi\rangle} - \overline{\langle \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f),\Psi\rangle} \\
&= \langle \Psi, \mathcal{L}^{-1}(\mathscr{F}_u^+) - \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f)\rangle.
\end{aligned}$$

Proof of (ii)). Let $\Psi \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ have compact support within $\mathbb{R}^{n+1}_+$. Using (4.4.17), we have that

$$\begin{aligned}
&\langle \Psi, \mathcal{S}^{\mathcal{L}}\partial_\nu u - \mathcal{D}^{\mathcal{L},+}\operatorname{Tr}_0 u\rangle \\
&= \langle \Psi, \mathcal{L}^{-1}(\mathscr{F}_u^+)\rangle - \langle \Psi, \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f)\rangle - \big[-\langle \Psi, u|_{\mathbb{R}^{n+1}_+}\rangle + \langle \Psi, \mathcal{L}^{-1}(\mathscr{F}_u^+)|_{\mathbb{R}^{n+1}_+}\rangle\big] \\
&= \langle \Psi, u - \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f)\rangle.
\end{aligned}$$

Proof of (iii)). Let $\Psi \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ have compact support within $\mathbb{R}^{n+1}_-$. Using (4.4.8) and (4.4.17), we have that

$$\begin{aligned}
\langle \Psi, \mathcal{D}^{\mathcal{L},-}(\operatorname{Tr}_0 u) + \mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},+}u)\rangle &= \langle \Psi, -\mathcal{L}^{-1}(\mathscr{F}_u^+)|_{\mathbb{R}^{n+1}_-} + \mathcal{L}^{-1}(\mathscr{F}_u^+) - \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f)\rangle \\
&= -\langle \Psi, \mathcal{L}^{-1}(\mathbb{1}_{\mathbb{R}^{n+1}_+}f)\rangle.
\end{aligned}$$

The proof of (iv)) is the same as (iii)), and is thus omitted. □

Let us now consider some adjoint relations for the double layer potential. First, for any $u \in Y^{1,2}(\mathbb{R}^{n+1})$, denote by $\mathscr{F}_u^{*+} \in (Y^{1,2}(\mathbb{R}^{n+1}))^*$ the functional given by $\langle \mathscr{F}_u^{*+}, v\rangle := B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[u,v]$ for $v \in Y^{1,2}(\mathbb{R}^{n+1})$.

Proposition 4.4.18. *We have the following identities.*

(i) For each $\varphi, \psi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$, the identity $\langle \partial_\nu^{\mathcal{L},+} \mathcal{D}^{\mathcal{L},+}\varphi, \psi\rangle = \langle \varphi, \partial_\nu^{\mathcal{L}^*,+}\mathcal{D}^{\mathcal{L}^*,+}\psi\rangle$ holds.

(ii) For each $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, $\varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$, $t \geq 0$, the adjoint relation

$$\begin{aligned}\langle \gamma, \operatorname{Tr}_t \mathcal{D}^{\mathcal{L},+}\varphi\rangle = -\langle \partial_\nu^{\mathcal{L}^*,-} \mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle = -\langle \partial_{\nu,-t}^{\mathcal{L}^*,-}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle \\ = \langle \partial_{\nu,-t}^{\mathcal{L}^*,+}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle \qquad (4.4.19)\end{aligned}$$

holds. In the case that $t = 0$, we may write

$$\langle \gamma, \operatorname{Tr}_0 \mathcal{D}^{\mathcal{L},+}\varphi\rangle = -\langle \gamma, \varphi\rangle + \langle \partial_\nu^{\mathcal{L}^*,+}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle. \qquad (4.4.20)$$

(iii) Fix $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^n)$. For each $t > 0$, and every $\zeta \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, we have the identity $\langle \operatorname{Tr}_t D_{n+1}\mathcal{D}^{\mathcal{L},+}\varphi, \zeta\rangle = \frac{d}{dt}\langle \operatorname{Tr}_t \mathcal{D}^{\mathcal{L},+}\varphi, \zeta\rangle = \langle \varphi, \partial_{\nu,-t}^{\mathcal{L}^*,-} D_{n+1}\mathcal{S}^{\mathcal{L}^*}\zeta\rangle_{L^2,L^2}$.

(iv) Fix $t > 0$. Let $\mathbf{g} = (\vec{g_\|}, g_\perp) : \mathbb{R}^n \to \mathbb{C}^{n+1}$ be such that $g_\|, g_\perp \in C_c^\infty(\mathbb{R}^n)$. In the sense of distributions, we have the adjoint relation

$$\langle \nabla \operatorname{Tr}_t D_{n+1}\mathcal{D}^{\mathcal{L},+}\varphi, \mathbf{g}\rangle_{\mathscr{D}',\mathscr{D}} = \langle \varphi, D_{n+1}\partial_{\nu,-t}^{\mathcal{L}^*,-}(\mathcal{S}^{\mathcal{L}^*}\nabla)\mathbf{g}\rangle_{L^2,L^2} \qquad (4.4.21)$$

Proof. Proof of (i). Let Φ, Ψ be extensions of φ, ψ respectively to $Y^{1,2}(\mathbb{R}^{n+1})$. Then,

$$\begin{aligned}\langle \partial_\nu^{\mathcal{L},+}\mathcal{D}^{\mathcal{L},+}\varphi, \psi\rangle &= B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\mathcal{D}^{\mathcal{L},+}\varphi, \Psi] \\ &= -B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\Phi, \Psi] + B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\mathcal{L}^{-1}(\mathscr{F}_\Phi^+), \Psi] \\ &= -\overline{B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\Psi, \Phi]} + \overline{B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\Psi, \mathcal{L}^{-1}(\mathscr{F}_\Phi^+)]} \\ &= \overline{B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\mathcal{D}^{\mathcal{L}^*,+}\psi, \Phi]} - \overline{B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[(\mathcal{L}^*)^{-1}(\mathscr{F}_\Psi^{*+}), \Phi]} + \overline{B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\Psi, \mathcal{L}^{-1}(\mathscr{F}_\Phi^+)]} \\ &= \langle \varphi, \partial_\nu^{\mathcal{L}^*,+}\mathcal{D}^{\mathcal{L}^*,+}\psi\rangle + \overline{\Big[- B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[(\mathcal{L}^*)^{-1}(\mathscr{F}_\Psi^{*+}), \Phi] + B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\Psi, \mathcal{L}^{-1}(\mathscr{F}_\Phi^+)]\Big]},\end{aligned}$$

where in the first equality we used the definition of the conormal derivative, in the second equality we used the definition of the double layer potential. Hence it suffices to show that $B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\Psi, \mathcal{L}^{-1}(\mathscr{F}_\Phi^+)] = B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[(\mathcal{L}^*)^{-1}(\mathscr{F}_\Psi^{*+}), \Phi]$. Simply note that

$$\begin{aligned}B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\Psi, \mathcal{L}^{-1}(\mathscr{F}_\Phi^+)] &= \langle \mathscr{F}_\Psi^{*+}, \mathcal{L}^{-1}(\mathscr{F}_\Phi^+)\rangle = \langle (\mathcal{L}^{-1})^*(\mathscr{F}_\Psi^{*+}), \mathscr{F}_\Phi^+\rangle \\ &= \overline{B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\Phi, (\mathcal{L}^{-1})^*(\mathscr{F}_\Psi^{*+})]} = B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[(\mathcal{L}^*)^{-1}(\mathscr{F}_\Psi^{*+}), \Phi],\end{aligned}$$

where in the first equality we used the definition of the functional $\mathscr{F}_{\Psi}^{*+}$, and in the third equality we used the definition of $\mathscr{F}_{\Phi}^{+}$. The desired identity follows.

Proof of (ii)). Let $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, $\varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$, and let $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$ be an extension of φ such that $\operatorname{Tr}_0 \Phi = \varphi$. By the definition of $\mathcal{D}^{\mathcal{L},+}\varphi$, we have that $\langle \gamma, \operatorname{Tr}_t \mathcal{D}^{\mathcal{L},+}\varphi\rangle = -\langle \gamma, \operatorname{Tr}_t \Phi\rangle + \langle \gamma, \operatorname{Tr}_t \mathcal{L}^{-1}(\mathscr{F}_{\Phi}^{+})\rangle$. By (4.4.4), we have that

$$
\begin{aligned}
\langle \gamma, \operatorname{Tr}_t \mathcal{L}^{-1}(\mathscr{F}_{\Phi}^{+})\rangle &= \langle (\operatorname{Tr}_t \circ \mathcal{L}^{-1})^* \gamma, \mathscr{F}_{\Phi}^{+}\rangle = \langle \mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \mathscr{F}_{\Phi}^{+}\rangle \\
&= \overline{\langle \mathscr{F}_{\Phi}^{+}, \mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma\rangle} = \overline{B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\Phi, \mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma]} = B_{\mathcal{L}^*,\mathbb{R}^{n+1}_+}[\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \Phi] \\
&= B_{\mathcal{L}^*}[\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \Phi] - B_{\mathcal{L}^*,\mathbb{R}^{n+1}_-}[\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \Phi] = \langle \gamma, \operatorname{Tr}_t \Phi\rangle - \langle \partial_\nu^{\mathcal{L}^*,-}\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle,
\end{aligned}
$$

where in the last equality we used (4.4.4) combined with (4.4.3) for the first term, and for the second term we used the definition of the conormal derivative and the fact that $\mathcal{L}\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*} = 0$ in $\mathbb{R}^{n+1}_-$. From this calculation, the first equality in (4.4.19) follows. The second and third equalities are straightforward consequences of Lemma 4.4.11. To see that (4.4.20) is true, simply observe that when $t = 0$, we have that $\mathcal{L}^*\mathcal{S}^{\mathcal{L}^*}\gamma = 0$ in $\mathbb{R}^{n+1}_+$ and in $\mathbb{R}^{n+1}_-$, whence we deduce that $\langle \partial_\nu^{\mathcal{L}^*,-}\mathscr{T}^{-0}\mathcal{S}^{\mathcal{L}^*}\gamma + \partial_\nu^{\mathcal{L}^*,+}\mathscr{T}^{-0}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle = B_{\mathcal{L}^*}[\mathcal{S}^{\mathcal{L}^*}\gamma, \Phi] = \langle \gamma, \operatorname{Tr}_0 \Phi\rangle$. Adding and subtracting $\langle \partial_\nu^{\mathcal{L}^*,+}\mathscr{T}^{-t}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle$ to the right-hand side of (4.4.19) now proves the claim.

Proof of (iii)). Let $t > 0$. By Proposition 4.4.7 (iv)), we have that $\mathcal{L}\mathcal{D}^{\mathcal{L},+}\varphi = 0$ in $\mathbb{R}^{n+1}_+$. Therefore, using Proposition 4.3.23 (iv)) we see that $\operatorname{Tr}_\tau D_{n+1}\mathcal{D}^{\mathcal{L},+}\varphi \in H^{\frac{1}{2}}(\mathbb{R}^n)$ for each $\tau > 0$. Similarly, we have that $\operatorname{Tr}_{-\tau} \nabla D_{n+1}\mathcal{S}^{\mathcal{L}^*}\zeta \in L^2(\mathbb{R}^n)$ for each $\tau > 0$. Using (4.3.26) and (ii)), we calculate that $\frac{d}{d\tau}(\operatorname{Tr}_t \mathcal{D}^{\mathcal{L},+}\varphi, \zeta)\big|_{\tau=t} = -\frac{d}{d\tau}(\varphi, \partial_{\nu,-t}^{\mathcal{L}^*,-}\mathcal{S}^{\mathcal{L}^*}\zeta)\big|_{\tau=t}$. Now we use the characterization of the conormal derivative, (4.4.12), to obtain

$$
\begin{aligned}
-\frac{d}{d\tau}(\varphi, \partial_{\nu,-t}^{\mathcal{L}^*,-}\mathcal{S}^{\mathcal{L}^*}\zeta)\big|_{\tau=t} &= -\frac{d}{d\tau}\big(\varphi, \big[e_{n+1}\cdot \operatorname{Tr}_{-\tau}(A^*\nabla + \overline{B_2})\mathcal{S}^{\mathcal{L}^*}\zeta\big]\big)_{2,2}\big|_{\tau=t} \\
&= \big(\varphi, \big[e_{n+1}\cdot \operatorname{Tr}_{-t}(A^*\nabla + \overline{B_2})D_{n+1}\mathcal{S}^{\mathcal{L}^*}\zeta\big]\big)_{2,2}.
\end{aligned}
$$

Finally, (iv)) follows from (iii)) similarly as in Proposition 4.4.2 (viii)). $\square$

Let us now establish standard jump relations.

Proposition 4.4.22 (Jump relations)**.** *Let $\varphi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$ and $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$.*

(i) The identity $\mathrm{Tr}_0\, \mathcal{D}^{\mathcal{L},+}\varphi + \mathrm{Tr}_0\, \mathcal{D}^{\mathcal{L},+}\varphi = -\varphi$ *holds in* $H_0^{\frac{1}{2}}(\mathbb{R}^n)$.

(ii) The identity $\partial_\nu^{\mathcal{L},+}\mathcal{S}^{\mathcal{L}}\gamma + \partial_\nu^{\mathcal{L},-}\mathcal{S}^{\mathcal{L}}\gamma = \gamma$ *holds in* $H^{-\frac{1}{2}}(\mathbb{R}^n)$.

(iii) The identity $\partial_\nu^{\mathcal{L},+}\mathcal{D}^{\mathcal{L},+}\varphi = \partial_\nu^{\mathcal{L},-}\mathcal{D}^{\mathcal{L},-}\varphi$ *holds in* $H^{-\frac{1}{2}}(\mathbb{R}^n)$.

(iv) The identity $\mathrm{Tr}_0(\mathcal{S}^{\mathcal{L}}\gamma|_{\mathbb{R}^{n+1}_+}) = \mathrm{Tr}_0(\mathcal{S}^{\mathcal{L}}\gamma|_{\mathbb{R}^{n+1}_-})$ *holds in* $H_0^{\frac{1}{2}}(\mathbb{R}^n)$.

Proof. The statement (iv)) is immediate from the fact that $\mathcal{S}^{\mathcal{L}}\gamma \in Y^{1,2}(\mathbb{R}^{n+1})$. The statement (ii)) follows from the definition of the conormal derivative and the fact that $\mathcal{L}\mathcal{S}^{\mathcal{L}}\gamma = 0$ in $\mathbb{R}^{n+1}\backslash\{t=0\}$.

Proof of (i)). Let $\gamma \in H^{-\frac{1}{2}}(\mathbb{R}^n)$, and let $\Phi \in Y^{1,2}(\mathbb{R}^{n+1})$ be any extension of φ. Using (4.4.20), we see that

$$\begin{aligned}\langle \gamma, \mathrm{Tr}_0[\mathcal{D}^{\mathcal{L},+}\varphi + \mathcal{D}^{\mathcal{L},-}\varphi]\rangle &= -2\langle\gamma,\varphi\rangle + \langle \partial_\nu^{\mathcal{L}^*,-}\mathcal{S}^{\mathcal{L}^*}\gamma + \partial_\nu^{\mathcal{L}^*,+}\mathcal{S}^{\mathcal{L}^*}\gamma, \varphi\rangle \\ &= -2\langle\gamma,\varphi\rangle + B_{\mathcal{L}^*}[\mathcal{S}^{\mathcal{L}^*}\gamma, \Phi] = -2\langle\gamma,\varphi\rangle + \langle\gamma, \mathrm{Tr}_0\,\Phi\rangle = -\langle\gamma,\varphi\rangle.\end{aligned}$$

Proof of (iii)). Let $\psi \in H_0^{\frac{1}{2}}(\mathbb{R}^n)$, and let $\Phi, \Psi \in Y^{1,2}(\mathbb{R}^{n+1})$ be extensions of φ, ψ respectively such that $\mathrm{Tr}_0\,\Phi = \varphi, \mathrm{Tr}_0\,\Psi = \psi$. Also recall that $\mathcal{L}\mathcal{D}^{\mathcal{L},+}\varphi = 0$ in $\mathbb{R}^{n+1}_+$, and $\mathcal{L}\mathcal{D}^{\mathcal{L},-}\varphi = 0$ in $\mathbb{R}^{n+1}_-$. Then,

$$\begin{aligned}&\langle \partial_\nu^{\mathcal{L},+}\mathcal{D}^{\mathcal{L},+}\varphi, \psi\rangle \\ &\quad= B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\mathcal{D}^{\mathcal{L},+}\varphi, \Psi] = -B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\Phi,\Psi] + B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\mathcal{L}^{-1}(\mathscr{F}_\Phi^+), \Psi] \\ &\quad= -B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\Phi,\Psi] + B_{\mathcal{L}}[\mathcal{L}^{-1}(\mathscr{F}_\Phi^+), \Psi] - B_{\mathcal{L},\mathbb{R}^{n+1}_-}[\mathcal{L}^{-1}(\mathscr{F}_\Phi^+), \Psi] \\ &= -B_{\mathcal{L},\mathbb{R}^{n+1}_+}[\Phi,\Psi] + \langle \mathscr{F}_\Phi^+, \Psi\rangle - B_{\mathcal{L},\mathbb{R}^{n+1}_-}[\mathcal{L}^{-1}(\mathcal{L}\Phi), \Psi] + B_{\mathcal{L},\mathbb{R}^{n+1}_-}[\mathcal{L}^{-1}(\mathscr{F}_\Phi^-), \Psi] \\ &\quad= -B_{\mathcal{L},\mathbb{R}^{n+1}_-}[\Phi,\Psi] + B_{\mathcal{L},\mathbb{R}^{n+1}_-}[\mathcal{L}^{-1}(\mathscr{F}_\Phi^-), \Psi] = B_{\mathcal{L},\mathbb{R}^{n+1}_-}[\mathcal{D}^{\mathcal{L},-}\varphi, \Psi] \\ &\qquad= \langle \partial_\nu^{\mathcal{L},-}\mathcal{D}^{\mathcal{L},-}\varphi, \psi\rangle.\end{aligned}$$

□

4.4.2 Initial L^2 estimates for the single layer potential

We now establish several estimates for the single layer potential. This will allow us to prove the square function estimates, via a Tb theorem, in the next section. We begin with a perturbation result.

Proposition 4.4.23 (Initial slice estimates)**.** *The following statements hold provided that* $\max\{\|B_1\|_n, \|B_2\|_n\}$ *is small enough, depending only on* n *and* C_A*.*

(i) For each $f \in C_c^\infty(\mathbb{R}^n)$*, each* $a > 0$*, and each* $m \geq 1$*, we have the estimate*

$$\fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla \partial_t^m \mathcal{S}^{\mathcal{L}} f|^2 \, dt \lesssim_m \|f\|_2^2. \tag{4.4.24}$$

(ii) For each $f \in C_c^\infty(\mathbb{R}^n)$*, each* $t \geq 0$*, and each* $m \geq 2$*, we have the estimate*

$$\|t^m \nabla \partial_t^m \mathcal{S}_t^{\mathcal{L}} f\|_{L^2(\mathbb{R}^n)} \lesssim_m \|f\|_2. \tag{4.4.25}$$

Proof. First we see that the second estimate is a consequence of the first by the Caccioppoli inequality on slices (4.3.22). In particular, we have that

$$\|t^m \nabla \partial_t^m \mathcal{S}_t^{\mathcal{L}} f\|_2^2 = \sum_{Q \in \mathbb{D}_t} \int_Q |t^m \nabla \partial_t^m \mathcal{S}_t^{\mathcal{L}} f|^2 \, dx \lesssim \fint_t^{2t} \int_{\mathbb{R}^n} |s^m \nabla \partial_s^{m-1} \mathcal{S}_s^{\mathcal{L}} f|^2 \, dx \, ds,$$

where $\mathbb{D}_t$ is a grid of n-dimensional cubes of side length t. Thus it suffices to show (i)).

To this end, we know from [AAA+11] that (i)) holds with $\mathcal{S}^{\mathcal{L}}$ replaced by $\mathcal{S}^{\mathcal{L}_0}$, where $\mathcal{L}_0 = -\Delta$. Thus, to prove (i)), we show that $\fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla \partial_t^m (\mathcal{S}^{\mathcal{L}} - \mathcal{S}^{\mathcal{L}_0}) f|^2 \, dt \lesssim \|f\|_2$. Observe that

$$\begin{aligned} \mathcal{S}^{\mathcal{L}} - \mathcal{S}^{\mathcal{L}_0} &= (\mathrm{Tr}_0 \circ ((\mathcal{L}^*)^{-1} - (\mathcal{L}_0^*)^{-1}))^* = (\mathrm{Tr}_0 \circ ((\mathcal{L}_0^*)^{-1}(\mathcal{L}_0^* - \mathcal{L}^*)(\mathcal{L}^*)^{-1}))^* \\ &= ((\mathcal{L}_0^* - \mathcal{L}^*)(\mathcal{L}^*)^{-1})^* \mathcal{S}^{\mathcal{L}_0} = \mathcal{L}^{-1}(\mathcal{L}_0 - \mathcal{L}) \mathcal{S}^{\mathcal{L}_0} \\ &= -\operatorname{div}(I - A) \nabla \mathcal{S}^{\mathcal{L}_0} - \mathcal{L}^{-1} \operatorname{div}(B_1 \mathcal{S}^{\mathcal{L}_0}) - \mathcal{L}^{-1} B_2 \cdot \nabla \mathcal{S}^{\mathcal{L}_0}. \end{aligned}$$

Now let $f \in C_c^\infty(\mathbb{R}^n)$. Then we have that

$$\begin{aligned} &\fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m (\mathcal{S}^{\mathcal{L}} - \mathcal{S}^{\mathcal{L}_0}) f|^2 \, dt \\ &\qquad \lesssim \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(I - A) \nabla \mathcal{S}^{\mathcal{L}_0} f)|^2 \, dt \\ &\qquad\quad + \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \, dt \\ &\qquad + \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} B_2 \cdot \nabla \mathcal{S}^{\mathcal{L}_0} f|^2 \, dt \; quad =: I + II + III. \end{aligned}$$

We prove only the bound $II \lesssim \|f\|_2^2$ as the bounds for I and III are entirely analogous, and we will indicate the small differences after we bound II. Let $\psi = \psi(t)$ be such that $\psi \in C_c^\infty(-a/5, a/5)$, $\psi \equiv 1$ on $(-a/10, a/10)$, $0 \le \psi \le 1$, $\frac{d^k}{dt^k}\psi \lesssim_k (1/a)^k$. Writing $1 = \psi + (1-\psi)$, we have that

$$\begin{aligned} II &\le \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \, dt \\ &\le \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(\psi B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \, dt \\ &\quad + \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}((1-\psi) B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \, dt =: II_1 + II_2. \end{aligned}$$

To bound II_1, we notice that if $g = \operatorname{div}(\psi B_1 \mathcal{S}^{\mathcal{L}_0} f)$, then $g \equiv 0$ on $\mathbb{R}^n \times (a/5, \infty)$. It follows that each $D_{n+1}^k \mathcal{L}^{-1} g = \mathcal{L}^{-1} D_{n+1}^k g$, $k = 0, 1, \dots m$ is a (null) solution in $\mathbb{R}^n \times (a/5, \infty)$. Let $\mathbb{D}_a$ be a grid of n-dimensional cubes with side length a. Applying the Caccioppoli inequality m times and using that $t \approx a$ on $(a, 2a)$, we see that

$$\begin{aligned} II_1 &\lesssim a^{2m-1} \int_a^{2a} \int_{\mathbb{R}^n} |\nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(\psi B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \\ &\lesssim a^{2m-1} \sum_{Q \in \mathbb{D}_a} \int_a^{2a} \int_Q |\nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(\psi B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \\ &\lesssim a^{-1} \sum_{Q \in \mathbb{D}_a} \int_{a/2}^{4a} \int_{2Q} |D_{n+1} \mathcal{L}^{-1} \operatorname{div}(\psi B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \\ &\lesssim a^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\nabla \mathcal{L}^{-1} \operatorname{div}(\psi B_1 \mathcal{S}^{\mathcal{L}_0} f)|^2 \\ &\lesssim a^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\psi B_1 \mathcal{S}^{\mathcal{L}_0} f|^2 \lesssim \fint_{-a/5}^{a/5} \int_{\mathbb{R}^n} |B_1 \mathcal{S}^{\mathcal{L}_0} f|^2 \lesssim \|f\|_2^2, \end{aligned}$$

where we used that $\sup_{t \neq 0} \|B_1 \mathcal{S}_t^{\mathcal{L}_0}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim \sup_{t \neq 0} \|\nabla \mathcal{S}_t^{\mathcal{L}_0}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C$ (see [AAA+11, Lemma 4.18]) and that $\nabla \mathcal{L}^{-1} \operatorname{div} : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$.

Now we deal with II_2. Set $g = (1-\psi) B_1 \mathcal{S}^{\mathcal{L}_0} f$. Then, we have that

$$D_{n+1}^m g = (1-\psi) B_1 D_{n+1}^m \mathcal{S}^{\mathcal{L}_0} f + \sum_{k=1}^m \psi^{(k)} B_1 D_{n+1}^{m-k} \mathcal{S}^{\mathcal{L}_0} f =: F_0 + \sum_{k=1}^m F_k.$$

where $\psi^{(k)} = \frac{d^k}{dt^k}\psi$. The triangle inequality yields that

$$II_2 \leq \sum_{k=0}^{m} \fint_a^{2a} \int_{\mathbb{R}^n} |t^m \nabla D_{n+1}^m \mathcal{L}^{-1} \operatorname{div}(F_k)|^2 \, dt =: \sum_{k=0}^{m} II_{2,k}.$$

For $II_{2,k}$, $k = 1, 2 \ldots m$, we use that $t \approx a$ in the region of integration, the properties of ψ, and that $\nabla \mathcal{L}^{-1} \operatorname{div} : L^2 \to L^2$ to obtain that

$$\begin{aligned} II_{2,k} &\lesssim a^{2m-1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\psi^{(k)} B_1 \partial_t^{m-k} \mathcal{S}^{\mathcal{L}_0} f|^2 \, dt \\ &\qquad \lesssim a^{2m-2k-1} \int\limits_{a/10 \leq |t| \leq a/5} \int_{\mathbb{R}^n} |B_1 \partial_t^{m-k} \mathcal{S}^{\mathcal{L}_0} f|^2 \, dt \\ &\qquad\qquad \lesssim \fint_{-a/5}^{a/5} \int_{\mathbb{R}^n} |t^{m-k} B_1 \partial_t^{m-k} \mathcal{S}^{\mathcal{L}_0} f|^2 \, dt \lesssim \|f\|_2^2, \end{aligned}$$

where we used [AAA+11, Lemma 2.10] in the last line. Finally, to handle $II_{2,0}$, we use that $(1-\psi) = 0$ if $|t| < a/10$, and that $\nabla \mathcal{L} \operatorname{div} : L^2 \to L^2$ to obtain that

$$\begin{aligned} II_{2,0} &\lesssim a^{2m-1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |(1-\psi) B_1 \partial_t^m \mathcal{S}^{\mathcal{L}_0} f|^2 \, dt \\ &\qquad \lesssim a^{2m-1} \int_{|t| > a/10} \int_{\mathbb{R}^n} |B_1 \partial_t^m \mathcal{S}^{\mathcal{L}_0} f|^2 \, dt \\ &\qquad\qquad \lesssim \int_{|t| > a/10} \int_{\mathbb{R}^n} |t^{m+1} B_1 \partial_t^m \mathcal{S}^{\mathcal{L}_0} f|^2 \, \frac{dt}{t} \lesssim \|f\|_2^2, \end{aligned}$$

where we used the estimate $|||t^{m+1} B_1 \partial_t^m \mathcal{S}^{\mathcal{L}_0} f|||_2^2 \lesssim |||t^{m+1} \partial_t^m \nabla \mathcal{S}^{\mathcal{L}_0} f|||_2^2 \lesssim \|f\|_2^2$ in the last line. To see this last estimate, we simply use the "travelling up" procedure for square functions (see Proposition 4.5.2 below) and that $\mathcal{L}_0 = \Delta$ has good square function estimates. We now observe that handling the term III amounts to replacing the use of the mapping property $\nabla \mathcal{L}^{-1} \operatorname{div} : L^2 \to L^2$ by the fact that $\nabla \mathcal{L}^{-1} B_2 : L^2 \to L^2$. The term I is handled exactly the same way, using the L^∞ bound for $(I - A)$, without appealing to the mapping properties of multiplication by B_1. □

Remark 4.4.26. Note that, from now on, it makes sense to write the objects appearing in (4.4.24) and (4.4.25) for f in $L^2(\mathbb{R}^n)$ after we have made extensions by continuity.

Before proceeding, we will need some identities improving on the duality results in

Section 4.4 for the single and double layers. To ease the notation, we will use $(G)_t$ to denote the trace at t of a function G defined in $\mathbb{R}^{n+1}_+$.

Proposition 4.4.27 (Further distributional identities of the layer potentials)**.** *For any $t \neq 0$ and $m \geq 1$, the following statements are true.*

(i) For any $f \in C_c^\infty(\mathbb{R}^n)$ and any $\vec{g} \in L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$, we have that

$$\frac{d^m}{dt^m} \langle \nabla \mathcal{S}_t^{\mathcal{L}} f, \vec{g} \rangle = \langle (D_{n+1}^m \nabla \mathcal{S}^{\mathcal{L}}[f])_t, \vec{g} \rangle.$$

(ii) For any $f \in L^2(\mathbb{R}^n)$ and any $\vec{g} \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$, we have that

$$\frac{d^m}{dt^m} \langle f, ((\mathcal{S}^{\mathcal{L}^*} \nabla)[\vec{g}])_{-t} \rangle = (-1)^m \langle f, (D_{n+1}^m (\mathcal{S}^{\mathcal{L}^*} \nabla)[\vec{g}])_{-t} \rangle.$$

(iii) If $m \geq 2$, then for every $f \in L^2(\mathbb{R}^n)$ and $\vec{g} \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}))$, we have the identity

$$\langle (D_{n+1}^m \nabla \mathcal{S}^{\mathcal{L}}[f])_t, \vec{g} \rangle = (-1)^m \langle f, (D_{n+1}^m (\mathcal{S}^{\mathcal{L}^*} \nabla)[\vec{g}])_{-t} \rangle.$$

Proof. Let us first show the identities with $f \in C_c^\infty(\mathbb{R}^n)$ and $\vec{g} \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$. For the first equality, note that $u := \mathcal{S}^{\mathcal{L}}[f] \in Y^{1,2}(\mathbb{R}^{n+1})$ and $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \setminus \{x_{n+1} = 0\}$. In particular, $\partial_t u \in W^{1,2}(\Sigma_a^b)$ for any $a < b$ such that $0 \notin [a, b]$, by Lemma 4.3.17. By iteration we have that $\partial_t^m \nabla u \in L^2(\Sigma_a^b)$. In particular, arguing as in Lemma 4.2.3, we realize that the map $t \mapsto \nabla u(\cdot, t)$ is smooth (with values in $L^2(\mathbb{R}^n; \mathbb{C}^{n+1})$). The first equality for $m = 1$ then boils down to proving the weak convergence of the difference quotients to the derivative in $L^2(\mathbb{R}^n)$; that is, showing that

$$\lim_{h \to 0} \frac{\nabla u(t+h) - \nabla u(t)}{h} = \partial_t \nabla u(t), \qquad \text{weakly in } L^2(\mathbb{R}^n).$$

But this follows from the smoothness of our map. The case of general m now follows by induction.

For the second equality, by definition we have that

$$((\mathcal{S}^{\mathcal{L}^*} \nabla)[\vec{g}])_s = -(\mathcal{S}^{\mathcal{L}^*}[\operatorname{div}_\| g_\|])_s - (D_{n+1} \mathcal{S}^{\mathcal{L}^*}[g_\perp])_s,$$

and since $\vec{g} \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$, we can apply the same argument as above to conclude that

$$\frac{d^m}{dt^m}\langle f, ((\mathcal{S}^{\mathcal{L}^*}\nabla)[g])_{-t}\rangle = (-1)^m \langle f, (D_{n+1}^m(\mathcal{S}^{\mathcal{L}^*}\nabla)[g_\perp])_{-t}\rangle.$$

The third equality now follows by duality: For $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$, we have that

$$\begin{aligned}\langle (D_{n+1}^m \nabla \mathcal{S}^{\mathcal{L}}[f])_t, \vec{g}\rangle = \frac{d^m}{dt^m}\langle \nabla \mathcal{S}_t^{\mathcal{L}} f, \vec{g}\rangle = \frac{d^m}{dt^m}\langle f, ((\mathcal{S}^{\mathcal{L}}\nabla)[\vec{g}])_{-t}\rangle \\ = (-1)^m \langle f, (D_{n+1}^m(\mathcal{S}^{\mathcal{L}}\nabla)[\vec{g}])_{-t}\rangle\end{aligned}$$

Finally, the identities are extended to the respective L^2 spaces via a straightforward density argument using Proposition 4.4.23. □

We now present an off-diagonal decay result.

Proposition 4.4.28 (Good off-diagonal decay)**.** *Let $Q \subset \mathbb{R}^n$ be a cube and $g \in L^2(Q)$ with $\operatorname{supp} g \subseteq Q$. If $p \in [2, p_+]$ is such that $|p-2|$ is small enough that Lemma 4.3.4 holds, we have that*

$$\Big(\int_{R_0} |t^m(\partial_t)^m \nabla \mathcal{S}_t^{\mathcal{L}} g(x)|^p \, dx\Big)^{\frac{1}{p}} \lesssim 2^{-(m+1)} t^m \ell(Q)^{-n(1/2-1/p)} \ell(Q)^{-m} \|g\|_{L^2(Q)},$$

provided $t \approx \ell(Q)$. Moreover, for any $k \geq 1$ and any $t \in \mathbb{R}$, the estimate

$$\begin{aligned}\Big(\int_{R_k} |t^m(\partial_t)^m \nabla \mathcal{S}_t^{\mathcal{L}} g(x)|^p \, dx\Big)^{\frac{1}{p}} \\ \lesssim 2^{nk\alpha} 2^{-k(m+1)} t^m \ell(Q)^{-n(1/2-1/p)} \ell(Q)^{-m} \|g\|_{L^2(Q)},\end{aligned}$$

where $\alpha = \alpha(p) = \frac{1}{p}(1 - \frac{p}{p_+})$ and the annular regions $R_k = R_k(Q)$ are defined by $R_0 := 2Q$, $R_k := 2^{k+1}Q \setminus 2^k Q$, for all $k \geq 1$. In particular, if $t \approx \ell(Q)$ we have that

$$\Big(\int_{R_k} |t^m(\partial_t)^m \nabla \mathcal{S}_t^{\mathcal{L}} g(x)|^p \, dx\Big)^{\frac{1}{p}} \lesssim 2^{nk\alpha} 2^{-k(m+1)} \ell(Q)^{-n(1/2-1/p)} \|g\|_{L^2(Q)}.$$

By a straightforward duality argument, from the above proposition we deduce

Corollary 4.4.29. *Define $\Theta_{t,m} := t^m \partial_t^m(\mathcal{S}_t \nabla)$. Let $g \in L^2(Q)$ and suppose that*

$p \in [2, p_+]$ is such that $|p-2|$ is small enough so that Lemma 4.3.4 holds. Then for $q = \frac{p}{p-1}$ and $k \geq 1$, we have that

$$\|\Theta_{t,m}(f\mathbb{1}_{R_k})\|_{L^2(Q)} \lesssim 2^{nk\alpha} 2^{-k(m+1)} t^m \ell(Q)^{-n(1/q-1/2)} \ell(Q)^{-m} \|f\|_{L^q(R_k)},$$

where $\alpha = \alpha(p)$ is as in Proposition 4.4.28. Moreover, if $t \approx \ell(Q)$, then for all $k \geq 0$,

$$\begin{aligned}\|\Theta_{t,m}(f\mathbb{1}_{R_k})\|_{L^2(Q)} &\lesssim 2^{nk\alpha} 2^{-k(m+1)} \ell(Q)^{-n(1/q-1/2)} \|f\|_{L^q(R_k)} \\ &\approx 2^{nk\alpha} 2^{-k(m+1)} t^{-n(1/q-1/2)} \|f\|_{L^q(R_k)}.\end{aligned}$$

Proof of Proposition 4.4.28. Notice that $g \in L^2(Q) \subset L^{2n/(n+1)}(Q) \subset H^{-\frac{1}{2}}(\mathbb{R}^n)$, so that $\mathcal{S}_t^{\mathcal{L}} g$ is well defined.

We treat first the case $k \geq 1$. Fix a small parameter $\delta = \delta(m) > 0$ and set $\tilde{R}_k = (2+\delta)^{k+1} Q \setminus (2-\delta)^k Q$ be a small (but fixed) dilation of R_k. We may use that $\partial_t^m \mathcal{S}_t g$ is a solution (see Proposition 4.3.16), a slight variant of Lemma 4.3.20 adapted to annular regions, and Proposition 4.3.9 to see that

$$\Big(\int_{R_k} |t^m (\partial_t)^m \nabla \mathcal{S}_t^{\mathcal{L}} g|^p\Big)^{\frac{1}{p}} \lesssim \frac{t^m}{(2^k \ell(Q))^{1+1/p}} \Big(\iint_{I_{k,1}} |\partial_s^m \mathcal{S}_s^{\mathcal{L}} g|^p\Big)^{1/p},$$

where $I_k := \{(y,s) \in \mathbb{R}^{n+1} : y \in \tilde{R}_{k,1},\, s \in (t - 2^k \ell(Q), t + 2^k \ell(Q)\}$ and $\tilde{R}_{k,j}$ is defined as $\tilde{R}_k$ but with $\delta/(m+2-j)$ instead of δ (so that, in particular, $\tilde{R}_{k,m+1} = \tilde{R}_k$). Now, applying the $(n+1)$-dimensional L^p Caccioppoli m times (see Proposition 4.3.9), we further obtain that

$$\Big(\int_{R_k} |t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} g|^p\Big)^{\frac{1}{p}} \lesssim \frac{t^m}{(2^k \ell(Q))^{m+1+1/p}} \Big(\iint_{I_{k,m+1}} |\mathcal{S}_s^{\mathcal{L}} g|^p\Big)^{1/p}.$$

Now, using Hölder's inequality in t and the mapping properties of $\mathcal{S}_t^{\mathcal{L}}$ we see that

$$\begin{aligned}\Big(\int_{R_k} |t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} g|^p\Big)^{\frac{1}{p}} &\lesssim \frac{t^m}{[2^k \ell(Q)]^{m+1}} \sup_{t \in (-2^k \ell(Q), 2^k \ell(Q))} \|\mathcal{S}_t^{\mathcal{L}} g\|_{L^p(\tilde{R}_k)} \\ &\lesssim \frac{t^m [2^k \ell(Q)]^{n\alpha}}{[2^k \ell(Q)]^{m+1}} \sup_{t \in (-2^k \ell(Q), 2^k \ell(Q))} \|\mathcal{S}_t^{\mathcal{L}} g\|_{L^{p_+}(\tilde{R}_k)} \lesssim \frac{t^m [2^k \ell(Q)]^{n\alpha}}{[2^k \ell(Q)]^{m+1}} \|g\|_{L^{p_-}(Q)} \\ &\lesssim \frac{t^m [2^k \ell(Q)]^{n\alpha}}{[2^k \ell(Q)]^{m+1}} |Q|^{\frac{1}{2n}} \|g\|_{L^2(Q)}.\end{aligned}$$

The case $k = 0$ is treated similarly, except that we impose the restriction $t \approx \ell(Q)$ to guarantee that we are far away from the support of g. □

For the most part, the case $q = p = 2$ in the above proposition will be enough for our purposes; however, the introduction of error terms in the Tb theorem below will necessitate a certain quasi-orthogonality result for which we use the case $p > 2 > q$.

Lemma 4.4.30 (Quasi-orthogonality). *Let $m \geq n$ and let $\mathcal{Q}_s$ be a CLP family (see Definition 4.2.26). Then there exist $\gamma, C > 0$ such that for all $s < t$, we have that*

$$\|\Theta_{t,m} B_1 I_1 \mathcal{Q}_s^2 g\|_2 \leq C\Big(\frac{s}{t}\Big)^{\gamma} \|\mathcal{Q}_s g\|_2 \tag{4.4.31}$$

for all $g \in L^2(\mathbb{R}^n)$, where I_1 is the standard fractional integral operator of order 1. Here, C and γ depend on m, n, C_A, and the constants in the definition of $\mathcal{Q}_s$.

Proof. Let us first note that if $\alpha(p)$ is given as in Proposition 4.4.28, then $\alpha(p) \leq 1/(2n)$. Therefore, for all $k \geq 0$ and Q with $\ell(Q) \approx t$, we have that

$$\begin{aligned}\|\Theta_{t,m}(f\mathbb{1}_{R_k})\|_{L^2(Q)} &\lesssim 2^{nk\alpha}2^{-k(m+1)}t^{-n(1/q-1/2)}\|f\|_{L^q(R_k)}\\ &\lesssim 2^{-k\beta}t^{-n(1/q-1/2)}\|f\|_{L^q(R_k)},\end{aligned} \tag{4.4.32}$$

for some $\beta \geq n/2 + 1$, where we use that $m \geq n$.

We first establish a variant of (4.4.31) with a collection of CLP families. Let $\zeta \in C_c^\infty(B(0, \frac{1}{100}))$ be real, radial and have zero average. Define $\mathcal{Q}_s^{(1)} f(x) := (\zeta_s * f)(x)$, where $\zeta_s(x) = s^{-n}\zeta(\frac{x}{s})$. Set $\mathcal{Q}_s^{(2)} f := s^2\Delta e^{s^2\Delta} f$. By re-normalizing ζ (multiplying by a constant) we may assume that

$$\int_0^\infty \mathcal{Q}_s^{(1)}\mathcal{Q}_s^{(2)}\frac{ds}{s} = I \tag{4.4.33}$$

in the strong operator topology of L^2. Indeed,

$$\begin{aligned}\mathscr{F}\Big(\int_0^\infty \mathcal{Q}_s^{(1)}\mathcal{Q}_s^{(2)} f\,\frac{ds}{s}\Big) &= -\int_0^\infty \hat{\zeta}(s|\xi|)s^2|\xi|^2 e^{-s^2|\xi|^2}\hat{f}(\xi)\,\frac{ds}{s}\\ &= -\hat{f}(\xi)\int_0^\infty \hat{\zeta}(s)s^2 e^{-s^2}\,\frac{ds}{s},\end{aligned}$$

where $\hat{\zeta}$ is the Fourier transform of ζ and we abused notation by regarding ζ and hence $\hat{\zeta}$ as a function of the radial variable. Then, to achieve the desired reproducing formula, (4.4.33), we may renormalize ζ so that $\int_0^\infty \hat{\zeta}(s) s^2 e^{-s^2} \frac{ds}{s} = -1$. Let $q < 2$ be such that the conclusion of Corollary 4.4.29 holds. We will show that for all $s < t$,

$$\|\Theta_{t,m} B_1 I_1 \mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} g\|_2 \lesssim \left(\frac{s}{t}\right)^{n(1/q-1/2)} \|\mathcal{Q}_s^{(3)} \vec{R} g\|_2, \tag{4.4.34}$$

where $\vec{R} = I_1 \nabla_{\|}$ is the vector valued Riesz transform on $\mathbb{R}^n$ and $\mathcal{Q}_s^{(3)} \vec{f} := s e^{s^2 \Delta} \operatorname{div}_{\|} \vec{f}$. Before proving (4.4.34), we establish a "local hypercontractivity" estimate. For $Q \subset \mathbb{R}^n$ a cube and $s < \ell(Q)$, we have that

$$\|\mathcal{Q}_s^{(1)} h\|_{L^{\frac{nq}{n-q}}(R_k(Q))} \lesssim s^{-n\left(\frac{1}{2} - \frac{n-q}{nq}\right)} \|h\|_{L^2(B_k)} \tag{4.4.35}$$

for all $k \geq 0$, where $R_0(Q) = 2Q$, $R_k(Q) = 2^{k+1}Q \setminus 2^k Q$ for $k \geq 1$, and $B^k(Q) = B(x_Q, 2^{k+2}\ell(Q)\sqrt{n})$. To verify (4.4.35), we use that $s < \ell(Q)$, Young's convolution inequality, and the properties of ζ_s.

Now we are ready to prove (4.4.34). Let $\mathbb{D}_t$ be a grid of cubes on $\mathbb{R}^n$ with side length t and set $F = I_1 g$. Consider the estimate

$$\begin{aligned}
\|\Theta_{t,m} B_1 I_1 \mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} g\|_2 &= \|\Theta_{t,m} B_1 \mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} F\|_2 \\
&= \Big(\sum_{Q \in \mathbb{D}_t} \int_Q |\Theta_{t,m} B_1 \mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} F|^2 \Big)^{1/2} \\
&\leq \sum_{k \geq 0} \Big(\sum_{Q \in \mathbb{D}_t} \int_Q \big|\Theta_{t,m}\big([B_1 \mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} F] \mathbb{1}_{R_k(Q)}\big)(x)\big|^2 \, dx \Big)^{1/2} \\
&\lesssim \sum_{k \geq 0} 2^{-\beta k} t^{-n(1/q-1/2)} \Big(\sum_{Q \in \mathbb{D}_t} \Big(\int_{R_k(Q)} |B_1 \mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} F|^q \Big)^{2/q} \Big)^{1/2} \\
&\lesssim \|B_1\|_n \sum_{k \geq 0} 2^{-\beta k} t^{-n(1/q-1/2)} \Big(\sum_{Q \in \mathbb{D}_t} \Big(\int_{R_k(Q)} |\mathcal{Q}_s^{(1)} \mathcal{Q}_s^{(2)} F|^{\frac{nq}{n-q}} \Big)^{\frac{2(n-q)}{nq}} \Big)^{1/2} \\
&\lesssim \sum_{k \geq 0} 2^{-\beta k} t^{-n(1/q-1/2)} s^{-n\left(\frac{1}{2} - \frac{n-q}{nq}\right)} \Big(\sum_{Q \in \mathbb{D}_t} \int_{B_k(Q)} |\mathcal{Q}_s^{(2)} F|^2 \Big)^{1/2} \\
&\lesssim \sum_{k \geq 0} 2^{-\beta k} t^{-n(1/q-1/2)} s^{-n\left(\frac{1}{2} - \frac{n-q}{nq}\right)} s \Big(\sum_{Q \in \mathbb{D}_t} \int_{B_k(Q)} |\mathcal{Q}_s^{(3)} \nabla_{\|} F|^2 \Big)^{1/2}
\end{aligned}$$

$$\lesssim \left(\frac{s}{t}\right)^{n(1/q-1/2)} \sum_{k\geq 0} 2^{-\beta k+\frac{nk}{2}} \Big(\sum_{Q\in\mathbb{D}_t} \int_Q \fint_{B_k(Q)} |\mathcal{Q}_s^{(3)} \nabla_{\|} F(x)|^2 \, dx \, dy \Big)^{1/2}$$
$$\lesssim \left(\frac{s}{t}\right)^{n(1/q-1/2)} \sum_{k\geq 0} 2^{-\beta k+\frac{nk}{2}} \Big(\int_{\mathbb{R}^n} \fint_{|x-y|<2^k t} |\mathcal{Q}_s^{(3)} \nabla_{\|} F(x)|^2 \, dx \, dy \Big)^{1/2}$$
$$\lesssim \left(\frac{s}{t}\right)^{n(1/q-1/2)} \| \mathcal{Q}_s^{(3)} \nabla_{\|} F\|_2 = \left(\frac{s}{t}\right)^{n(1/q-1/2)} \| \mathcal{Q}_s^{(3)} \vec{R} g\|_2,$$

where first we used that $I_1 g = F$, then Minkowski's inequality in the second line, (4.4.32) in the third line, Hölder's inequality in the fourth line, (4.4.35) in the fifth line, and the mapping properties of the Hardy-Littlewood maximal function in the last line. The above estimate proves (4.4.34).

Now we are ready to pass to an arbitrary CLP family $\mathcal{Q}_s$. We may obtain, using the Cauchy-Schwarz inequality and (4.4.33), that

$$\|\Theta_{t,m} B_1 I_1 \mathcal{Q}_s^2 g\|_2 = \int_{\mathbb{R}^n} \Big| \int_0^\infty \Theta_{t,m} B_1 I_1 \mathcal{Q}_\tau^{(1)} \mathcal{Q}_\tau^{(2)} \mathcal{Q}_s^2 g(x) \, \frac{d\tau}{\tau} \Big| \, dx$$
$$\leq C_\gamma \int_{\mathbb{R}^n} \int_0^\infty \max\left(\frac{s}{\tau}, \frac{\tau}{s}\right)^\gamma |\Theta_{t,m} B_1 I_1 \mathcal{Q}_\tau^{(1)} \mathcal{Q}_\tau^{(2)} \mathcal{Q}_s^2 g(x)|^2 \, \frac{d\tau}{\tau} \, dx =: I + II + III,$$

where I, II, III are, respectively, the integrals over the intervals $\tau < s < t$, $s \leq \tau \leq t$, and $s < t < \tau$. On the other hand, note that the kernel of $\mathcal{Q}_s^{(3)} \vec{R}$ is, up to a constant multiple, the inverse Fourier transform of $s|\xi|e^{-s^2|\xi|^2}$. Therefore, if we set $\mathcal{Q}_s^{(4)} = \mathcal{Q}_s^{(3)} \vec{R}$, then we have that

$$\max\left\{ \|\mathcal{Q}_\tau^{(4)} \mathcal{Q}_s f\|_2, \|\mathcal{Q}_\tau^{(2)} \mathcal{Q}_s f\|_2 \right\} \lesssim \min\left(\frac{\tau}{s}, \frac{s}{\tau}\right)^{2\gamma} \|f\|_2, \tag{4.4.36}$$

for some $\gamma > 0$ (and hence all smaller γ). For convenience, set $\sigma = n(1/q - 1/2)$ and we assume that γ above is such that $\gamma < 2\sigma$. By (4.4.34) and (4.4.36), we have that

$$I \lesssim \int_0^s \left(\frac{s}{\tau}\right)^\gamma \left(\frac{\tau}{t}\right)^{2\sigma} \|\mathcal{Q}_\tau^{(4)} \mathcal{Q}_s^2 h\|_2^2 \frac{d\tau}{\tau} \lesssim \left(\frac{s}{t}\right)^{2\sigma} \|Q_s h\|_2^2\,,$$

and observe that $\tau < s$ in the present scenario. Similarly, we have that

$$II \lesssim \int_s^t \left(\frac{\tau}{s}\right)^\gamma \left(\frac{\tau}{t}\right)^{2\sigma} \left(\frac{s}{\tau}\right)^{2\gamma} \|\mathcal{Q}_s h\|_2^2 \frac{d\tau}{\tau} \lesssim \left(\frac{s}{t}\right)^\gamma \|\mathcal{Q}_s h\|_2^2\,,$$

since in particular, $\gamma < 2\sigma$. Finally, by (4.4.25) and the mapping $B_1 I_1 : L^2(\mathbb{R}^n) \to$

$L^2(\mathbb{R}^n)$, we have that $\Theta_{t,m} B_1 I_1 \mathcal{Q}_\tau^{(1)} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ uniformly in t and τ, and thus it follows that

$$III \lesssim \int_t^\infty \left(\frac{\tau}{s}\right)^\gamma \|Q_\tau^{(2)} Q_s h(x)\|_2^2 \frac{d\tau}{\tau} \lesssim \int_t^\infty \left(\frac{\tau}{s}\right)^\gamma \left(\frac{s}{\tau}\right)^{2\gamma} \|Q_s h\|_2^2 \frac{d\tau}{\tau}$$
$$\lesssim \left(\frac{s}{t}\right)^\gamma \|Q_s h\|_2^2,$$

where we used (4.4.36). □

We conclude this section with the following proposition, which summarizes the off-diagonal decay given by Proposition 4.4.28 and Corollary 4.4.29.

Proposition 4.4.37. *For $m \in \mathbb{N}$, $m \geq \frac{n+1}{2} + 2$, the operators $t^m \partial_t^m (\mathcal{S}_t \nabla)$, $t^m \partial_t^{m+1} \mathcal{S}_t$ and Θ_t' defined by*

$$\Theta_t' \vec{g}(x) := (t^m \partial_t^m \mathcal{S}_t \nabla \widetilde{A} \vec{g} + t^m \partial_t^m \mathcal{S}_t [B_{2\|} g])(x)$$

have good off diagonal-decay in the sense of Definition 4.2.22 with the implicit constants depending on n, m, C_A, provided that $\max\{\|B_1\|_n, \|B_2\|_n\} < \varepsilon_0$, where ε_0 depends on n, C_A.

Proof. By Corollary 4.4.29 with $p = 2$, for any cube $Q \subset \mathbb{R}^n$ and $k \geq 2$ we have that

$$\|\Theta_{t,m}(f \mathbb{1}_{R_k})\|_{L^2(Q)}^2 \lesssim 2^{-k} \left(\frac{t}{2^k \ell(Q)}\right)^{2m} \|f\|_{L^2(R_k)}^2,$$

where $R_k = R_k(Q) = 2^{k+1}Q \setminus 2^k Q$. Thus, for all $t \in (0, C\ell(Q))$, it follows that

$$\|\Theta_{t,m}(f \mathbb{1}_{R_k})\|_{L^2(Q)}^2 \lesssim 2^{-kn} \left(\frac{t}{2^k \ell(Q)}\right)^{2m-(n-1)} \|f\|_{L^2(R_k)}^2,$$

so that if $m \geq \frac{n+1}{2}$, we obtain the estimate

$$\|\Theta_{t,m}(f \mathbb{1}_{R_k})\|_{L^2(Q)}^2 \lesssim 2^{-kn} \left(\frac{t}{2^k \ell(Q)}\right)^2 \|f\|_{L^2(R_k)}^2. \tag{4.4.38}$$

This bound provides the desired good off-diagonal decay for $t^m \partial_t^m (\mathcal{S}_t \nabla)$, $t^m \partial_t^{m+1} \mathcal{S}_t$ and $t^m \partial_t^m \mathcal{S}_t \nabla \widetilde{A} \vec{g}$ in the sense of Definition 4.2.22. To obtain the good off-diagonal decay for the remainder of Θ_t', $t^m \partial_t^m (\mathcal{S}_t B_{2\|})$, we return to the proof of Proposition 4.4.28 and make a slight modification. Let η be a smooth cut-off adapted to R_k; that is, $\eta \equiv 1$ on R_k, $\eta \in C_c^\infty(\widetilde{R}_k)$ and $|\nabla \eta| \lesssim \frac{1}{\ell(Q)}$, where $\widetilde{R}_k$ is as in Proposition 4.4.28. Then for $g \in L^2(Q)$ with $\operatorname{supp} g \subseteq Q$, from Hölder's inequality and the Sobolev embedding on $\mathbb{R}^n$ we have that

$$
\begin{aligned}
\|t^m \partial_t^m B_{2\|} \mathcal{S}_t g\|_{L^2(R_k)} &\lesssim \|\eta t^m \partial_t^m \mathcal{S}_t g\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\lesssim \|(\nabla \eta) t^m \partial_t^m \mathcal{S}_t g\|^2_{L^2(\widetilde{R}_k)} + \|t^m \partial_t^m \nabla \mathcal{S}_t g\|^2_{L^2(\widetilde{R}_k)} \\
&\lesssim \|t^{m-1} \partial_t^m \mathcal{S}_t g\|^2_{L^2(\widetilde{R}_k)} + \|t^m \partial_t^m \nabla \mathcal{S}_t g\|^2_{L^2(\widetilde{R}_k)} \\
&\lesssim \|(\Theta_{t,m-1})^* g\|^2_{L^2(\widetilde{R}_k)} + \|(\Theta_{t,m})^* g\|^2_{L^2(\widetilde{R}_k)}.
\end{aligned}
$$

Dualizing these estimates, the off-diagonal decay for $t^m \partial_t^m (\mathcal{S}_t B_{2\|})$ follows from the off-diagonal decay in (4.4.38), provided that $m \geq \frac{n+1}{2} + 1$. $\square$

Before continuing on to the next section we make two remarks.

Remarks 4.4.39. (i) In the next section, we will use the off diagonal decay of the operators in Proposition 4.4.37 or *similar* ones. The proof of good off-diagonal decay for these operators is entirely analogous to those above.

(ii) As seen above, there may be some loss of t-derivatives (and hence decay) in our operators when we obtain certain estimates. Therefore, when proving the first square function estimate (Theorem 4.5.1), we ensure that $m \geq n + 10 > \frac{n+1}{2} + 10$ so that Lemma 4.4.30 and Proposition 4.4.37 hold.

4.5 Square function bounds via Tb Theory

The goal of this section is to prove Theorem 4.1.1.

4.5.1 Reduction to high order t-derivatives

We will adapt the methods of [GdlHH16, HMM15b] to prove the square function bound in Theorem 4.1.1 for m large:

Theorem 4.5.1 (Square function bound for high $t-$derivatives)**.** *For each* $m \in \mathbb{N}$ *with*

$m \geq n+10$, we have the estimate

$$\iint_{\mathbb{R}^{n+1}_+} \left|t^m (\partial_t)^{m+1} \mathcal{S}_t^{\mathcal{L}} f(x)\right|^2 \frac{dx\, dt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)},$$

where C depends on m, n, C_A, provided that $\max\{\|B_1\|_n, \|B_2\|_n\}$ is sufficiently small depending on n, C_A. Under the same hypotheses, the analogous bounds hold for $\mathcal{L}$ replaced by $\mathcal{L}^*$, and for $\mathbb{R}^{n+1}_+$ replaced by $\mathbb{R}^{n+1}_-$.

Let us see that we may reduce the proof of Theorem 4.1.1 to that of Theorem 4.5.1. First, it is a fact that square function estimates for solutions u of $\mathcal{L}u = 0$ "travel up" the $t-$derivatives:

Lemma 4.5.2 (Square function bound "travels up" t-derivatives)**.** *Fix $m, k \in \mathbb{N}$ with $m > k \geq 1$. Suppose that $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ solves $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense. Then there exists a constant C depending only on m, n, C_A, and $\max\{\|B_1\|_n, \|B_2\|_n\}$, such that $|||t^m \partial_t^{m-1} \nabla u||| \leq C|||t^k \partial_t^k u|||$.*

The proof of the previous lemma is very straightforward (decompose into Whitney cubes and then use the Caccioppoli inequality), and thus omitted.

Now, the following proposition (and Lemma 4.5.2[2]) immediately allow us to reduce proof of Theorem 4.1.1 to that of Theorem 4.5.1, and is a partial converse to Lemma 4.5.2. Recall that $L^2(\mathbb{R}^n) \subset H^{-\frac{1}{2}}(\mathbb{R}^n)$.

Proposition 4.5.3 (Square function bound "travels down" t-derivatives)**.** *The following estimates hold, where the implicit constants depend on m, k, and C_A.*

(i) For each $f \in L^2(\mathbb{R}^n)$ and each $m \geq 1$, $|||t^m \partial_t^m \nabla \mathcal{S} f||| \lesssim_m |||t^{m+1} \partial_t^{m+1} \nabla \mathcal{S} f||| + \|f\|_2$.

(ii) For each $f \in L^2(\mathbb{R}^n)$ and each $m > k \geq 1$,

$$|||t^k \partial_t^k \nabla \mathcal{S} f||| \lesssim_m |||t^m \partial_t^{m+1} \mathcal{S} f||| + \|f\|_2. \tag{4.5.4}$$

Proof. One may obtain (ii) as a consequence of $i)$ via induction on m, using Caccioppoli's inequality on Whitney boxes after increasing the number of t derivatives appropriately. So it suffices to prove $i)$. Fix $m \in \mathbb{N}$, $N > 0$ large, $\epsilon > 0$ small and let $f \in L^2(\mathbb{R}^n)$. Let

[2] Lemma 4.5.2 is used to show that ε_0 can be chosen independently of m.

$\psi \in C_c^\infty(0,\infty)$ be a non-negative function which satisfies

$$\psi \equiv 1 \quad \text{on } \left(\epsilon, \tfrac{1}{\epsilon}\right), \qquad\qquad \psi \equiv 0 \quad \text{on } \left(0, \tfrac{\epsilon}{2}\right) \cup \left(\tfrac{2}{\epsilon}, \infty\right),$$
$$|\psi'| \leq \tfrac{4}{\epsilon} \quad \text{on } \left(\tfrac{\epsilon}{2}, \epsilon\right), \qquad\qquad |\psi'| \leq 2\epsilon \quad \text{on } \left(\tfrac{1}{\epsilon}, \tfrac{2}{\epsilon}\right).$$

Since $\mathcal{S}f \in Y^{1,2}(\mathbb{R}^{n+1})$ and $\mathcal{L}\mathcal{S}f = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense, then $\partial_t^m \mathcal{S}f \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$ and $\mathcal{L}\partial_t^m\mathcal{S}f = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense. Observe that

$$\int_{B(0,N)} \int_{\epsilon}^{1/\epsilon} t^{2m-1} |\partial_t^m \nabla \mathcal{S}f|^2 \, dt \leq \int_{B(0,N)} \int_{\epsilon/2}^{2/\epsilon} t^{2m-1} |\partial_t^m \nabla \mathcal{S}f|^2 \psi \, dt,$$

and notice per our observations in Proposition 4.4.23 that the right-hand side above is finite. Now,

$$\begin{aligned}
\int_{B(0,N)} \int_{\epsilon/2}^{2/\epsilon} t^{2m-1} |\partial_t^m \nabla \mathcal{S}f|^2 \psi \, dt &= \int_{B(0,N)} \int_{\epsilon/2}^{2/\epsilon} t^{2m-1} \partial_t^m \nabla \mathcal{S}f \overline{\partial_t^m \nabla \mathcal{S}f} \psi \, dt \\
&= -\frac{1}{2m} \int_{B(0,N)} \int_{\epsilon/2}^{2/\epsilon} t^{2m} \partial_t (\partial_t^m \nabla \mathcal{S}f \overline{\partial_t^m \nabla \mathcal{S}f} \psi) \, dt \\
&\leq \frac{1}{m} \int_{B(0,N)} \int_{\epsilon/2}^{2/\epsilon} t^{2m} |\partial_t^{m+1} \nabla \mathcal{S}f| |\partial_t^m \nabla \mathcal{S}f| \psi \, dt \\
&\quad + \frac{1}{m} \int_{B(0,N)} \Big[\fint_{\epsilon/2}^{\epsilon} t^{2m} |\partial_t^m \nabla \mathcal{S}f|^2 \, dt + \fint_{1/\epsilon}^{2/\epsilon} t^{2m} |\partial_t^m \nabla \mathcal{S}f|^2 \, dt \Big].
\end{aligned}$$

The last two terms are controlled by (4.4.24). As for the first term, note that $2m = \frac{2m-1}{2} + \frac{2m+1}{2}$, and we use Cauchy's inequality and absorb one of the resulting summands to the left-hand side. Sending $N \to \infty$ and $\epsilon \searrow 0$ yields the desired result. $\square$

Combining Lemma 4.5.26 below and Theorem 4.5.1, we will also obtain the following result.

Theorem 4.5.5 (Square function bound for $\mathcal{S}\nabla$)**.** *For each $m \in \mathbb{N}$, with $m \geq n+10$,*

$$\iint_{\mathbb{R}^{n+1}_+} \left| t^m (\partial_t)^m (\mathcal{S}_t \nabla) \vec{f}(x) \right|^2 \frac{dx\,dt}{t} \lesssim \|\vec{f}\|_{L^2(\mathbb{R}^n)}, \tag{4.5.6}$$

where C depends on m, n, C_A, provided that $\max\{\|B_1\|_n, \|B_2\|_n\}$ is sufficiently small depending on m, n, C_A. These results hold for $\mathcal{L}^*$ and in the lower half space as the *hypotheses are symmetric.*

4.5.2 Setup for the Tb argument and testing functions

Having reduced matters to proving Theorem 4.5.1, we fix $m \in \mathbb{N}$ with $m \geq n + 10$. We define the space H to be the subspace of $L^2(\mathbb{R}^n)^n$ consisting of the gradients of $Y^{1,2}(\mathbb{R}^n)$-functions. That is, $H = \{h' : h' = \nabla F,\, F \in Y^{1,2}(\mathbb{R}^n)\}$. For $h' \in H$ and $h^0 \in L^2(\mathbb{R}^n)$, we set $h = (h', h^0)$ and define for each $t \in \mathbb{R}\backslash\{0\}$,

$$\Theta_t^0 h^0 := t^m \partial_t^{m+1} \mathcal{S}_t h^0, \qquad \text{and}$$
$$\Theta_t' h' := t^m \partial_t^m (\mathcal{S}_t \nabla) \tilde{A} h' + t^m (\partial_t)^m \mathcal{S}_t (B_{2\|} \cdot h'),$$

where we recall that $\tilde{A}$ is the $(n+1) \times n$ submatrix of A consisting of the first n columns of A. We let $\Theta_t := (\Theta_t', \Theta_t^0) : H \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which acts on $h = (h', h^0)$ via the identity $\Theta_t h = \Theta_t' h' + \Theta_t^0 h^0$.

For each $t > 0$, we also define an auxiliary operator $\Theta_t^{(a)} : L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) \to L^2(\mathbb{R}^n)$ which acts on $g = (g', g^0)$ via $\Theta_t^{(a)} g = t^m (\partial_t)^m (\mathcal{S}_t \nabla)(g', g^0)$. This auxiliary operator will play the role of an error term that allows us to integrate by parts. Accordingly, define $\widehat{\Theta}_t$ acting on functions $h = (h', h^0, h'') \in H \times L^2(\mathbb{R}^n, \mathbb{C}) \times L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ via

$$\widehat{\Theta}_t h(x) = \Theta_t(h', h_0)(x) + \Theta_t^{(a)} h''(x).$$

We need to define appropriate testing functions for our family $\{\Theta_t\}$. Let $\tau \in (0, 1/40)$ be a small parameter to be chosen later, and let $\widetilde{\Psi}$ be a smooth cut-off function in $\mathbb{R}^{n+1}$ with the following properties:

$$\widetilde{\Psi} \in C_c^\infty\big(\big[-\tfrac{1}{1000}, \tfrac{1}{1000}\big]^n \times \big[-\tfrac{\tau}{2}, \tfrac{\tau}{2}\big]\big), \qquad \widetilde{\Psi} \equiv 1 \text{ on } \big[-\tfrac{1}{2000}, \tfrac{1}{2000}\big]^n \times \big[-\tfrac{\tau}{4}, \tfrac{\tau}{4}\big]$$
$$0 \leq \widetilde{\Psi} \leq 1, \qquad |\nabla \widetilde{\Psi}| \lesssim 1/\tau.$$

Let $\Psi := c_{n,\tau} \widetilde{\Psi}$ where $c_{n,\tau}$ is chosen so that $\|\Psi\|_1 = 1$. Hence Ψ is a normalization of $\widetilde{\Psi}$. For any cube $Q \subset \mathbb{R}^n$, we define the measurable functions

$$\Psi_Q(X) := \frac{1}{\ell(Q)^{n+1}} \Psi\Big(\frac{1}{\ell(Q)}[X - (x_Q, 0)]\Big), \quad \big(\text{note that } \|\Psi_Q\|_1 = 1\big),$$

$$\Psi_Q^{\pm}(y,s) := \Psi_Q\Big(y\ ,\ s \mp \frac{3}{2}\tau\ell(Q)\Big),$$

and $\Psi_Q^{s'}(y,s) := \Psi_Q(y, s+s')$, for each $s' \in \mathbb{R}$. Let us make a few observations about $\widetilde{\Psi}$ and Ψ. The fact that

$$\mathbb{1}_{\left[-\frac{1}{2000},\frac{1}{2000}\right]^n \times \left[-\frac{\tau}{4},\frac{\tau}{4}\right]} \leq \widetilde{\Psi} \leq \mathbb{1}_{\left[-\frac{1}{1000},\frac{1}{1000}\right]^n \times \left[-\frac{\tau}{2},\frac{\tau}{2}\right]}$$

forces that $c_{n,\tau} \approx \frac{1}{\tau}$ and that $\|\widetilde{\Psi}\|_{2_*} \approx \tau^{\frac{1}{2_*}}$. Consequently, $\|\Psi\|_{2_*} \approx \tau^{-1+1/2_*}$, and

$$\|\Psi_Q\|_{2_*} \approx \tau^{-1+1/2_*}[\ell(Q)^{n+1}]^{-1+1/2_*} = [\tau\ell(Q)^{n+1}]^{-1/2+1/(n+1)}.$$

Of course, the same L^{2_*} estimate holds for $\Psi_Q^{\pm}$ and $\Psi_Q^{s'}$. Now, we define for any cube and $s' \in \mathbb{R}$ the quantities

$$F_Q^{\pm} := \mathcal{L}^{-1}(\Psi_Q^{\pm}), \qquad F_Q := F_Q^+ - F_Q^-, \qquad F_Q^{s'} := \mathcal{L}^{-1}(\Psi_Q^{s'}).$$

By our previous observations and the fact that $L^{2_*}(\mathbb{R}^{n+1})$ embeds continuously into $(Y^{1,2}(\mathbb{R}^{n+1}))^*$, we easily see that for any cube Q and any $s' \in \mathbb{R}$, the estimate

$$\max\big\{\|\nabla F_Q\|_2, \|\nabla F_Q^{\pm}\|_2, \|\nabla F_Q^{s'}\|_2\big\} \lesssim [\tau\ell(Q)^{n+1}]^{-1/2+1/(n+1)} \tag{4.5.7}$$

holds. Notice that we have

$$\Psi_Q^+(y,s) - \Psi_Q^-(y,s) = -\int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \partial_{s'}\Psi(y, s+s')\,ds' = -\int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \partial_s \Psi_Q^{s'}(y,s)\,ds'.$$

Therefore, the identity

$$\begin{aligned} F_Q = -\int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \mathcal{L}^{-1}(D_{n+1}\Psi_Q^{s'})\,ds' &= -\int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \partial_t \mathcal{L}^{-1}(\Psi_Q^{s'})\,ds' \\ &= -\int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \partial_t F_Q^{s'}\,ds' \end{aligned} \tag{4.5.8}$$

is valid in $Y^{1,2}(\mathbb{R}^{n+1})$. For convenience, we write $(\nabla_{y,s}u)(y,0) := (\nabla_{y,s}u(y,s))\big|_{s=0}$. We are now ready to define our testing functions $b_Q = (b_Q', b_Q^0)$. Let b_Q^0 be defined via

$b_Q^0(y) := |Q|(\partial_\nu^{\mathcal{L},-} F_Q)(y,0)$, where

$$\partial_\nu^{\mathcal{L},-} u(y,0) = e_{n+1} \cdot [A(\nabla_{y,s} u)(y,0) - B_1 u(y,0)]$$
$$= e_{n+1} \cdot [A(\nabla_{y,s} u)(y,0)] - (B_1)_\perp u(y,0).$$

We define b'_Q via $b'_Q := |Q| \nabla_\| F_Q(y,0)$, while we define the auxiliary testing function $b_Q^{(a)}$ via $b_Q^{(a)} := |Q| B_1 F_Q(y,0)$.

We will define a measure for each cube Q that corresponds to a smoothened characteristic function. We do this exactly as in [GdlHH16]. Let > 0 to be chosen. For each cube, we let $d\mu_Q = \phi_Q \, dx$, where $\phi_Q : \mathbb{R}^n \to [0,1]$ is a smooth bump function supported in $(1+)Q$ with $\phi_Q \equiv 1$ on $(1/2)Q$. Clearly, we can choose ϕ_Q so that $\phi_Q \gtrsim$ on Q and $\|\nabla \phi_Q\|_{L^\infty} \lesssim 1/\ell(Q)$. We also let $\Phi_Q : \mathbb{R}^{n+1} \to [0,1]$ be a smooth extension of ϕ_Q; that is, $\Phi_Q(y,0) = \phi_Q(y)$, with Φ_Q supported in $I_{(1+)Q}$ and $\Phi_Q \equiv 1$ on $I_{(1/2)Q}$, where for any cube $Q \subset \mathbb{R}^n$, we let $I_Q = Q \times (-\ell(Q), \ell(Q))$ denote the "double Carleson box" associated to Q. We may also ensure that $\|\nabla \Phi_Q\|_{L^\infty(\mathbb{R}^{n+1})} \lesssim 1/\ell(Q)$.

4.5.3 Properties of the testing functions

The testing functions defined above enjoy the following essential properties which justify their use in the Tb argument.

Proposition 4.5.9 (Properties of the testing functions)**.** *Let $b_Q = (b'_Q, b_Q^0)$, $\hat{b}_Q$, and $\widehat{\Theta}_t$ be as above. For any $\eta > 0$, there exists $\tau \in (0,1)$ depending on n, C_A, η, and $C_0 = C_0(m,\tau)$, and there exists a measure μ_Q as described above, such that for each cube Q, the estimates*

$$\int_{\mathbb{R}^n} |b_Q|^2 \leq C_0 |Q| \tag{4.5.10}$$

$$\int_0^{\ell(Q)} \int_Q \left|\widehat{\Theta}_t \hat{b}_Q(x)\right|^2 \frac{dx\,dt}{t} \leq C_0 |Q| \tag{4.5.11}$$

$$\frac{1}{2} \leq \mathfrak{Re}\Big(\frac{1}{\mu_Q(Q)} \int_Q b_Q^0 \, d\mu\Big) \tag{4.5.12}$$

$$\Big|\frac{1}{\mu_Q(Q)} \int_Q b'_Q \, d\mu_Q\Big| \leq \frac{\eta}{2}, \tag{4.5.13}$$

hold, provided that $\max\{\|B_1\|_n, \|B_2\|_n\} = \varepsilon_m < \tau$.

We note that while the smallness of $\varepsilon_m = \max\{\|B_1\|_n, \|B_2\|_n\}$ apparently depends on m at this point, we may prove Theorem 4.5.1 for a fixed sufficiently large m, and then use Lemma 4.5.2 and Proposition 4.5.3 to remove any dependence on m in the bound for $\max\{\|B_1\|_n, \|B_2\|_n\}$. For now, throughout the Tb argument, we shall continue to use ε_m to denote this quantity.

We will establish several preliminary lemmas in anticipation of the proof of the above proposition.

Lemma 4.5.14 (Estimate of the L^2 norm of b_Q)**.** *The estimate*

$$\int_{\mathbb{R}^n} |b_Q|^2 \lesssim \tau^{-2+2/(n+1)}|Q|,$$

holds, where the implicit constant depends on n *and* C_A.

Proof. Set $a := \frac{\tau\ell(Q)}{1000}$ and observe that F_Q solves $\mathcal{L}F_Q = 0$ in the strip $\{(x,t) : |t| < 50a\}$. Let $\mathbb{G}_a$ be the grid of pairwise disjoint n-dimensional cubes with sides of length a parallel to the coordinate axes, and for each $P \in \mathbb{G}_a$, define the $(n+1)-$dimensional box $P^* := 2P \times [-2\ell(P), 2\ell(P)]$. Applying Lemma 4.3.20 and the estimate (4.5.7), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla F_Q(\cdot,0)|^2 &= \sum_{P\in\mathbb{G}_a}\int_P |\nabla F_Q(\cdot,0)|^2 \lesssim \frac{1}{a}\sum_{P\in\mathbb{G}_a}\iint_{P^*}|\nabla F_Q|^2 \\ &\lesssim \frac{1}{a}\|\nabla F_Q\|_2^2 \lesssim \frac{1}{a}[\tau\ell(Q)^{n+1}]^{-1+2/(n+1)} \lesssim \tau^{-2+2/(n+1)}|Q|^{-1},\end{aligned}$$

where we used that $a \approx \tau\ell(Q)$ and the bounded overlap of $\{P^*\}_{P\in\mathbb{G}_a}$. Upon multiplying the above inequality by $|Q|^2$, we have the desired estimate up to controlling $\| \, |Q|(B_1)_\perp F_Q(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}$. We have already shown that $\|\nabla_\| F_Q(\cdot,0)\|_{L^2(\mathbb{R}^n)} < \infty$, and from Lemma 4.2.3 and Lemma 4.3.17, we have that $F_Q(\cdot,0) \in L^{2^*}(\mathbb{R}^n)$, so that $F(\cdot,0) \in Y^{1,2}(\mathbb{R}^n)$. From this, we can deduce the estimate $\|F_Q(\cdot,0)\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \lesssim \|\nabla_\| F_Q(\cdot,0)\|_{L^2(\mathbb{R}^n)}$. Consequently, we may use the estimate for $\|\nabla F_Q(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}$ obtained above and Hölder's inequality to show that

$$\int_{\mathbb{R}^n} |(B_1)_\perp F_Q(\cdot,0)|^2 \le \|B_1\|_n^2 \|F_Q(\cdot,0)\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \lesssim \varepsilon_m^2 \|\nabla F_Q(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}$$

$$\lesssim \varepsilon_m^2 \tau^{-2+2/(n+1)} |Q|^{-1}.$$

Upon multiplying the previous estimates by $|Q|^2$, we easily obtain the claimed inequality from the ellipticity of A. $\square$

The next lemma says that we have a Carleson estimate by including the error term.

Lemma 4.5.15 (Good behavior of $\hat{b}_Q$ vis-à-vis Carleson norm of $\hat{\Theta}_t$)**.** *Let b'_Q, b^0_Q, and $b^{(a)}_Q$ be as above. Then, if $\hat{b}_Q = (b'_Q, b^0_Q, b^{(a)}_Q)$, we have the estimate*

$$\int_0^{\ell(Q)} \int_Q \left| \widehat{\Theta}_t \hat{b}_Q(x) \right|^2 \frac{dx\,dt}{t} \leq C|Q|\tau^{-\beta},$$

where $\beta = 2 + 2m - 2/(n+1) > 0$, and C depends on m, n, and C_A.

Proof. First, let us show the identity

$$\widehat{\Theta}_t \hat{b}_Q(x) = |Q| t^m (\partial_t)^{m+1} F_Q^-, \qquad \text{on } \mathbb{R}^{n+1}_+. \tag{4.5.16}$$

By (an analogue of) Theorem 4.4.16 (iii)), to show the above identity, it suffices to show that for each $t > 0$, the representation

$$\hat{\Theta}_t \hat{b}_Q = |Q| t^m \partial_t^{m+1} \big(\mathcal{S}_t^{\mathcal{L}} (\partial_\nu^{\mathcal{L},-} F_Q) + \mathcal{D}_t^{\mathcal{L},+} (\mathrm{Tr}_0\, F_Q) \big)$$

holds in $L^2(\mathbb{R}^n)$. For notational convenience, we will write $F_Q^0 := \mathrm{Tr}_0\, F_Q$. By definition, we have that for any $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\langle \hat{\Theta}_t \hat{b}_Q, f \rangle = \langle |Q| t^m (D_{n+1}^{m+1} \mathcal{S}^{\mathcal{L}})_t [\partial_\nu^{\mathcal{L},-} F_Q], f \rangle \\
&+ \langle |Q| t^m (D_{n+1}^m (\mathcal{S}^{\mathcal{L}} \nabla)[\tilde{A} \nabla_\| F_Q^0 + B_1 F_Q^0])_t, f \rangle + \langle |Q| t^m (D_{n+1}^m \mathcal{S}^{\mathcal{L}} [B_{2\|} \cdot \nabla_\| F_Q^0])_t, f \rangle \\
&= \langle |Q| t^m (D_{n+1}^{m+1} \mathcal{S}^{\mathcal{L}})_t [\partial_\nu^{\mathcal{L},-} F_Q], f \rangle \\
&+ (-1)^m \langle \tilde{A} \nabla_\| F_Q^0 + B_1 F_Q^0, |Q| t^m (D_{n+1}^m \nabla \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle \\
&+ (-1)^m \langle B_{2\|} \cdot \nabla_\| F_Q^0, |Q| t^m (D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle.
\end{aligned}$$

Therefore, it suffices to show that

$$\langle |Q| t^m (D_{n+1}^{m+1} \mathcal{D}^{\mathcal{L},+}[F_Q^0])_t, f \rangle = (-1)^m \langle \tilde{A} \nabla_\| F_Q^0 + B_1 F_Q^0, |Q| t^m (D_{n+1}^m \nabla \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle$$

$$+ (-1)^m \langle B_{2\|} \cdot \nabla_\| F_Q^0, |Q| t^m (D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle$$
$$=: (-1)^m |Q| t^m I_t.$$

We rewrite I_t as follows, using Proposition 4.3.19, and the fact that $F_Q^0 \in W^{1,2}(\mathbb{R}^n)$,

$$\begin{aligned}
I_t &= \langle \tilde{A} \nabla_\| F_Q^0 + B_1 F_Q^0, (\nabla D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle + \langle B_{2\|} \cdot \nabla_\| F_Q^0, (D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle \\
&= \langle \nabla_\| F_Q^0, \big((A^* \nabla D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_\| \big)_{-t} + \overline{B_{2\|}} (D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle \\
&\qquad + \langle F_Q^0, \overline{B_1} (\nabla D_{n+1}^m \mathcal{S}^{\mathcal{L}^*}[f])_{-t} \rangle \\
&= (-1)^{m+1} \Big\langle F_Q^0, D_{n+1}^{m+1} \big(\vec{A}_{n+1,\cdot}^* \nabla (\mathcal{S}^{\mathcal{L}^*}[f])_{-s} \big)_{s=t} + D_{n+1}^{m+1} \big(\overline{B_{2\perp}} (\mathcal{S}^{\mathcal{L}^*}[f])_{-s} \big)_{s=t} \Big\rangle \\
&= (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} \Big|_{s=t} \big\langle F_Q^0, \vec{A}_{n+1,\cdot}^* \nabla (\mathcal{S}^{\mathcal{L}^*}[f])_{-s} + \overline{B_{2\perp}} (\mathcal{S}^{\mathcal{L}^*}[f])_{-s} \big\rangle \\
&= (-1)^{m+1} \frac{d^{m+1}}{dt^{m+1}} \Big|_{s=t} \langle F_Q^0, \partial_{\nu,-s}^{\mathcal{L}^*,-} (\mathcal{S}^{\mathcal{L}^*}[f]) \rangle = (-1)^{m+2} \frac{d^{m+1}}{dt^{m+1}} \Big|_{s=t} \langle \mathcal{D}_s^{\mathcal{L},+}[F_Q^0], f \rangle \\
&= (-1)^{m+2} \langle (D_{n+1}^{m+1} \mathcal{D}^{\mathcal{L},+}[F_Q^0])_t, f \rangle,
\end{aligned}$$

where we used (i) in Lemma 4.4.11 in the fifth equality, we used (ii) of Proposition 4.4.18 in the sixth equality, and we justify the handling of the t-derivatives via Proposition 4.4.27. This concludes the proof of the identity (4.5.16).

Now, we let $a = \frac{\tau \ell(Q)}{1000}$ as before, and note that $(\partial_t)^{m+2} F_Q^-$ is a solution in the half space $\{(x,t) : t > 50a\}$. For $P \in \mathbb{G}_a$ and $t \geq 0$, we set

$$P_t^* = 2P \times \big(t - \tfrac{a}{20}, t + \tfrac{a}{20}\big), \quad \text{and} \quad P_t^{**} = 4P \times \big(t - \tfrac{a}{5}, t + \tfrac{a}{5}\big).$$

Then using (4.3.21) and then Proposition 4.3.9 repeatedly ($m+1$ times) , we obtain for $t \in (0, \ell(Q)]$

$$\begin{aligned}
\int_Q \big|\widehat{\Theta}_t \hat{b}_Q\big|^2 &\leq \int_{\mathbb{R}^n} \big| |Q| t^m (\partial_t)^{m+1} F_Q^-(\cdot,t) \big|^2 = t^{2m} |Q|^2 \sum_{P \in \mathbb{G}_a} \int_P \big| (\partial_t)^{m+1} F_Q^-(\cdot,t) \big|^2 \\
&\lesssim t^{2m} |Q|^2 a^{-1} \sum_{P \in \mathbb{G}_a} \iint_{P_t^*} |(\partial_t)^{m+1} F_Q^-|^2 \lesssim t^{2m} |Q|^2 a^{-1-2m} \sum_{P \in \mathbb{G}_a} \iint_{P_t^{**}} |\partial_t F_Q^-|^2 \\
&\lesssim t^{2m} |Q|^2 a^{-1-2m} \|\nabla F_Q^-\|_2^2 \lesssim t^{2m} |Q|^2 a^{-1-2m} [\tau \ell(Q)^{n+1}]^{-1+2/(n+1)} \\
&\lesssim |Q| \tau^{-\beta} \Big(\frac{t}{\ell(Q)} \Big)^{2m},
\end{aligned}$$

where we used the bounded overlap of $\{P_t^{**}\}_{P\in\mathbb{G}_a}$. Hence, we see that

$$\int_0^{\ell(Q)}\int_Q\left|\widehat{\Theta}_t\hat{b}_Q(x)\right|^2\frac{dx\,dt}{t}\lesssim|Q|\tau^{-\beta}\int_0^{\ell(Q)}\Big(\frac{t}{\ell(Q)}\Big)^{2m}\frac{dx\,dt}{t}\quad\lesssim|Q|\tau^{-\beta}.$$

□

Observe that Lemma 4.5.14 and the properties of μ_Q allow us to establish that

$$\int_{\mathbb{R}^n\setminus Q}|b_Q|\,d\mu_Q\leq|(1+)Q\setminus Q|^{1/2}\|b_Q\|_{L^2(\mathbb{R}^n)}\lesssim^{1/2}\tau^{-1+1/(n+1)}|Q|.\tag{4.5.17}$$

Let us furnish a smallness estimate for b'_Q.

Lemma 4.5.18 (Almost atomic behavior of b'_Q)**.** *Let b'_Q and μ_Q be as above. Then*

$$\Big|\int_{\mathbb{R}^n}b'_Q\,d\mu_Q\Big|\lesssim|Q|\tau^{1/2+1/(n+1)},\tag{4.5.19}$$

where the implicit constant depends on n and C_A. In particular,

$$\Big|\frac{1}{\mu_Q(Q)}\int_Q b'_Q\,d\mu_Q\Big|\lesssim\tau^{1/2+1/(n+1)}+^{1/2}\tau^{-1+1/(n+1)}.\tag{4.5.20}$$

Proof. We first show how to derive (4.5.20) from the first inequality. We have that

$$\Big|\int_Q b'_Q\,d\mu_Q\Big|\leq\Big|\int_{\mathbb{R}^n}b'_Q\,d\mu_Q\Big|+\int_{\mathbb{R}^n\setminus Q}|b_Q|\,d\mu_Q,$$

so that (4.5.20) readily follows from (4.5.19), (4.5.17), and the fact that $\mu_Q(Q)\geq(1/2)^n|Q|$. It remains to show (4.5.19). To this end, we utilize the properties of ϕ_Q, (4.5.8), (4.3.21) and Hölder's inequality to see that

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}b'_Q\,d\mu_Q\Big|=|Q|\Big|\int_{\mathbb{R}^n}\nabla_{\|}F_Q(\cdot,0)\phi_Q\Big|&=|Q|\Big|\int_{\mathbb{R}^n}F_Q(\cdot,0)\nabla\phi_Q\Big|\\&\lesssim\ell(Q)^{n-1}\int_{(1+)Q\setminus(1/2)Q}|F_Q(\cdot,0)|\\&\lesssim\ell(Q)^{n-1}\int_{(1+)Q\setminus(1/2)Q}\Big|\int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)}\partial_tF_Q^{s'}(y,0)\,ds'\Big|\,dy\end{aligned}$$

$$\lesssim \ell(Q)^{n-1} \int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \int_{(1+)Q\setminus(1/2)Q} \left|\partial_t F_Q^{s'}(y,0)\right| dy\, ds'$$

$$\lesssim \ell(Q)^{n-1} \frac{\ell(Q)^{n/2}}{\ell(Q)^{1/2}} \int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \Big(\iint_{I_{2Q}\setminus I_{(1/4)Q}} |\nabla F_Q^{s'}(Y)|^2\, dY\Big)^{1/2} ds'$$

$$\lesssim |Q|\tau^{1/2+1/(n+1)},$$

where we used (4.5.7) in the last line and, in order to use (4.3.21), we used that for $s \in (-\frac{3}{2}\tau\ell(Q), \frac{3}{2}\tau\ell(Q))$ each $F_Q^{s'}$ is a solution in $I_{2Q} \setminus I_{(1/4)Q}$. □

The last preliminary lemma we will need establishes a coercivity estimate for b_Q^0.

Lemma 4.5.21 (Coercivity of b_Q^0)**.** *Let b_Q^0 and $d\mu_Q = \phi_Q\, dx$ as above. Suppose that $\varepsilon_m > 0$ is a small number depending on m. Then, if $\max\{\|B_1\|_n, \|B_2\|_n\} \leq \varepsilon_m$, the estimate*

$$\Re e\Big(\frac{1}{\mu_Q(Q)} \int_Q b_Q^0\, d\mu_Q\Big) \geq \Big(1 - C\big[\tau^{1/2+1/(n+1)} + \varepsilon_m \tau^{-1/2+1/(n+1)} +^{1/2} \tau^{-1+1/(n+1)}\big]\Big),$$

holds, where C depends on m, n, and C_A.

Proof. By the definitions of μ_Q, b_Q^0, and the conormal derivative, we observe that

$$\int_{\mathbb{R}^n} b_Q^0\, d\mu_Q = \int_{\mathbb{R}^n} b_Q^0 \phi_Q = |Q| \int_{\mathbb{R}^n} (\partial_\nu^{\mathcal{L},-} F_Q)(y,0)\phi_Q(y)\, dy$$

$$= |Q|\Big(-\langle \Phi_Q, \mathcal{L}F_Q\rangle_{\mathbb{R}^{n+1}_-} + \iint_{\mathbb{R}^{n+1}} A\nabla F_Q \cdot \nabla\Phi_Q + (B_1 F_Q)\cdot\nabla\Phi_Q + (B_2\cdot\nabla F_Q)\Phi_Q\Big)$$

$$= |Q|(I + II).$$

Since $\operatorname{supp} \Psi_Q^+ \cap \mathbb{R}^{n+1}_- = \varnothing$, $\Phi_Q \equiv 1$ on $\operatorname{supp} \Psi_Q^-$, and $\int_{\mathbb{R}^{n+1}_-} \Psi_Q^- = 1$, we have that

$$I = -\langle \Phi_Q, \mathcal{L}F_Q\rangle_{\mathbb{R}^{n+1}_-} = -\iint_{\mathbb{R}^{n+1}_-} (-\Psi_Q^-) = 1.$$

To bound II, we write $II = II_1 + II_2 + II_3$, where the II_i correspond to each of the summands in the integral defining II. For the term, II_1, we use essentially the same estimates as in the previous lemma. In particular we use the properties of Φ_Q, Hölder's inequality, the Caccioppoli inequality, and (4.5.8) to obtain that

$$
\begin{aligned}
|II_1| &\le \iint_{\mathbb{R}^{n+1}} \Big| A\nabla F_Q \cdot \nabla \Phi_Q \Big| \lesssim \frac{1}{\ell(Q)} \iint_{I_{(1+)Q}\setminus I_{(1/2)Q}} |\nabla F| \\
&\lesssim \ell(Q)^{(n-1)/2} \Big(\iint_{I_{(1+)Q}\setminus I_{(1/2)Q}} |\nabla F|^2 \Big)^{1/2} \lesssim \ell(Q)^{\frac{(n-3)}{2}} \Big(\iint_{I_{(1+)Q}\setminus I_{(1/2)Q}} |F|^2 \Big)^{1/2} \\
&\lesssim \ell(Q)^{\frac{(n-3)}{2}} \Big(\iint_{I_{(1+)Q}\setminus I_{(1/2)Q}} \Big| \int_{-\frac{3}{2}\tau\ell(Q)}^{\frac{3}{2}\tau\ell(Q)} \partial_t F_Q^{s'}(Y)\, ds' \Big|^2 dY \Big)^{1/2} \lesssim \tau^{1/2+1/(n+1)}.
\end{aligned}
$$

To bound II_2, we use the estimate $\|B_1 F_Q\|_2 \lesssim \varepsilon_m \|\nabla F_Q\|_2$ and (4.5.7) to see that

$$
\begin{aligned}
|II_2| &\le \iint_{I_{2Q}} |(B_1 F)\cdot \nabla \Phi_Q| \lesssim \frac{1}{\ell(Q)} \iint_{I_{2Q}} |B_1 F_Q| \\
&\lesssim \frac{\ell(Q)^{\frac{n+1}{2}}}{\ell(Q)} \Big(\iint_{I_{2Q}} |B_1 F_Q|^2 \Big)^{1/2} \lesssim \varepsilon_m \ell(Q)^{\frac{n-1}{2}} \|\nabla F_Q\|_2 \lesssim \varepsilon_m \tau^{-1/2+1/(n+1)}.
\end{aligned}
$$

To bound II_3 we use Hölder's inequality, $\|B_2\|_n \le \varepsilon_m$, and (4.5.7) as follows:

$$
\begin{aligned}
|II_3| &\le \int_{-2\ell(Q)}^{2\ell(Q)} \int_{2Q} |\nabla F_Q B_2| \le \varepsilon_m \int_{-2\ell(Q)}^{2\ell(Q)} \Big(\int_{2Q} |\nabla F_Q|^{\frac{n}{n-1}} \Big)^{\frac{n-1}{n}} \\
&\lesssim \varepsilon_m \ell(Q)^{\frac{n-2}{2}} \int_{-2\ell(Q)}^{2\ell(Q)} \Big(\int_{2Q} |\nabla F_Q|^2 \Big)^{\frac{1}{2}} \lesssim \varepsilon_m \ell(Q)^{\frac{n-1}{2}} \Big(\int_{-2\ell(Q)}^{2\ell(Q)} \int_{2Q} |\nabla F_Q|^2 \Big)^{\frac{1}{2}} \\
&\lesssim \varepsilon_m \ell(Q)^{\frac{n-1}{2}} \|\nabla F_Q\|_2 \lesssim \varepsilon_m \tau^{-1/2+1/(n+1)}.
\end{aligned}
$$

Combining the previous estimates gives that

$$
\mathfrak{Re}\Big(\int_{\mathbb{R}^n} b_Q^0 \, d\mu_Q \Big) \ge |Q| \Big(1 - C\big[\tau^{1/2+1/(n+1)} + \varepsilon_m \tau^{-1/2+1/(n+1)}\big] \Big).
$$

This estimate, in concert with (4.5.17) and the fact that $\mu_Q(Q) \le 1$, ends the proof. □

With ε_m and at our disposal, we collapse the dependence of parameters to only τ, leaving freedom to take ε_m even smaller. We ensure that $\varepsilon_m < \tau$ and set $= \tau^3$. Under these choices, we are ready to present the

Proof of Proposition 4.5.9. When the choices $\varepsilon_m < \tau$ and $= \tau^3$ are used in Lemma 4.5.21, we have that

$$\Re e\Big(\frac{1}{\mu_Q(Q)}\int_Q b_Q^0\,d\mu\Big) \geq 1 - C\tau^{1/2+1/(n+1)},$$

where C depends on n and C_A. Accordingly, we may pick τ small enough so that (4.5.12) holds. The choice $= \tau^3$ used in (4.5.20) gives that

$$\Big|\frac{1}{\mu_Q(Q)}\int_Q b_Q'\,d\mu_Q\Big| \leq C\tau^{1/2+1/(n+1)},$$

where C depends on n and C_A. Hence, we may guarantee that (4.5.13) holds by choosing τ small depending on C and η. Having chosen τ so that (4.5.12) and (4.5.13) hold, (4.5.10) and (4.5.11) follow from Lemma 4.5.14 and Lemma 4.5.15 respectively. $\square$

4.5.4 Control of the auxiliary square functions

As a last preliminary step to presenting the proof of the square function bound, we elucidate how to control the error terms involving $\Theta_t^{(a)}$ and Θ_t'.

Proposition 4.5.22 (Control of error terms)**.** *Let T_t be either Θ_t' or $\Theta_t^{(a)}$. Then, for each fixed $t > 0$, $T_t 1$ is well defined as an element of $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, we have the estimates*

$$|||T_t|||_{op} \leq C|||\Theta_t^0|||_{op} + 1, \tag{4.5.23}$$

and

$$\|T_t 1\|_{\mathcal{C}} \leq C\|\Theta_t^0 1\|_{\mathcal{C}} + 1. \tag{4.5.24}$$

where C depends on m, n, and C_A, provided that $\max\{\|B_1\|_n, \|B_2\|_n\}$ *is sufficiently small depending on m, n, and C_A.*

Remark 4.5.25. We will operate under the assumption that $T_t 1$ and $\Theta_t^0 1$ have finite $\|\cdot\|_{\mathcal{C}}$ norm. Indeed, otherwise for $\gamma > 0$, we replace $T_t 1$ by $(T_t 1)_\gamma = (T_t 1)\mathbb{1}_{\gamma < t \leq 1/\gamma}$ and analogously for $\Theta_t^0 1$, and we observe that these truncated versions will always have finite $\|\cdot\|_{\mathcal{C}}$ norm under our hypotheses.

Proposition 4.5.22 will be a direct consequence of the following lemma.

Lemma 4.5.26 (Control of gradient field terms)**.** *Let* $\widetilde{\Theta}_t := t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| $ *for* $m \in \mathbb{N}$, $m \geq n+10$*. Then*

$$|||\widetilde{\Theta}_t|||_{op} \lesssim |||\Theta_t^0|||_{op} + 1, \tag{4.5.27}$$

and

$$\|\widetilde{\Theta}_t 1\|_{\mathcal{C}} \lesssim \|\Theta_t^0 1\|_{\mathcal{C}} + 1, \tag{4.5.28}$$

where the constants depends on m*,* n*, and* C_A*, provided that* $\max\{\|B_1\|_n, \|B_2\|_n\}$ *is sufficiently small depending on* m*,* n*, and* C_A*.*

Proof. We note that (4.5.28) follows from Lemma 4.2.23, (4.5.27) and Proposition 4.4.37. The proof will follow the general scheme of [HMM15b, Lemma 3.1], with modifications due to the first order terms. Write $L_\| := \operatorname{div}_x A_\| \nabla_\|$ where $A_\| = (A_{i,j})_{1 \leq i,j \leq n}$. By the Hodge decomposition for the operator $L_\|$, to prove (4.5.27) it is enough to show that

$$\iint_{\mathbb{R}^{n+1}_+} \left| t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} (\nabla_\| \cdot A_\| \nabla_\| F)(x) \right|^2 \frac{dx\, dt}{t} \lesssim (1 + |||\Theta_t^0|||_{op}^2), \tag{4.5.29}$$

for all $F \in Y^{1,2}(\mathbb{R}^n)$ with $\|\nabla_\| F\|_{L^2} \lesssim 1$ (dependence on C_A). We write

$$\begin{aligned}
&t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| \nabla_\| F \\
&\quad = \left\{ t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| - t^m \left(\partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| \right) P_t \right\} \nabla_\| F + t^m \left(\partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| \right) P_t \nabla_\| F \\
&\qquad =: R_t(\nabla_\| F) + t^m \left(\partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| \cdot A_\| \right) P_t \nabla_\| F,
\end{aligned}$$

where $t^m(\partial_t^m \mathcal{S}_t \nabla_\| A_\|)$ is the (vector-valued) operator $t^m \partial_t^m \mathcal{S}_t \nabla_\|$ applied to $A_\|$, the latter understood as a vector function with components in $L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^n)$, and P_t is a nice approximate identity constructed as follows. Let $\zeta_t(x) = t^{-n} \zeta\left(\frac{|x|}{t}\right)$, where $\zeta \in C_c^\infty(B(0,1/2))$ is radial with $\int_{\mathbb{R}^n} \zeta = 0$ and $\mathcal{Q}_t f(x) = (\zeta_t * f)(x)$ satisfies the Calderón reproducing formula

$$\int_0^\infty \mathcal{Q}_t^2 \frac{dt}{t} = I, \quad \text{in the strong operator topology on } L^2.$$

Then $\mathcal{Q}_s$ is a CLP family (see Definition 4.2.26) and we set $P_t := \int_t^\infty \mathcal{Q}_s^2 \frac{ds}{s}$. Then P_t is a nice approximate identity; that is, $P_t = (\varphi_t * f)(x)$ where $\varphi_t = t^{-n} \varphi\left(\frac{|\cdot|}{t}\right)$ and $\varphi \in C_c^\infty(B(0,1))$ is a radial function with $\int_{\mathbb{R}^n} \varphi = 1$.

The term $t^m \partial_t^m \mathcal{S}_t \nabla_\| \cdot A_\| P_t \nabla_\| F$ is the 'main term' and we will apply the techniques of the solution to the Kato problem [AHL$^+$02a] to handle its contribution. For now, we focus on the remainder term $R_t(\nabla_\| F)$, which takes a bit of exposition due to the number of terms arising from the lower order terms in the differential operator $\mathcal{L}$. To this end, we write

$$R_t = t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| - t^m \big(\partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\|\big) P_t$$
$$= \big\{t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| P_t - t^m \big(\partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\|\big) P_t\big\} + t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \nabla_\| A_\| (I - P_t) =: R_t^{[1]} + R_t^{[2]}.$$

Observe that $R_t^{[1]} 1 = 0$, $R_t^{[1]}$ has sufficient off-diagonal decay (Proposition 4.4.37) and uniform L^2 boundedness (Proposition 4.4.23), and $\|R_t^{[1]} \nabla_x\|_{2\to 2} \leq C/t$. Then the square function bound

$$\iint_{\mathbb{R}^{n+1}_+} |R_t^{[1]} \nabla_\| F|^2 \, \frac{dx\, dt}{t} \lesssim \|\nabla_\| F\|_2^2$$

follows from Lemma 4.2.25 as desired. To control R_t it remains to control $R_t^{[2]}$. Set $Z_t := I - P_t$ and define $\vec{b} := (A_{n+1,1}, \ldots, A_{n+1,n})$. By using integration by parts on slices (Proposition 4.3.19) and Proposition 4.2.27, we obtain that

$$t^m \partial_t^m \mathcal{S}_t \nabla_\| A_\| Z_t \nabla_\| F = t^m \partial_t^m \mathcal{S}_t \nabla_\| A_\| \nabla_\| Z_t F$$
$$= t^m \partial_t^{m+1} (\mathcal{S}_t \nabla) \cdot \vec{A}_{\cdot, n+1} Z_t F \quad - t^m \partial_t^{m+1} \mathcal{S}_t (\vec{b} \nabla Z_t F) \quad + t^m \partial_t^m (\mathcal{S}_t \nabla) B_1 Z_t F$$
$$- t^m \partial_t^m \mathcal{S}_t (B_{2\|} \nabla_\| Z_t F) \quad + t^m \partial_t^{m+1} \mathcal{S}_t (B_{2\perp} Z_t F) \qquad =: J_1 + J_2 + J_3 + J_4 + J_5.$$

Note that, using Plancherel's theorem, we have that

$$\iint_{\mathbb{R}^{n+1}_+} |t^{-1}(I - P_t) F(x)|^2 \, \frac{dx\, dt}{t} \lesssim \|\nabla_\| F\|_2^2. \tag{4.5.30}$$

Since $t^{m+1} \partial_t^{m+1} (\mathcal{S}_t \nabla) : L^2 \to L^2$ uniformly in t, we easily obtain the associated square function bound for J_1. To bound J_2, we write

$$J_2 = -t^m \partial_t^{m+1} \mathcal{S}_t \big(\vec{b} \cdot \nabla_\| (I - P_t) F\big)$$
$$= -t^m \partial_t^{m+1} \mathcal{S}_t \vec{b} \cdot \nabla_\| F$$
$$+ \{t^m \partial_t^{m+1} \mathcal{S}_t \vec{b} P_t - (t^m \partial_t^{m+1} \mathcal{S}_t \vec{b}) P_t\} \nabla_\| F + (t^m \partial_t^{m+1} \mathcal{S}_t \vec{b}) P_t \nabla_\| F$$

$$=: J_{2,1} + J_{2,2} + J_{2,3}.$$

For $J_{2,1}$, we see that $J_{2,1} = \Theta_t^0 \vec{b} \nabla_\| F$, whence

$$\iint_{\mathbb{R}^{n+1}_+} |t^m \partial_t^{m+1} \mathcal{S}_t \vec{b} \nabla_\| F|^2 \, \frac{dx\, dt}{t} \lesssim |||\Theta_t^0|||_{op}^2 \|\nabla_\| F\|_2^2.$$

Similarly, by Lemma 4.2.23 and Carleson's Lemma, we have that

$$\iint_{\mathbb{R}^{n+1}_+} |(t^m \partial_t^{m+1} \mathcal{S}_t \vec{b}) P_t \nabla_\| F|^2 \, \frac{dx\, dt}{t} \lesssim |||\Theta_t^0|||_{op}^2 \|\nabla_\| F\|_2^2,$$

so that the contribution from $J_{2,3}$ has the desired control. Notice that $J_{2,2}$ is of the form $R_t \nabla_\| F$ where $R_t 1 = 0$, $R_t : L^2 \to L^2$ and $\|R_t \nabla_x\|_{L^2 \to L^2} \le C/t$ and R_t good off-diagonal decay. Thus, the desired square function bound for term $J_{2,2,}$ follows immediately from Lemma 4.2.25.

For term J_3, let g be such that $I_1 g = F$ and $\|g\|_2 \approx \|\nabla_\| F\|_2$. Then using $t^m \partial_t^m (\mathcal{S}_t \nabla) = \Theta_t^{(a)}$, we have by Proposition 4.4.30 that

$$\|\Theta_t^{(a)} B_1 I_1 \mathcal{Q}_s^2 g\|_{L^2(\mathbb{R}^n)} \lesssim \left(\frac{s}{t}\right)^\gamma \|\mathcal{Q}_s g\|_{L^2(\mathbb{R}^n)}$$

for some $\gamma > 0$ independent of g. Then by standard estimates we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{n+1}_+} \left| t^m \partial_t^m (\mathcal{S}_t \nabla) B_1 (I - P_t) F \right|^2 \frac{dx\, dt}{t} \\
&\qquad = \iint_{\mathbb{R}^{n+1}_+} \left| t^m \partial_t^m (\mathcal{S}_t \nabla) B_1 I_1 (I - P_t) g \right|^2 \frac{dx\, dt}{t} \\
&\qquad \lesssim \iint_{\mathbb{R}^{n+1}_+} \left| t^m \partial_t^m (\mathcal{S}_t \nabla) B_1 I_1 \Big(\int_0^t \mathcal{Q}_s^2 g \frac{ds}{s} \Big) \right|^2 \frac{dx\, dt}{t} \\
&\qquad \lesssim_\gamma \iint_{\mathbb{R}^{n+1}_+} \int_0^t \left(\frac{t}{s}\right)^{\gamma/2} \left| t^m \partial_t^m (\mathcal{S}_t \nabla) B_1 I_1 \mathcal{Q}_s^2 g \right|^2 \frac{ds}{s} \frac{dx\, dt}{t} \\
&\lesssim \int_0^\infty \int_s^\infty \left(\frac{s}{t}\right)^{\gamma/2} \|\mathcal{Q}_s g\|_2^2 \frac{dt}{t} \frac{ds}{s} \lesssim_\gamma \int_0^\infty \|\mathcal{Q}_s g\|_2^2 \frac{ds}{s} \lesssim \|g\|_2^2 \approx \|\nabla_\| F\|_2^2,
\end{aligned}$$

where in the fourth inequality we used Cauchy's inequality in the $\frac{ds}{s}$ integral noting that $\int_0^t (s/t)^\gamma \frac{ds}{s} \lesssim C_\gamma$, and we used the square function estimate for the CLP family $\mathcal{Q}_s$ (see Definition 4.2.26). This takes care of the contribution from J_3.

Next, we handle J_4. We write J_4 as the sum of its pieces, as follows:

$$J_4 = -t^m \partial_t^m \mathcal{S}_t B_{2\|} \nabla_\| (I - P_t) F$$
$$= -t^m \partial_t^m \mathcal{S}_t B_{2\|} \nabla_\| F + t^m \partial_t^m \mathcal{S}_t B_{2\|} \nabla_\| P_t F = J_{4,1} + J_{4,2}.$$

For $J_{4,1}$, we observe that

$$J_{4,1} = -t^m \partial_t^m \mathcal{S}_t B_{2\|} \nabla_\| F = -t^m \partial_t^m \mathcal{S}_t \operatorname{div}_\| \nabla_\| I_2 B_{2\|} \nabla_\| F = -\widetilde{\Theta}(\nabla_\| I_2 B_{2\|} F)$$

and notice that $\|\nabla_\| I_2 B_{2\|} F\|_2 \lesssim \|B_2\|_n \|\nabla_\| F\|_2$. Therefore,

$$\iint_{\mathbb{R}^{n+1}_+} |t^m \partial_t^m \mathcal{S}_t B_{2\|} \nabla_\| F|^2 \, \frac{dx\, dt}{t} \lesssim |||\widetilde{\Theta}_t|||_{op}^2 \|B_2\|_n^2 \|\nabla_\| F\|_2^2,$$

and hence $J_{4,1}$ can be hidden in (4.5.29) when $\|B_2\|_n$ is small. For $J_{4,2}$, we write

$$J_{4,2} = \left\{ t^m \partial_t^m \mathcal{S}_t B_{2\|} P_t - (t^m \partial_t^m \mathcal{S}_t B_{2\|}) P_t \right\} \nabla_\| F + (t^m \partial_t^m \mathcal{S}_t B_{2\|}) P_t \nabla_\| F$$
$$= \widetilde{R}_t \nabla_\| F + (t^m \partial_t^m \mathcal{S}_t B_{2\|}) P_t \nabla_\| F.$$

We may handle $\widetilde{R}_t \nabla_\| F$ using Lemma 4.2.25, as $\widetilde{R}_t$ satisfies the required hypotheses (see Propositions 4.4.23 and 4.4.37). We see, in a similar fashion to $J_{4,1}$, that $t^m \partial_t^m \mathcal{S}_t B_{2\|} = \widetilde{\Theta}_t \nabla_\| I_2 B_{2\|}$, and $\|\nabla_\| I_2 B_{2\|}\|_{BMO} \lesssim \|B_2\|_n^2$. Noting that $\widetilde{\Theta}_t 1 = 0$, it follows from Lemma 4.2.23 and Carleson's Lemma that

$$\iint_{\mathbb{R}^{n+1}_+} \left| (t^m \partial_t^m \mathcal{S}_t B_{2\|}) P_t F \right|^2 \frac{dx\, dt}{t} \lesssim (1 + |||\widetilde{\Theta}_t|||_{op}^2) \|B_2\|_n^2 \|\nabla_\| F\|_2^2,$$

which can be hidden in (4.5.29) when $\|B_2\|_n$ is sufficiently small.

Finally, to handle J_5, rewrite it as $J_5 = t^{m+1} \partial_t^{m+1} \mathcal{S}_t B_{2\perp} (\frac{1}{t}[I - P_t] F)$. Since $t^{m+1} \partial_t^{m+1} \mathcal{S}_t B_{2\perp} : L^2 \to L^2$ uniformly in t, we may handle this term exactly as J_1 by using (4.5.30).

Having handled the remainder R_t, we have reduced matters to showing that the square function bound

$$\iint_{\mathbb{R}^{n+1}_+} |t^m (\partial_t^m \mathcal{S}_t \nabla_\| \cdot A_\|)(x) P_t \nabla_\| F(x)|^2 \, \frac{dx\, dt}{t} \lesssim \|\nabla_\| F\|_2^2$$

holds for all $F \in Y^{1,2}(\mathbb{R}^n)$ with $\|\nabla_\| F\|_2 \leq 1$. By Carleson's Lemma, it is enough to show that

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_{\mathbb{R}^n} |t^m (\partial_t^m \mathcal{S}_t \nabla_\| \cdot A_\|)(x)|^2 \, \frac{dx\, dt}{t} \leq C. \tag{4.5.31}$$

In order to obtain (4.5.31), we appeal to the technology of the solution of the Kato problem [AHL+02a], and follow the argument of [HMM15b]. By [AHL+02a], for each dyadic cube Q there exists a mapping $F_Q : \mathbb{R}^n \to \mathbb{C}^n$ such that

(i) $\displaystyle\int_{\mathbb{R}^n} |\nabla_\| F_Q|^2 \leq C|Q|$

(ii) $\displaystyle\int_{\mathbb{R}^n} |L_\| F_Q|^2 \leq \frac{|Q|}{\ell(Q)^2}$

(iii) $\displaystyle\sup_Q \int_0^{\ell(Q)} \fint_Q |\vec{\zeta}(x,t)|^2 \, \frac{dx\, dt}{t} \lesssim C \sup_Q \int_0^{\ell(Q)} \fint_Q |\vec{\zeta}(x,t) E_t \nabla_\| F_Q|^2 \, \frac{dx\, dt}{t}$

for each $\vec{\zeta} : \mathbb{R}^{n+1}_+ \to \mathbb{C}^n$, where E_t denotes the dyadic averaging operator; that is, if $Q(x,t)$ is the minimal dyadic cube containing $x \in \mathbb{R}^n$ with side length at least t, then $E_t g(x) = \fint_{Q(x,t)} g$. Here, we note that $\nabla_\| F_Q$ is the Jacobian of F_Q and $\vec{\zeta} E_t \nabla_\| F_Q$ is a vector. Given such a family $\{F_Q\}_Q$, we see that by applying property (iii) with $\vec{\zeta}(x,t) = T_t A_\|$, where $T_t := t^m \partial_t^m (\mathcal{S}_t \nabla_\|)$ it is enough to show that

$$\int_0^{\ell(Q)} \int_Q |(T_t A_\|)(x) E_t \nabla_\| F_Q(x)|^2 \, \frac{dx\, dt}{t} \lesssim (1 + |||\Theta_t^0|||_{op}^2)|Q|.$$

Following [AT98, CM86], we write that

$$\begin{aligned}
(T_t A_\|) E_t \nabla_\| F_Q &= \{(T_t A_\|) E_t - T_t A_\|\} \nabla_\| F_Q + T_t A_\| \nabla_\| F_Q \\
&= T_t A_\| (E_t - P_t) \nabla_\| F_Q + \{(T_t A_\|) P_t - T_t A_\|\} \nabla_\| F_Q + T_t A_\| \nabla_\| F_Q \\
&=: R_t^{(1)} \nabla_\| F_Q + R_t^{(2)} \nabla_\| F_Q + T_t A_\| \nabla_\| F_Q.
\end{aligned}$$

Observe that $R_t^{(2)} = -R_t$ from above, and we have already shown that $|||R_t|||_{op} \lesssim (1 + |||\Theta_t^0|||_{op})$[3], so that the desired bound holds from property $i)$ of F_Q. For the last term, we have that $T_t A_\| \nabla_\| F_Q = t^m \partial_t^m \mathcal{S}_t L_\| F_Q$, and we know that $t^{m-1} \partial_t^m \mathcal{S}_t : L^2 \to L^2$ uniformly in t. Thus, by property (ii) of F_Q, we have that

[3] We have shown that $|||R_t|||_{op} \lesssim (1 + |||\Theta_t^0|||_{op}) + \epsilon |||\widetilde{\Theta}_t|||_{op}$, where ϵ is at our disposal by the smallness of $\max\{\|B_1\|_n, \|B_2\|_n\}$, and this is enough for our purposes.

$$\int_0^{\ell(Q)}\int_Q |(T_t A_\| F_Q)(x)|^2 \,\frac{dx\,dt}{t} \le \int_0^{\ell(Q)}\int_{\mathbb{R}^n} |t^{m-1}\partial_t^m \mathcal{S}_t L_\| F(x)|^2 t\,dx\,dt$$
$$\lesssim \frac{|Q|}{\ell(Q)^2}\int_0^{\ell(Q)} t\,dt \lesssim |Q|,$$

which shows the desired bound for this term.

To bound the contribution from $R_t^{(1)}$, we note that $T_t : L^2 \to L^2$ uniformly in t and

$$\iint_{\mathbb{R}^{n+1}_+} |(E_t - P_t)g(x)|^2 \,\frac{dx\,dt}{t} \lesssim \|g\|_2^2$$

for $g \in L^2(\mathbb{R}^n)$. Therefore,

$$\int_0^{\ell(Q)}\int_Q |R_t^{(1)}\nabla_\| F_Q|^2 \,\frac{dx\,dt}{t} \le \int_0^{\ell(Q)}\int_{\mathbb{R}^n} |T_t A_\|(E_t - P_t)\nabla_\| F_Q|^2 \,\frac{dx\,dt}{t}$$
$$\lesssim \int_0^{\ell(Q)}\int_{\mathbb{R}^n} |(E_t - P_t)\nabla_\| F_Q|^2 \,\frac{dx\,dt}{t}$$
$$\lesssim \|\nabla_\| F\|_2^2 \lesssim C|Q|,$$

where we used the ellipticity of A in the second inequality, and property $i)$ of F_Q in the last inequality. This controls the contribution from $R_t^{(1)}$ and finishes the proof of the Lemma. $\square$

We move on to the

Proof of Proposition 4.5.22. To see that $|||\Theta_t^{(a)}|||_{op} \lesssim 1 + |||\Theta_t^0|||_{op}$, and that $\|\Theta_t^{(a)} 1\|_{\mathcal{C}} \lesssim 1 + \|\Theta_t^0\|_{\mathcal{C}}$, we simply notice that $\Theta_t^{(a)} = (\Theta_t^0, t^m \partial_t^m(\mathcal{S}_t \nabla_\|))$ so that the desired bounds follow directly from the previous lemma.

We are left with showing the bounds in Proposition 4.5.22 for $T_t = \Theta_t'$. We note immediately that (4.5.24) will follow from (4.5.23) and Lemma 4.2.23. Therefore, it is enough to show (4.5.23). In fact, by Lemma 4.5.26, it suffices to show that $|||\Theta_t'|||_{op} \lesssim |||\widetilde{\Theta}_t|||_{op} + |||\Theta_t^0|||_{op}$. For $g \in L^2(\mathbb{R}^n, \mathbb{C}^n)$, we have that

$$\Theta_t' g = t^m \partial_t^m \mathcal{S}_t(B_{2\|} g) + t^m \partial_t^m (\mathcal{S}_t \nabla) \cdot \tilde{A} g$$
$$= t^m \partial_t^m \mathcal{S}_t(B_{2\|} g) + t^m \partial_t^m (\mathcal{S}_t \nabla_\|) \cdot A_\| g - t^m \partial_t^{m+1} \mathcal{S}_t \vec{b} g,$$

where $\vec{b} = (A_{n+1,j})_{1\le j\le n}$. The ellipticity of A gives immediately that $|||t^m\partial_t^m(\mathcal{S}_t\nabla_\|)A_\||||_{op} \lesssim |||\widetilde{\Theta}|||_{op}$, and $|||t^m\partial_t^{m+1}\mathcal{S}_t\vec{b}|||_{op} \lesssim |||\widetilde{\Theta}|||_{op}$. It remains to handle the first term. Observe that $B_{2\|}g = \operatorname{div}_\| \nabla_\| I_2 B_{2\|}g = \operatorname{div}_\| \vec{R}I_1B_{2\|}g$, where $\vec{R}$ is the vector-valued Riesz tranform. It follows that $B_{2\|}g = \operatorname{div}_\| \vec{G}$ with $\|\vec{G}\|_2 \lesssim \|B_2\|_n\|g\|_2$, and hence

$$|||t^m\partial_t^m S_tB_2|||_{op} \lesssim |||\widetilde{\Theta}_t|||_{op}\|B_2\|_n,$$

which yields the desired bound. □

4.5.5 Proof of the square function bound

We finally turn to the proof of Theorem 4.5.1 (and hence, by our reduction, the proof of Theorem 4.1.1). Our method follows the lines of [GdlHH16], circumventing some difficulties by introducing $\Theta_t^{(a)}$ and $b_Q^{(a)}$.

Proof of Theorem 4.5.1. Let C_1 be a constant, depending on m, n, and C_A, for which the inequalities (4.5.23) and (4.5.24) hold. We choose η in Proposition 4.5.9 as $\eta := 1/(2C_1+4)$. By the generalized Christ-Journé $T1$ theorem for square functions, (see [GdlHH16, Theorem 4.3]) to prove the theorem it is enough[4] to show that

$$\|\Theta_t^0 1\|_{\mathcal{C}} \le C. \tag{4.5.32}$$

As in [GdlHH16], we want to reduce the above estimate to one of the form

$$\iint_{R_Q} \left|(\Theta_t 1)A_t^{\mu_Q} b_Q\right|^2 \frac{dx\,dt}{t} \le C|Q|,$$

where $A_t^{\mu_Q}$ is an averaging operator adapted to μ_Q (and hence Q) we will introduce later and R_Q is the Carleson region $Q\times(0,\ell(Q))$. The argument up until this reduction, namely (4.5.40), is almost exactly as in [GdlHH16]. Define $\zeta(x,t) := \Theta_t 1(x)$, $\zeta^0(x,t) := \Theta_t^0 1(x)$, and $\zeta'(x,t) := \Theta_t' 1(x)$, where these objects make sense as elements

[4]The careful reader will notice that we have verified the hypotheses of [GdlHH16, Theorem 4.3] above aside from the quasiorthogonality estimate [GdlHH16, equation (4.4)]. This estimate is slightly misstated in [GdlHH16], where h should be replaced by $\mathcal{Q}_s h$ and we verify this below when dealing with the term labeled J_1.

of $L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ by Lemma 4.2.24 and Proposition 4.4.37. Consider the cut-off surfaces

$$F_1 := \{(x,t) \in \mathbb{R}^{n+1}_+ : |\zeta^0(x,t)| \leq \sqrt{\eta}|\zeta'(x,t)|\},$$

$$F_2 := \{(x,t) \in \mathbb{R}^{n+1}_+ : |\zeta^0(x,t)| > \sqrt{\eta}|\zeta'(x,t)|\}.$$

We easily have that $\|\zeta^0\|_{\mathcal{C}} \leq \|\zeta^0 \mathbb{1}_{F_1}\|_{\mathcal{C}} + \|\zeta^0 \mathbb{1}_{F_2}\|_{\mathcal{C}}$. By definition of F_1, Proposition 4.5.22, and the fact that $\eta < 1/(2C_1)$, we realize that

$$\|\zeta^0 \mathbb{1}_{F_1}\|_{\mathcal{C}} \leq \eta\|\zeta^1\|_{\mathcal{C}} \leq C_1\eta(1 + \|\zeta^0\|_{\mathcal{C}}) \leq \frac{1}{2}(1 + \|\zeta^0\|_{\mathcal{C}}).$$

Consequently, $\|\zeta^0\|_{\mathcal{C}} \leq 1 + 2\|\zeta^0 \mathbb{1}_{F_2}\|_{\mathcal{C}}$, and recall that we may work with truncated versions of each of ζ, ζ^0, ζ' so that all quantities are finite. Accordingly, we have reduced the proof of (4.5.32) to showing that

$$\|\zeta^0 \mathbb{1}_{F_2}\|_{\mathcal{C}} \leq C. \tag{4.5.33}$$

By (4.5.12) and (4.5.13) we have that

$$\begin{aligned}\frac{1}{2}|\zeta^0| \leq \left|\zeta^0 \cdot \frac{1}{\mu_Q(Q)}\int_Q b^0_Q\, d\mu_Q\right| &\leq \left|\zeta \cdot \frac{1}{\mu_Q(Q)}\int_Q b_Q\, d\mu_Q\right| + \left|\zeta' \cdot \frac{1}{\mu_Q(Q)}\int_Q b'_Q\, d\mu_Q\right| \\ &\leq \left|\zeta \cdot \frac{1}{\mu_Q(Q)}\int_Q b_Q\, d\mu_Q\right| + \frac{\eta}{2}|\zeta'|,\end{aligned}$$

for every dyadic cube $Q \subset \mathbb{R}^n$. Therefore, for every such $Q \subset \mathbb{R}^n$, the estimates

$$\frac{1}{2}|\zeta^0| \leq \left|\zeta \cdot \frac{1}{\mu_Q(Q)}\int_Q b_Q\, d\mu_Q\right| + \frac{\sqrt{\eta}}{2}|\zeta^0|, \qquad \text{and}$$

$$|\zeta| \leq |\zeta^0| + |\zeta'| \leq (1 + \eta^{-\frac{1}{2}})|\zeta^0| \leq 2\eta^{-1/2}|\zeta^0|$$

hold in F_2. Combining the previous three estimates, we have that for $(x,t) \in F_2$ and every dyadic cube Q

$$\frac{\sqrt{\eta}}{2}(1 - \sqrt{\eta})\frac{1}{2}|\zeta(x,t)| \leq (1 - \sqrt{\eta})\frac{1}{2}|\zeta^0(x,t)| \leq \left|\zeta \cdot \frac{1}{\mu_Q(Q)}\int_Q b_Q\, d\mu_Q\right|. \tag{4.5.34}$$

At this juncture, we make the observation that, in order to obtain (4.5.33), it suffices to show that for some $\alpha > 0$ chosen small enough, we have that

$$\|\zeta^0 \mathbb{1}_{F_2} \mathbb{1}_{\Gamma_\nu^\alpha}(\zeta)\|_{\mathcal{C}} \leq C, \tag{4.5.35}$$

with C independent of ν, where Γ_ν^α is an arbitrary cone of aperture α; that is,

$$\Gamma_\nu^\alpha := \{z \in \mathbb{C}^2 : |(z/|z|) - \nu)| < \alpha\},$$

for $\nu \in \mathbb{C}^2$ a unit vector. It is clear that if we establish (4.5.35), then (4.5.33) follows by summing over a collection of cones covering $\mathbb{C}^2$. In light of this, we fix such a cone Γ_ν^α with α to be chosen. By (4.5.34) and the fact that $\eta < 1/4$ we have that for each $(x,t) \in F_2$ with $\zeta(x,t) \in \Gamma_\nu^\alpha$ and every dyadic cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned}
\frac{\sqrt{\eta}}{8} &\leq \Big|\frac{\zeta(x,t)}{|\zeta(x,t)|} \cdot \frac{1}{\mu_Q(Q)} \int_Q b_Q \, d\mu\Big| \\
&\leq \Big|\Big(\frac{\zeta(x,t)}{|\zeta(x,t)|} - \nu\Big) \cdot \frac{1}{\mu_Q(Q)} \int_Q b_Q \, d\mu\Big| + \Big|\nu \cdot \frac{1}{\mu_Q(Q)} \int_Q b_Q \, d\mu\Big| \\
&\leq C_0 \alpha + \Big|\nu \cdot \frac{1}{\mu_Q(Q)} \int_Q b_Q \, d\mu\Big|,
\end{aligned}$$

where in the last step, we used Schwarz's inequality, the fact that

$$1/C_0 \leq d\mu/dx = \phi_Q \leq 1 \text{ on } Q,$$

and (4.5.10). Since α is at our disposal, we may choose $\alpha < \frac{\sqrt{\eta}}{16 C_0}$, so that

$$\frac{\sqrt{\eta}}{16} =: \theta \leq \Big|\nu \cdot \frac{1}{\mu_Q(Q)} \int_Q b_Q \, d\mu\Big|. \tag{4.5.36}$$

Next, we observe that in order to obtain (4.5.36) we needed $(x,t) \in F_2$ with $\zeta(x,t) \in \Gamma_\nu^\alpha$. This means that (4.5.36) holds whenever

$$\iint_{R_Q} |\zeta^0(x,t)|^2 \mathbb{1}_{F_2}(x,t) \mathbb{1}_{\Gamma_\nu^\alpha}(\zeta(x,t)) \, \frac{dx\, dt}{t} \neq 0.$$

Consequently, when proving (4.5.35) we can always assume that (4.5.36) holds.

Now, fix any dyadic cube Q such that (4.5.36) holds and, following [GdlHH16], use a stopping time procedure to extract a family $\mathcal{F} = \{Q_j\}$ of non-overlapping dyadic subcubes of Q which are maximal with respect to the property that at least one of the following conditions holds:

$$\frac{1}{\mu_Q(Q_j)} \int_{Q_j} |b_Q| \, d\mu_Q > \frac{\theta}{4\alpha} \qquad \text{(type } I\text{)}$$
$$\left| \nu \cdot \frac{1}{\mu_Q(Q_j)} \int_{Q_j} b_Q \, d\mu_Q \right| \leq \frac{\theta}{2} \qquad \text{(type } II\text{)}.$$

If some Q_j happens to satisfy both the type I and type II conditions we (arbitrarily) assign it to be of type II. We will write $Q_j \in \mathcal{F}_I$ or $Q_j \in \mathcal{F}_{II}$ to mean that a cube is of type I or of type II respectively. This stopping time argument produces an 'ample sawtooth' with desirable bounds in the following sense.

Claim 4.5.37 (Ample sawtooth). There exists $\beta > 0$, uniform in Q, such that

$$\sum_{Q_j \in \mathcal{F}} |Q_j| \leq (1-\beta)|Q|, \tag{4.5.38}$$

provided that $\alpha > 0$ is small enough (depending on allowable constants). Moreover,

$$|\zeta(x,t)|^2 \mathbb{1}_{\Gamma^\alpha}(\zeta(x,t)) \leq C_\theta |\zeta(x,t) A_t^{\mu_Q} b_Q(x)|^2, \quad \text{for } (x,t) \in E_Q^*, \tag{4.5.39}$$

where $E_Q^* := R_Q \setminus (\cup_{Q_j \in \mathcal{F}} R_{Q_j})$. Here $A_t^{\mu_Q}$ is the 'dyadic averaging operator adapted to the measure μ_Q', that is, $A_t^\mu f(x) = \frac{1}{\mu_Q(Q(x,t))} \int_{Q(x,t)} f \, d\mu_Q$, where $Q(x,t)$ denotes the smallest dyadic cube, of side length at least t, that contains x.

We postpone the proof of the claim for a bit. The ampleness condition (4.5.38) allows us to use the "John-Nirenberg lemma for Carleson measures" to replace R_Q in the definition of $\|\cdot\|_{\mathcal{C}}$ by E_Q^*. This is done via an induction argument; see for instance, [Hof10, Lemma 1.37]. Thus, we have by (4.5.39) that

$$\begin{aligned} \|\zeta^0 \mathbb{1}_{F_2} \mathbb{1}_{\Gamma_\nu^\alpha}(\zeta)\|_{\mathcal{C}} &\lesssim_\beta \sup_Q \frac{1}{|Q|} \iint_{E_Q^*} |\zeta^0(x,t)|^2 \mathbb{1}_{F_2}(x,t) \mathbb{1}_{\Gamma_\nu^\alpha}(\zeta(x,t)) \frac{dx\,dt}{t} \\ &\lesssim \sup_Q \frac{1}{|Q|} \iint_{R_Q} |\zeta(x,t) A_t^{\mu_Q} b_Q(x)|^2 \frac{dx\,dt}{t}, \end{aligned}$$

where we used that $|\zeta^0| \leq |\zeta|$ in the first line and replaced E_Q^* by the larger set R_Q after using (4.5.39) in the second line. As we had reduced the proof of the theorem to showing the estimate (4.5.35), it is enough to show that

$$\sup_{Q} \frac{1}{|Q|} \iint_{R_Q} |\zeta(x,t) A_t^{\mu_Q} b_Q(x)|^2 \, \frac{dx\,dt}{t} \leq C. \tag{4.5.40}$$

To this end, we fix a dyadic cube Q and write

$$\zeta A_t^{\mu_Q} b_Q = [(\Theta_t 1) A_t^{\mu} - \Theta_t] b_Q + \Theta_t b_Q =: R_t b_Q + \Theta_t b_Q = I + II.$$

First we handle term II, which is (almost) good by design. We write

$$II = \Theta_t b_Q = \hat{\Theta}_t \hat{b}_Q - \Theta_t^{(a)} b_Q^{(a)} =: II_1 + II_2.$$

By (4.5.11), the contribution from the term II_1 in (4.5.40) is controlled by C_0. Moreover, by Proposition 4.5.22 we have that

$$\begin{aligned}\iint_{R_Q} |\Theta_t^{(a)} b_Q^{(a)}|^2 \, \frac{dx\,dt}{t} &\leq C_1 \|b_Q^{(a)}\|^2_{L^2(\mathbb{R}^n)} (1 + \|\Theta_t^0 1\|_{\mathcal{C}}) \\ &\leq C_1 C_0 |Q| \, \|B_1\|_n^2 (1 + \|\Theta_t^0 1\|_{\mathcal{C}}),\end{aligned}$$

so that the contribution of II_2 can be hidden in (4.5.32), provided that $\|B_1\|_n$ is sufficiently small (depending on η, α). Here, we used that $b_Q^{(a)}(y) = |Q| B_1 F_Q(y,0)$, so that

$$\|b_Q^{(a)}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |Q|^2 |B_1 F_Q(\cdot,0)|^2 \leq \|B_1\|_n^2 |Q|^2 \int_{\mathbb{R}^n} |\nabla F_Q(\cdot,0)|^2 \leq C_0 \|B_1\|_n^2 \, |Q|.$$

It remains to obtain a desirable bound for I. Let $\{\mathcal{Q}_s\}_{s>0}$ be a CLP family (see Definition 4.2.26). By a standard orthogonality argument and (4.5.10), it is enough to show that for some $\beta_0 > 0$ and all $t \in (0, \ell(Q))$, the estimate

$$\int_Q |R_t \mathcal{Q}_s^2 h|^2 \lesssim \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\beta_0} \int_{\mathbb{R}^n} |\mathcal{Q}_s h|^2 \tag{4.5.41}$$

holds for all $h \in H \times L^2(\mathbb{R}^n)$.

We remind the reader that $H := \{h' : h' = \nabla F, F \in Y^{1,2}(\mathbb{R}^n)\}$ and that $b_Q \in H \times L^2(\mathbb{R}^n)$. Before proving (4.5.41), we make a small technical point. Having fixed Q, we let $\tilde{\mu}_Q$ be a measure on $\mathbb{R}^n$ defined by $\tilde{\mu}_Q := \mu_Q|_Q + \frac{1}{C_0} dx|_{\mathbb{R}^n \setminus Q}$, and set $E_t = A_t^{\tilde{\mu}_Q}$.

Notice that for $(x,t) \in Q \times (0, \ell(Q))$, $A_t^{\tilde{\mu}_Q}$ acts exactly as $A_t^{\mu_Q}$. Thus, in order to prove (4.5.41), we may replace R_t by $\widetilde{R}_t$, where $\widetilde{R}_t := [(\Theta_t 1)E_t - \Theta_t]$. Notice that we may apply Lemma 4.2.24 to Θ_t, since Θ_t has good off-diagonal decay (see Proposition 4.4.37) and satisfies uniform L^2 bounds on slices (see Proposition 4.4.23). Thus, $(\Theta_t 1)$ is well defined as an element of L^2_{loc} and, since E_t is a self-adjoint averaging operator, we have that

$$\sup_{t>0} \|(\Theta_t 1)E_t\|_{L^2 \to L^2} \leq C. \tag{4.5.42}$$

We break (4.5.41) into cases.

Case 1: $t \leq s$**.** In this case, we see by (4.5.42) and properties of Θ_t that $\widetilde{R}_t 1 = 0$, $\|\widetilde{R}_t\|_{L^2 \to L^2} \leq C$ and $\widetilde{R}_t$ has good off-diagonal decay. Hence, it follows from Lemma 4.2.25 that

$$\|\widetilde{R}_t \mathcal{Q}_s^2 h\|_{L^2(\mathbb{R}^n)} \lesssim t\|\nabla \mathcal{Q}_s^2 h\|_{L^2(\mathbb{R}^n)} \lesssim \tfrac{t}{s}\|s\nabla \mathcal{Q}_s \mathcal{Q}_s h\|_{L^2(\mathbb{R}^n)} \lesssim \tfrac{t}{s}\|\mathcal{Q}_s h\|_{L^2(\mathbb{R}^n)},$$

which shows (4.5.41) with $\beta_0 = 2$ in this case.

Case 2: $t > s$**.** In this case, we break $\widetilde{R}_t$ into its two separate operators. One can verify that $\|E_t \mathcal{Q}_s\|_{L^2 \to L^2} \lesssim (\frac{s}{t})^\gamma$ for some $\gamma > 0$. Since E_t is a projection operator, we have that $E_t = E_t^2$ and hence by (4.5.42), we see that

$$\|(\Theta_t 1)E_t \mathcal{Q}_s^2 h\|_2 = \|(\Theta_t 1)E_t[E_t \mathcal{Q}_s^2 h]\|_2 \lesssim \|E_t \mathcal{Q}_s^2 h\|_2 \lesssim (\tfrac{s}{t})^\gamma \|\mathcal{Q}_s h\|_2,$$

which shows that the contribution of $(\Theta_t 1)E_t \mathcal{Q}_s^2$ to (4.5.41) when $t > s$ is as desired with $\beta_0 = 2\gamma$.

We are left with handling $\Theta_t \mathcal{Q}_s^2 h$. Since $h = (h', h^0) \in H \times L^2(\mathbb{R}^n)$, we write $h = (\nabla_\| F, h^0)$, with $F \in Y^{1,2}(\mathbb{R}^n)$ (note $\nabla_\| = \nabla$ here). Then we may write

$$\begin{aligned}\Theta_t \mathcal{Q}_s h &= \Theta_t^0 \mathcal{Q}_s^2 h^0 + \Theta_t' \mathcal{Q}_s^2 \nabla_\| F \\ &= \Theta_t^0 \mathcal{Q}_s^2 h^0 + [\Theta_t' \mathcal{Q}_s^2 \nabla_\| F + \Theta_t^{(a)} B_1 \mathcal{Q}_s^2 F] - \Theta_t^{(a)} B_1 \mathcal{Q}_s^2 F \\ &= J_1 + J_2 + J_3.\end{aligned}$$

To handle J_1, we write $\mathcal{Q}_s = s \operatorname{div}_\| s\nabla_\| e^{s^2\Delta}$, so that

$$J_1 = \Theta_t^0 \mathcal{Q}_s^2 h^0 = t^m (\partial_t)^{m+1} \mathcal{S}_t^{\mathcal{L}} \mathcal{Q}_s \mathcal{Q}_s h^0 = \tfrac{s}{t} t^{m+1} (\partial_t)^{m+1} \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_\| s\nabla_\| e^{s^2\Delta} \mathcal{Q}_s h^0.$$

Note that by (4.4.25) we have that $t^{m+1}(\partial_t)^{m+1}\mathcal{S}_t^{\mathcal{L}}\,\mathrm{div}_\parallel$ and $s\nabla_\parallel e^{s^2\Delta}$ are bounded operators on $L^2(\mathbb{R}^n)$. Therefore, we have that $\|\Theta_t^0\mathcal{Q}_s^2h^0\|_2 \lesssim \frac{s}{t}\|\mathcal{Q}_sh^0\|_2$, and the contribution of J_1 to (4.5.41) when $t > s$ is as desired with $\beta_0 = 2$.

For the term J_2, first we use Proposition 4.2.27 to justify that there exists $g \in L^2(\mathbb{R}^n)$ such that $\mathcal{Q}_sF = I_1g$, where $I_1 = (-\Delta)^{-1/2}$ is the Riesz potential of order 1, and satisfying $\|g\|_2 \approx \|\nabla_\parallel\mathcal{Q}_sF\|_2 = \|\mathcal{Q}_s\nabla_\parallel F\|_2 = \|\mathcal{Q}_sh'\|_2$ (every $F \in Y^{1,2}(\mathbb{R}^n)$ arises as the Riesz potential of a function g in $L^2(\mathbb{R}^n)$). Then, we may use integration by parts on slices (Proposition 4.3.19) to compute that

$$\begin{aligned} J_2 &= t^m(\partial_t)^m\mathcal{S}_t^{\mathcal{L}}\big(B_{2\parallel}\nabla_\parallel\mathcal{Q}_s^2F\big)+t^m(\partial_t)^m(\mathcal{S}_t^{\mathcal{L}}\nabla)\tilde{A}\nabla_\parallel\mathcal{Q}_s^2F+t^m(\partial_t)^m(\mathcal{S}_t^{\mathcal{L}}\nabla)B_1\mathcal{Q}_s^2F \\ &= -t^m(\partial_t)^{m+1}(\mathcal{S}_t^{\mathcal{L}}\nabla)\vec{A}_{\cdot,n+1}\mathcal{Q}_sI_1g + t^m(\partial_t)^{m+1}\mathcal{S}_t^{\mathcal{L}}B_{2\perp}\mathcal{Q}_sI_1g = J_{2,1} + J_{2,2}. \end{aligned}$$

Since $\|s^{-1}\mathcal{Q}_sI_1\|_{L^2\to L^2} \leq C$ and $t^{m+1}(\partial_t)^{m+1}(\mathcal{S}_t^{\mathcal{L}}\nabla) : L^2 \to L^2$, we obtain that the contribution of $J_{2,1}$ to (4.5.41) when $t > s$ is as desired with $\beta_0 = 2$. Similarly, $t^m(\partial_t)^{m+1}\mathcal{S}_t^{\mathcal{L}}B_{2\perp} : L^2 \to L^2$, so that the contribution of $J_{2,2}$ to (4.5.41) when $t > s$ is as desired with $\beta_0 = 2$.

We are left with controlling the contribution of

$$J_3 = \Theta_t^{(a)}B_1\mathcal{Q}_s^2F = t^m\partial_t^m(\mathcal{S}_t^{\mathcal{L}}\nabla)B_1\mathcal{Q}_sF = \Theta_{t,m}B_1I_1g,$$

where $F = I_1g$, $F \in Y^{1,2}$ and $g \in L^2$ with $\|g\|_2 \approx \|\nabla_\parallel F\|_2$. By Proposition 4.4.30, for all $s < t$ we have that

$$\|\Theta_t^{(a)}B_1\mathcal{Q}_s^2F\|_{L^2(\mathbb{R}^n)} \lesssim \left(\frac{s}{t}\right)^\gamma\|\mathcal{Q}_sg\|_{L^2(\mathbb{R}^n)}.$$

Then we may control this term in (4.5.41) with g in place of $h = \nabla_\parallel F$, which is sufficient as $\|g\|_2 \lesssim \|\nabla_\parallel F\|_2$.

The proof of the theorem is finished modulo the

Proof of Claim 4.5.37. We first verify (4.5.39). Observe, by the maximality of the family Q_j, that for any dyadic subcube Q' of Q which is not contained in any Q_j, we have the inequalities opposite to the type I and type II inequalities, with Q' in place of Q_j. Thus,

$$\frac{\theta}{2} \leq \left|\nu\cdot A_t^{\mu_Q}b_Q(x)\right| \quad \text{and} \quad |A_t^{\mu_Q}b_Q(x)| \leq \frac{\theta}{4\alpha}$$

for all $(x,t) \in E_Q^*$. It follows that if $z \in \Gamma_\nu^\alpha$ and $(x,t) \in E_Q^*$, we have the bound

$$\begin{aligned}\frac{\theta}{2} \leq \left|\nu \cdot A_t^{\mu_Q} b_Q(x)\right| \leq \left|(z/|z|) \cdot A_t^{\mu_Q} b_Q(x)\right| + \left|(z/|z| - \nu) \cdot A_t^{\mu_Q} b_Q(x)\right| \\ \leq \left|(z/|z|) \cdot A_t^{\mu_Q} b_Q(x)\right| + \frac{\theta}{4},\end{aligned}$$

where we used the definition of Γ_ν^α in the last line. The above estimate yields (4.5.39) with $C_\theta = (\frac{4}{\theta})^2$ by setting $z = \zeta(x,t)$.

Now we establish (4.5.38). Set $E := Q \setminus (\cup_{Q_j \in \mathcal{F}} Q_j)$ and $B_I := \cup_{Q_j \in \mathcal{F}_I} Q_j$. By definition of $\mathcal{F}_I$ and the fact that $1/C_0 \leq d\mu/dx \leq 1$ on Q, we have that $B_I \subset \left\{\mathscr{M}(b_Q) > \frac{\theta}{4C_0\alpha}\right\}$ where $\mathscr{M}$ is the uncentered Hardy-Littlewood maximal function on $\mathbb{R}^n$ (taken over cubes). The weak-type $(2,2)$ inequality for the Hardy-Littlewood maximal function and (4.5.10) yield the estimate

$$|B_I| \leq CC_0^2 \Big(\frac{\alpha}{\theta}\Big)^2 \int_{\mathbb{R}^n} |b_Q|^2 \leq CC_0^3 \Big(\frac{\alpha}{\theta}\Big)^2 |Q|.$$

From this estimate, (4.5.10), (4.5.36), the definition of type II cubes, and Hölder's inequality we obtain

$$\begin{aligned}\theta\mu_Q(Q) &\leq \left|\nu \cdot \int_Q b_Q \, d\mu_Q\right| \\ &\leq \left|\nu \cdot \int_E b_Q \, d\mu_Q\right| + \int_{B_I} |b_Q| \, d\mu + \sum_{Q_j \in \mathcal{F}_{II}} \left|\nu \cdot \int_{Q_j} b_Q \, d\mu_Q\right| \\ &\leq |E|^{1/2} \|b_Q\|_{L^2(\mathbb{R}^n)} + |B_I|^{1/2} \|b_Q\|_{L^2(\mathbb{R}^n)} + \frac{\theta}{2} \sum_{Q_j \in \mathcal{F}_{II}} \mu_Q(Q_j) \\ &\leq C|E|^{1/2}|Q|^{1/2} + C_\theta \alpha |Q| + \frac{\theta}{2}\mu_Q(Q).\end{aligned}$$

Choosing $\alpha > 0$ small enough and using the fact that $(1/2)^n|Q| \leq \mu_Q(Q) \leq |Q|$, the above estimate implies that $|Q| \leq C_\theta |E|$, which yields the claim with $\beta = 1/C_\theta$. $\square$

Thus we conclude the proof of Theorem 4.5.1. $\square$

4.6 Control of slices via square function estimates

We are able to use the square function estimate obtained in the previous section to immediately improve our $L^2 \to L^2$ boundedness results of $t-$derivatives of the single layer potential. More precisely, in the following lemma, we extend estimate (4.4.25) (previously valid for $m \geq 2$), to the case $m = 1$, given sufficient smallness of $\max\{\|B_1\|_n, \|B_2\|_n\}$.

Lemma 4.6.1 (Stronger $L^2 \to L^2$ estimate)**.** *The estimate*

$$\|t\nabla\partial_t\mathcal{S}_t^{\mathcal{L}}f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

holds, provided that $\max\{\|B_1\|_n, \|B_2\|_n < \varepsilon_0$ *and* $\varepsilon_0 > 0$ *is small enough so that* (4.5.6) *holds for* $m = n + 10$.

We may use Lemma 4.6.1 to obtain the "travel down" procedure for $\nabla\mathcal{S}^{\mathcal{L}}\nabla$.

Lemma 4.6.2 ($L^2 \to L^2$ estimates for $S_t\nabla$)**.** *The following statements are true.*

(i) For each $\mathbf{f} \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ *and each* $t \neq 0$ *we have that*

$$\|t^k\partial_t^k(\mathcal{S}_t^{\mathcal{L}}\nabla)\mathbf{f}\|_2 \lesssim \|\mathbf{f}\|_2, \qquad k \geq 1, \tag{4.6.3}$$

$$\|t^k\partial_t^{k-1}\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla)\mathbf{f}\|_2 \lesssim \|\mathbf{f}\|_2, \qquad k \geq 2, \tag{4.6.4}$$

provided that $\max\{\|B_1\|_n, \|B_2\|_n < \varepsilon_0$ *is small. Therefore, for each* $m > k \geq 2$,

$$\left|\left|\left|t^k\partial_t^{k-1}\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla)\mathbf{f}\right|\right|\right| \lesssim_m |||t^m\partial_t^m(\mathcal{S}_t^{\mathcal{L}}\nabla)\mathbf{f}||| + \|\mathbf{f}\|_2, \tag{4.6.5}$$

provided that $\max\{\|B_1\|_{L^n(\mathbb{R}^n)}, \|B_2\|_{L^n(\mathbb{R}^n)}\}$ *is small.*

(ii) The estimate (4.6.5) holds for $k = 1$ *if the operator* ∇ *acting on* $(\mathcal{S}_t^{\mathcal{L}}\nabla)$ *is replaced by* ∂_t.

We proceed with the

Proof of Theorem 4.1.2. Let $\mathbf{h} \in C_c^\infty(\mathbb{R}^n)^{n+1}$ and fix $\tau > 0$. Notice that by Lemma 4.2.3, the pairing $(\mathbf{h}, \operatorname{Tr}_t \nabla u)_{2,2}$ is meaningful. Let $R \gg \tau$, $\psi \in C_c^\infty(\mathbb{R})$ satisfy $\psi \equiv 1$

on $[\tau, R]$, $\psi \equiv 0$ on $[2R, \infty)$, $|\psi| \leq 1$ and $|\psi'| \leq \frac{2}{R}$. We have the following estimates:

$$
\begin{aligned}
|I| := \Big| \int_{\mathbb{R}^n} \mathbf{h} \cdot \int_R^{2R} \psi' \nabla u \Big| &\leq \int_{\mathbb{R}^n} \int_R^{2R} |\mathbf{h}||\psi'||\nabla u| \\
&\leq \frac{2}{\sqrt{R}} \|\mathbf{h}\|_{L^2(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^{n+1}_+)} \longrightarrow 0 \quad \text{as } R \to \infty, \qquad (4.6.6)
\end{aligned}
$$

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^n} \mathbf{h} \cdot t \operatorname{Tr}_{t+\tau} \nabla \partial_t u \Big| &\leq \|\mathbf{h}\|_{L^2(\mathbb{R}^n)} \frac{t}{t+\tau} \|(t+\tau) \operatorname{Tr}_{t+\tau} \nabla \partial_t u\|_{L^2(\mathbb{R}^n)} \\
&\leq \frac{t}{\tau} \|\mathbf{h}\|_{L^2(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^{n+1}_+)} \longrightarrow 0 \quad \text{as } t \searrow 0, \qquad (4.6.7)
\end{aligned}
$$

$$
\begin{aligned}
|II| := \Big| \int_{\mathbb{R}^n} \int_{R-\tau}^{2R-\tau} \mathbf{h} \cdot t\psi'(t+\tau) \operatorname{Tr}_{t+\tau} \partial_t \nabla u \, dt \Big| &\leq 2 \fint_R^{2R} \int_{\mathbb{R}^n} t|\mathbf{h}||\partial_t \nabla u| \, dt \\
&\leq 2\|\mathbf{h}\|_{L^2(\mathbb{R}^n)} \sup_{t \in (R,2R)} \|t \operatorname{Tr}_t \partial_t \nabla u\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|\mathbf{h}\|_{L^2(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^{n+1}_{R/2})} \longrightarrow 0 \quad \text{as } R \to \infty, \qquad (4.6.8)
\end{aligned}
$$

where in (4.6.7) we used (4.3.25), and in (4.6.8) we used (4.3.24) and the absolute continuity of the integral. We now perform two integration by parts in the following calculation, recalling that $\psi(2R) = 0$ so that the arising boundary terms vanish.

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{h} \cdot \operatorname{Tr}_\tau \nabla u &= \int_{\mathbb{R}^n} \mathbf{h} \cdot \psi(\tau) \operatorname{Tr}_\tau \nabla u - \int_{\mathbb{R}^n} \mathbf{h} \cdot \psi(2R) \operatorname{Tr}_{2R} \nabla u \\
&= - \int_{\mathbb{R}^n} \mathbf{h} \cdot \int_\tau^{2R} \psi \partial_t \nabla u \; - \; \int_{\mathbb{R}^n} \mathbf{h} \cdot \int_R^{2R} \psi' \nabla u \\
= \int_{\mathbb{R}^n} \int_0^{2R-\tau} \mathbf{h} \cdot t \operatorname{Tr}_t \mathscr{T}^\tau \psi \partial_t^2 \nabla \mathscr{T}^\tau u \, dt &+ \int_{\mathbb{R}^n} \int_{R-\tau}^{2R-\tau} \mathbf{h} \cdot t \operatorname{Tr}_t \mathscr{T}^\tau \psi' \partial_t \nabla \mathscr{T}^\tau u \, dt - I \\
&= \int_{\mathbb{R}^n} \int_0^{2R-\tau} \mathbf{h} \cdot t \operatorname{Tr}_t \mathscr{T}^\tau \psi \partial_t^2 \nabla \mathscr{T}^\tau u \, dt + II - I
\end{aligned}
$$

where in the third equality we used (4.6.7) already. Note that the terms I, II drop to 0 as $R \to \infty$ by the estimates (4.6.6) and (4.6.8). For technical reasons, let us integrate by parts one more time. The boundary term that is introduced is again controlled as in (4.6.7) and (4.6.8) because we may apply the results of Proposition 4.3.23 to $\partial_t^2 \mathscr{T}^\tau u$.

Hence we have that

$$\int_{\mathbb{R}^n}\int_0^{2R-\tau} \mathbf{h}\cdot t\,\mathrm{Tr}_t\,\mathscr{T}^\tau\psi\partial_t^2\nabla\mathscr{T}^\tau u\,dt$$
$$= -\frac{1}{2}\int_{\mathbb{R}^n}\int_0^{2R-\tau}\mathbf{h}\cdot t^2\,\mathrm{Tr}_t\,\mathscr{T}^\tau\psi\partial_t^3\nabla\mathscr{T}^\tau u\,dt + III, \quad (4.6.9)$$

where $|III| \to 0$ as $R\to\infty$. Intuitively, we would like to introduce Green's formula at this point, but we want the "input" in the layer potentials to still depend on t for when we later dualize to control our integral by square function estimates. Let us now do a change of variables $t\mapsto 2t$, and carefully track the use of the chain rule:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n}\int_0^{2R-\tau}\mathbf{h}\cdot t^2\,\mathrm{Tr}_t\,\mathscr{T}^\tau\psi\partial_t^3\nabla\mathscr{T}^\tau u\,dt\\
&\qquad = 4\int_{\mathbb{R}^n}\int_0^{R-\frac{\tau}{2}}\mathbf{h}\cdot t^2\,\mathscr{T}^\tau\psi(2t)\partial_{2t}^3\nabla_{x,2t}\mathscr{T}^\tau u(\cdot,2t)\,dt\\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^n}\Big[\int_0^{R-\frac{\tau}{2}}\vec{h_\parallel}\cdot t^2\,\mathscr{T}^\tau\psi(2t)\partial_t^3\nabla_\parallel\mathscr{T}^\tau u(\cdot,2t)\,dt\\
&\qquad\qquad + \frac{1}{2}h_\perp\int_0^{R-\frac{\tau}{2}}t^2\,\mathscr{T}^\tau\psi(2t)\partial_t^4\mathscr{T}^\tau u(x,2t)\,dt\Big].
\end{aligned}$$

We now consider $s\in\mathbb{R}$ and write $2t = t+s|_{s=t}$. If F is a differentiable function in t, the chain rule tells us that $\partial_t F(t+s) = \partial_s F(t+s)$. By this change of variables, and the above identity, we compute that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n}\int_0^{2R-\tau}\mathbf{h}\cdot t^2\,\mathrm{Tr}_t\,\mathscr{T}^\tau\psi\partial_t^3\nabla\mathscr{T}^\tau u\,dt\\
&\qquad = 4\int_0^{R-\frac{\tau}{2}}t^2\,\mathscr{T}^{t+\tau}\psi(t)\Big[\int_{\mathbb{R}^n}\mathbf{h}\cdot\mathrm{Tr}_t\,\nabla_{x,t}D_{n+1}^3\mathscr{T}^s\mathscr{T}^\tau u(x,t)\Big]_{s=t}\,dt.\\
&\qquad\quad = 4\int_0^{R-\frac{\tau}{2}}t^2\,\mathscr{T}^{t+\tau}\psi(t)\Big[\int_{\mathbb{R}^n}\mathbf{h}\cdot\mathrm{Tr}_t\,\nabla D_{n+1}\mathscr{T}^\tau\Big(D_{n+1}^2\mathscr{T}^s u\Big)\Big]_{s=t}\,dt.
\end{aligned}$$

We now apply Green's formula, Theorem 4.4.16 (ii)). The function $v := D_{n+1}^2\mathscr{T}^s u$ belongs to $W^{1,2}(\mathbb{R}^{n+1}_+)\subset Y^{1,2}(\mathbb{R}^{n+1}_+)$ and solves $\mathcal{L}v = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense. Therefore the identity $v = -\mathcal{D}^{\mathcal{L},+}(\mathrm{Tr}_0\,v) + \mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},+}v)$ holds in $Y^{1,2}(\mathbb{R}^{n+1}_+)$, for any

$s > 0$. But by the results of Proposition 4.3.23, for each $t > 0$ we have the identity

$$\mathrm{Tr}_t \nabla D_{n+1} \mathscr{T}^\tau v = \mathrm{Tr}_t \nabla D_{n+1} \mathscr{T}^\tau \big(-\mathcal{D}^{\mathcal{L},+}(\mathrm{Tr}_0\, v) + \mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},+} v)\big)$$

in $L^2(\mathbb{R}^n)$, for any $s > 0$ and $t > 0$. As such, per our calculations we have the identity

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n}\int_0^{2R-\tau} \mathbf{h}\cdot t^2\,\mathrm{Tr}_t\,\mathscr{T}^\tau\psi\partial_t^3\nabla\mathscr{T}^\tau u\,dt \\
&\qquad = -4\int_0^{R-\frac{\tau}{2}} t^2\mathscr{T}^{t+\tau}\psi(t)\Big[\int_{\mathbb{R}^n}\mathbf{h}\cdot\mathrm{Tr}_t\,\nabla D_{n+1}\mathscr{T}^\tau\mathcal{D}^{\mathcal{L},+}(\mathrm{Tr}_0\, v)\Big]_{s=t}dt \\
&\qquad + 4\int_0^{R-\frac{\tau}{2}} t^2\mathscr{T}^{t+\tau}\psi(t)\Big[\int_{\mathbb{R}^n}\mathbf{h}\cdot\mathrm{Tr}_t\,\nabla D_{n+1}\mathscr{T}^\tau\mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},+}v)\Big]_{s=t}dt = IV + V.
\end{aligned}$$

Now we make use of the adjoint relations (4.4.5), (4.4.21) and (4.3.26) to dualize IV and V. Indeed, we see that

$$\begin{aligned}
\int_{\mathbb{R}^n}\mathbf{h}\cdot\mathrm{Tr}_t\,\nabla D_{n+1}\mathscr{T}^\tau\mathcal{D}^{\mathcal{L},+}(\mathrm{Tr}_0\, v) &= \overline{\Big(D_{n+1}\partial_{\nu,-t-\tau}^{\mathcal{L}^*,-}(\mathcal{S}^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}, \mathrm{Tr}_0\, v\Big)_{2,2}} \\
&= \overline{\Big(D_{n+1}e_{n+1}\cdot\mathrm{Tr}_{-t-\tau}\Big[A^*\nabla+\overline{B_2}\Big](\mathcal{S}^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}, \mathrm{Tr}_s\, D_{n+1}^2 u\Big)_{2,2}},
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^n}\mathbf{h}\cdot\mathrm{Tr}_t\,\nabla D_{n+1}\mathscr{T}^\tau\mathcal{S}^{\mathcal{L}}(\partial_\nu^{\mathcal{L},+}v) &= \overline{\Big(\mathrm{Tr}_{-t-\tau}\,D_{n+1}(\mathcal{S}^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}, \partial_\nu^{\mathcal{L},+}v\Big)_{2,2}} \\
&= \overline{\Big(\mathrm{Tr}_{-t-\tau}\,D_{n+1}(\mathcal{S}^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}, -e_{n+1}\cdot\mathrm{Tr}_s\Big[A\nabla+B_1\Big]D_{n+1}^2 u\Big)_{2,2}}.
\end{aligned}$$

Therefore, using the Cauchy-Schwartz inequality, we estimate that

$$\begin{aligned}
|IV| &\le 4\int_0^{R-\frac{\tau}{2}} t^2\int_{\mathbb{R}^n}\Big|\mathrm{Tr}_{-t-\tau}\,D_{n+1}\Big[A^*\nabla+\overline{B_2}\Big](\mathcal{S}^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}\Big|\Big|\mathrm{Tr}_t\,D_{n+1}^2 u\Big|\,dt \\
&\lesssim |||t^2\partial_t\nabla(S^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}|||_-|||t\partial_t^2 u||| \lesssim \|\mathbf{h}\|_2|||t\partial_t^2 u|||, \quad (4.6.10)
\end{aligned}$$

$$\begin{aligned}
|V| &\le 4\int_0^{R-\frac{\tau}{2}} t^2\int_{\mathbb{R}^n}\Big|\mathrm{Tr}_{-t-\tau}\,D_{n+1}(\mathcal{S}^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}\Big|\Big|\mathrm{Tr}_t\Big[A\nabla+B_1\Big]D_{n+1}^2 u\Big|\,dt \\
&\lesssim |||t\partial_t(S^{\mathcal{L}^*}\nabla)\overline{\mathbf{h}}|||_-|||t^2\partial_t^2\nabla u||| \lesssim \|\mathbf{h}\|_2|||t^2\partial_t^2\nabla u|||, \quad (4.6.11)
\end{aligned}$$

where we used the square function estimate (4.5.6) and the "travel-down" procedure (4.6.5). Now send $R \to \infty$, which sends $|I|, |II|, |III| \to 0$. By the bounds (4.6.10), (4.6.11), and Lemma 4.5.2, the desired bound for the gradient follows.

To obtain the bound for the $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ norm, we use Lemma 4.2.3 to ensure that at each horizontal slice, the $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ norm of a $Y^{1,2}(\mathbb{R}^{n+1}_+)$ solution is finite. Then we may apply the Sobolev embedding, whence the desired result follows. $\square$

The method of proof of Theorem 4.1.2 is robust, in the sense that we may loosen the condition that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$, provided that u is such that the square function in the right-hand side of (4.1.3) is finite, and that the gradient of u decays to 0 in the sense of distributions for large t. More precisely, we have

Theorem 4.6.12 (A more general $\mathrm{Tr} < S$ result)**.** *Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense, and $\nabla u(\cdot, t)$ converges to 0 in the sense of distributions as $t \to \infty$ (we refer to this last condition as the decaying condition). Furthermore, assume that $|||t\nabla D_{n+1}u||| < \infty$. Then, for every $\tau > 0$, the following statements are true.*

(i) If $\mathcal{L}1 \neq 0$ in $\mathbb{R}^{n+1}_+$, then

$$\|\operatorname{Tr}_\tau u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + \|\operatorname{Tr}_\tau \nabla u\|_{L^2(\mathbb{R}^n)} \lesssim \int_\tau^\infty \int_{\mathbb{R}^n} t|D^2_{n+1}u|^2\, dxdt \lesssim |||tD^2_{n+1}u|||. \quad (4.6.13)$$

(ii) If $\mathcal{L}1 = 0$ in $\mathbb{R}^{n+1}_+$, then there exists a constant $c \in \mathbb{C}$ such that $v := u - c$ (which is again a solution) satisfies estimate (4.6.13).

The proof of this theorem is omitted as it is very similar to the proof of Theorem 4.1.2 as soon as we have the following technical result.

Proposition 4.6.14 (Solutions with gradient decay)**.** *Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ is a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ and that $\mathcal{L}1 \neq 0$ on some box $I = Q \times (t_1, t_2) \subset \mathbb{R}^{n+1}_+$. Further, assume that $\sup_{t>0} \|\nabla u(t)\|_{L^2(\mathbb{R}^n)} < \infty$, and that $\lim_{t\to\infty} \|\nabla u(t)\|_{L^2(\mathbb{R}^n)} = 0$ (see (4.2.2)). Then $u(t) \in Y^{1,2}(\mathbb{R}^n)$ for every $t > 0$.*

Proof. **Step 1.** There exists a constant $c \in \mathbb{C}$ such that for all $t > 0$, $u(\cdot, t) - c \in Y^{1,2}(\mathbb{R}^n)$.

To see this, first note that by the Sobolev embedding, there exists a function $f : (0, +\infty) \to \mathbb{C}$ such that for each $t > 0$, $u(\cdot, t) - f(t) \in Y^{1,2}(\mathbb{R}^n)$. We must show that f is identically a constant. Since (see the proof of Theorem 1.78 in [MZ97]) for each $t > 0$ we have that $f(t) = \lim_{R\to\infty} \fint_{B(0,R)} u(\cdot, t)$, it can be shown by the Sobolev embedding and considering the difference quotient $\frac{u(\cdot,t+h)-u(\cdot,t)}{h}$ that f is differentiable and that $f'(t) \equiv 0$ for all $t > 0$. It follows that f is a constant, as desired.

Step 2. For the box $I \subset \mathbb{R}^{n+1}_+$ as in the hypotheses, it holds that

$$\iint_I |u^R|^{2^*} \to 0 \qquad \text{as } R \to \infty,$$

where $u^R(\cdot, \cdot) = u(\cdot, \cdot + R)$.

This is the crucial step. We set $p = 2^*$ and $u^R_I = |I|^{-1} \int_I u^R$ for ease of notation. By the Poincaré-Sobolev inequality, we see that

$$\|u^R - u^R_I\|_{L^p(I)} \lesssim \|\nabla u^R\|_{L^2(I)} \to 0, \quad \text{as } R \to \infty, \tag{4.6.15}$$

where we used the definition of u^R and the decaying condition of the gradient. In particular, we have that $u^R - u^R_I \to 0$ in $Y^{1,2}(I)$, so that $\mathcal{L}(u^R - u^R_I) \to 0$ in I, which implies that for every $\varphi \in C^\infty_c(I)$, the limit

$$-u^R_I \iint_I B_1 \cdot \overline{\nabla\varphi} = \iint_I \left[(A\nabla(u^R - u^R_I) + B_1(u^R - u^R_I)) \cdot \overline{\nabla\varphi} + B_2 \cdot \nabla(u^R - u^R_I)\overline{\varphi}\right] \to 0$$

holds. Since $\mathcal{L}1 \neq 0$ in I, for some $\varphi_0 \in C^\infty_c(I)$ we have that $\int_I B_1 \cdot \overline{\nabla\varphi_0} \neq 0$, whence $u^R_I \to 0$ as $R \to \infty$. The claim now follows by using this result in (4.6.15). Notice that this argument holds just as well for any box J containing I, in particular it holds for $\frac{3}{2}I$.

Step 3. For $Q \subset \mathbb{R}^n$, $t \in (t_1, t_2)$ as in the hypotheses, we have that

$$\int_Q |\operatorname{Tr}_t u^R|^p \to 0, \qquad \text{as } R \to \infty.$$

This is a consequence of Step 2 and the definition of the trace: For any $\phi \in C^\infty_c(Q)$ and $\eta \in C^\infty_c(t_1, t_2)$ with $\eta(s) = 1$ near t, we set $\Phi := \phi\eta \in C^\infty_c(I)$ and estimate

$$\begin{aligned} |(\operatorname{Tr}_t u^R, \phi)| &= \Big| \iint_{\mathbb{R}^{n+1}_+} (D_{n+1}u^R\Phi + u^R D_{n+1}\Phi) \Big| \\ &\leq \|D_{n+1}u^R\|_{Y^{1,2}(I)} \|\Phi\|_{L^{p'}(I)} + \|u^R\|_{Y^{1,2}(I)} \|D_{n+1}\Phi\|_{L^{p'}(I)} \end{aligned}$$

$$\lesssim_{\eta,\eta'} (\|D_{n+1}u^R\|_{Y^{1,2}(I)} + \|u^R\|_{Y^{1,2}(I)})\|\phi\|_{L^{p'}(I)}.$$

The claim now follows by the Caccioppoli inequality; to wit,

$$\|D_{n+1}u^R\|_{L^p(I)} + \|\nabla D_{n+1}u^R\|_{L^2(I)} \lesssim_{|I|} \sup_{s>t_2+R} \|\nabla u(s)\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{ as } R \to \infty,$$

using the fact that $p < \frac{2n}{n-2}$.

We now conclude the proof: By Step 1, we can place $\operatorname{Tr}_s(u-c) \in Y^{1,2}(\mathbb{R}^n)$ for all $s > 0$. By Sobolev's inequality and the hypotheses, $\|\operatorname{Tr}_s u - c\|_{Y^{1,2}(\mathbb{R}^n)} \to 0$ as $s \to \infty$. On the other hand, by Step 3, we have that $\operatorname{Tr}_s u \to 0$ in $L^p(Q)$, so that $c = 0$ and the desired result follows. $\square$

A quick application of Theorem 4.6.12 to the improvement of (4.6.4) will be useful for the Dirichlet problem:

Corollary 4.6.16 (Improvement to slice estimate)**.** *The estimate (4.6.4) holds true for $k = 1$. In particular, (4.6.5) holds true for $k = 1$ as well.*

We can also, very similarly, prove

Theorem 4.6.17 (L^2-sup on slices)**.** *Suppose that $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$, $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$, and that u converges to 0 in the sense of distributions. Furthermore, assume that $|||t\nabla u||| < \infty$. Then, for every $\tau > 0$,*

$$\|\operatorname{Tr}_\tau u\|_{L^2(\mathbb{R}^n)} \lesssim \int_\tau^\infty \int_{\mathbb{R}^n} t|\nabla u|^2\, dxdt \lesssim |||t\nabla u|||$$

where the implicit constant is independent of τ and u.

Chapter 5

Critical Perturbations for Second Order Elliptic Operators. Part II: Existence, Uniqueness, and bounds on the non-tangential maximal function

The research in these chapters was done in collaboration with S. Bortz, S. Hofmann, J. L. Luna García, and S. Mayboroda. The results and proofs in these chapters will also appear in the doctoral thesis of J. L. Luna García.

5.1 Introduction

This chapter continues the research begun in Chapter 4, and briefly described in Section 1.2.3, where we study the L^2 Dirichlet, Neumann and Regularity problems for critical perturbations of second-order divergence-form equations by lower order terms. Relevant literature review lies in Section 1.3.2. We borrow completely the notation and setting of Section 4.1 from Chapter 4, which is also summarized in Section 1.2.3 of Chapter 1. In particular, we consider the operators $\mathcal{L}$ given in (1.2.10), the assumptions on the coefficients are described in Section 4.1, and the statements of the boundary value problems are given in (1.2.13), (1.2.14), and (1.2.15).

Let us immediately remark that, under the hypotheses on the coefficients (which we borrow from Section 4.1), the potential term V may be absorbed into the drift terms B_1, B_2 by writing $V = -\operatorname{div}\nabla(-\Delta)^{-1/2}(-\Delta)^{-1/2}V$, and as a consequence, we will not explicitly mention the potential term in any of our estimates. A more detailed account of this reduction can be found in Lemma 4.2.17.

Chapter 4 (itself based on the paper [BHL+b]) established L^2 square function and "slice" estimates for layer potential operators. The following theorem summarizes the results from the previous chapter.

Theorem 5.1.1 ([BHL+b]). *Let*

$$\mathcal{L} := -\operatorname{div}(A\nabla + B_1) + B_2 \cdot \nabla + V$$

where A, B_1, B_2, V are as above. There exists $\tilde{\rho}_1 > 0$ depending on dimension and the ellipticity of A such that if

$$\max\left\{\|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{n/2}(\mathbb{R}^n)}\right\} < \tilde{\rho}_1,$$

then the following estimates hold for the single and double layer potentials.

(i)

$$\iint_{\mathbb{R}^{n+1}_+} \left|t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} f(x)\right|^2 \frac{dx\,dt}{t} \leq C_m \|f\|^2_{L^2(\mathbb{R}^n)}, \quad \textit{for each } m \in \mathbb{N},$$

(ii)

$$\sup_{\tau>0} \|\operatorname{Tr}_\tau \mathcal{S} f\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} + \sup_{\tau>0} \|\operatorname{Tr}_\tau \nabla \mathcal{S} f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

(iii)

$$\iint_{\mathbb{R}^{n+1}_+} \left|t^m \partial_t^{m-1} \nabla \mathcal{D}_t^{\mathcal{L}} f(x)\right|^2 \frac{dx\,dt}{t} \leq C_m \|f\|^2_{L^2(\mathbb{R}^n)} \quad \textit{for each } m \in \mathbb{N},$$

(iv)

$$\sup_{\tau>0} \|\operatorname{Tr}_\tau \mathcal{D} f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

Here, C depends on dimension and ellipticity, while C_m depends on m, dimension, and ellipticity.

Items (iii) and (iv) were not treated explicitly in Chapter 4. However, using the identity in Lemma 5.2.50, and the square function estimates for the single layer obtained in Theorem 4.1.1, estimate (iii) follows. Finally, (iv) then follows from (iii) and Theorem 4.6.17.

In fact, the analysis in Chapter 4 (primarily these estimates) along with the existence and uniqueness sections of this work are enough to prove the existence and uniqueness for solutions with square function estimates. On the other hand, we desire to have the more "natural" non-tangential maximal function estimates for the single and double layer potentials, under (essentially) the same hypothesis as in Theorem 5.1.1. This is where we place a significant amount of our effort in this chapter. We prove the following.

Theorem 5.1.2. *Let*

$$\mathcal{L} := -\operatorname{div}(A\nabla + B_1) + B_2 \cdot \nabla + V$$

where A, B_1, B_2, V are as in Section 4.1. There exists $\rho_1 \in (0, \tilde{\rho}_1)$ depending on dimension and the ellipticity of A such that if

$$\max\left\{\|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\} < \rho_1,$$

then the following estimates hold for the single and double layer potentials.

1. $\|\widetilde{N}(\nabla \mathcal{S} f)\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$,
2. $\|\widetilde{N}(\mathcal{D} f)\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$.

Here, the constant C depends only on dimension and ellipticity, and $\widetilde{N}$ is the modified non-tangential maximal function (see Definition 5.2.5 below).

The idea to proving Theorem 5.1.2 starts with proving a weak-L^p bound ($L^{p,\infty}$ bound) for the non-tangential maximal function in terms of the L^p norms of the vertical and conical square functions (see Lemma 5.6.2). Then, interpolation will show that Theorem 5.1.2 holds provided that the vertical and conical maximal functions are bounded in L^p for an open interval (in p) around 2. The starting point for obtaining such bounds for the square functions is to prove *general* bounds for operators with sufficient off-diagonal decay which satisfy a local reverse-Hölder inequality using the extrapolation theory from weighted norm inequalities [CMP11]. Arguments similar to ours have been used in [Pri19] to treat square function estimates for operators built out of the heat or Poisson semigroups associated to an elliptic operator; however, in our case we must grapple with the added difficulty of having very mild off-diagonal decay. On the other hand, the local energy inequality for the equation (the Caccioppoli inequality) allows us to obtain the necessary off-diagonal decay for related operators with added (transversal) derivatives. Having done so, our remaining task is to "remove" these additional derivatives, a process which we call "traveling down". Due to its definition, this process for the vertical square function is a relatively simple integration by parts computation. For the conical square function, the additional spatial average impedes the simple integration by parts and our argument for this object requires the boundedness of the non-tangential maximal function with the *same* number of derivatives. Luckily, our Lemma 5.6.2, when combined with Proposition 5.6.1, gives that the non-tangential maximal function bounds (for this family of operators) depend on square functions with *more*[1] derivatives. This allows us to employ a two-step induction scheme where one alternates between bounding the L^p norm for a non-tangential maximal function by the L^p norm of square functions (with more derivatives) and then bounding the L^p norm of the conical square function by the L^p norm of a non-tangential maximal function (with the same number of derivatives). Thus, in finitely many steps, we remove these additional derivatives. (Recall we can start this process, that is, obtain L^p bounds for the vertical and conical square functions, by introducing enough transversal derivatives.)

With the square function and non-tangential maximal function bounds for layer potentials in hand, we turn our attention to the solvability of the boundary value problems $(\mathrm{D})_2$, $(\mathrm{N})_2$, and $(\mathrm{R})_2$ stated in (1.2.13), (1.2.14), and (1.2.15) respectively. As a consequence of the Theorem 5.1.4 below and previously known results (see below), we have the following result (which is a slight variant of Theorem 1.2.16).

Theorem 5.1.3. *Let $\mathcal{L}_0$ be a divergence form operator of the form*

$$\mathcal{L}_0 := -\operatorname{div} A\nabla,$$

where A is either Hermitian, block form or constant, and suppose that B_1, B_2, V verify the assumptions set forth in Section 4.1. Then there exists $\rho_0 > 0$ depending only on

[1]Note that in Lemma 5.6.2, we may use that $\|\mathbb{V}(\Theta_{t,1}f)\|_{L^p(\mathbb{R}^n)} \lesssim_m \|\mathbb{V}(\Theta_{t,m+1}f)\|_{L^p(\mathbb{R}^n)}$, by the aforementioned integration by parts argument. The subscript m refers to the number of transversal derivatives.

dimension and ellipticity such that if

$$\mathcal{L}_1 = -\operatorname{div}((A+M)\nabla + B_1) + B_2 \cdot \nabla + V,$$

and

$$\max\left\{\|M\|_{L^\infty(\mathbb{R}^n)}, \|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\} < \rho_0$$

then $(\mathrm{D})_2, (\mathrm{N})_2, (\mathrm{R})_2$ *are uniquely*[2] *solvable*[3] *for the operator* $\mathcal{L}_1$.

Our most general theorem concerning these boundary value problems, is as follows.

Theorem 5.1.4. *Let* $\mathcal{L}$ *be an operator of the form (1.2.10), where* A, B_1, B_2, V *are as in Section 4.1, and*

$$\max\left\{\|B_1\|_{L^n(\mathbb{R}^n)},\ \|B_2\|_{L^n(\mathbb{R}^n)},\ \|V\|_{L^{n/2}(\mathbb{R}^n)}\right\} < \rho_1,$$

where ρ_1 *is as in Theorem 5.1.2. Suppose further that the associated boundary operators*

$$\mathcal{S}_0^{\mathcal{L}_0} : L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n), \pm\frac{1}{2}I + \tilde{K}^{\mathcal{L}_0} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$
$$\mp\frac{1}{2}I + K^{\mathcal{L}_0} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

are all invertible (see Section 5.8). Then the boundary value problems $(\mathrm{D})_2, (\mathrm{N})_2, (\mathrm{R})_2$ *are uniquely*[4] *solvable for the operator* $\mathcal{L}_0$.

Moreover, there exists $\rho = \rho(\mathcal{L}_0) > 0$ *such that if*

$$\mathcal{L}_1 = -\operatorname{div}(\widetilde{A}\nabla + \widetilde{B}_1) + \widetilde{B}_2 \cdot \nabla + \widetilde{V}$$

with $\widetilde{A}, \widetilde{B}_1, \widetilde{B}_2, \widetilde{V}$ *as in Section 4.1 and satisfying*

$$\max\left\{\|\widetilde{A}-A\|_{L^\infty(\mathbb{R}^n)}, \|\widetilde{B}_1 - B_1\|_{L^n(\mathbb{R}^n)},\ \|\widetilde{B}_2 - B_2\|_{L^n(\mathbb{R}^n)},\ \|\widetilde{V}-V\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\} \leq \rho,$$

then the associated boundary operators, $\mathcal{S}_0^{\mathcal{L}_0}, \pm\frac{1}{2}I + \tilde{K}^{\mathcal{L}_0}, \mp\frac{1}{2}I + K^{\mathcal{L}_0}$, *are invertible and the boundary value problems* $(\mathrm{D})_2, (\mathrm{N})_2, (\mathrm{R})_2$ *are uniquely solvable for the operator*

[2] See Remark 5.1.5.

[3] Solvability throughout this chapter means that we have accompanying L^2 bounds for the non-tangential maximal function and square function, in terms of the data.

[4] Again, see remark 5.1.5.

$\mathcal{L}_1$.

Here, the constant $\rho(\mathcal{L}_0)$ is chosen with two constraints. The first is to ensure that $\widetilde{A}$ has ellipticity constant less than twice that for A and

$$\max\left\{\|\widetilde{B}_1\|_{L^n(\mathbb{R}^n)},\ \|\widetilde{B}_2\|_{L^n(\mathbb{R}^n)},\ \|\widetilde{V}\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}\right\} < \rho_1'$$

where ρ_1' is as in Theorem 5.1.2 for matrices with ellipticity twice that of A. The second constraint depends on the operator norms of the inverses of $\mathcal{S}_0^{\mathcal{L}_0}$, $\pm\frac{1}{2}I+\tilde{K}^{\mathcal{L}_0}$, $\mp\frac{1}{2}I+K^{\mathcal{L}_0}$.

Remark 5.1.5. Uniqueness, under the background hypothesis of invertible layer potentials and sufficient smallness of the lower order terms, is established among what we call "good $\mathcal{D}$ solutions" (in the case of $(\mathrm{D})_2$) and "good $\mathcal{N}/\mathcal{R}$ solutions" (in the case of $(\mathrm{N})_2$ and $(\mathrm{R})_2$). We show that non-tangential maximal function estimates or square function estimates imply that solutions are "good". For instance, (under the aforementioned background hypothesis) suppose that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ solves

$$(\widetilde{\mathrm{D}}2)\begin{cases}\mathcal{L}u = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ \lim_{t\to 0} u(\cdot,t) = f & \text{strongly in } L^2(\mathbb{R}^n)\end{cases}$$

for some $f \in L^2$ and that u has *one* of the following properties:

- u is a good $\mathcal{D}$ solution,
- $\|Nu\|_{L^2(\mathbb{R}^n)} < \infty$ or
- $\iint_{\mathbb{R}^{n+1}_+} t|\nabla u(x,t)|^2\,dx\,dt < \infty$.

Then u is the unique such solution and has the other two properties. In the case of the Neumann problem our solutions are unique modulo constants if the operator $\mathcal{L}$ annihilates constants.

Let us mention a few applications of our theorems. For the magnetic Schrödinger operator $-(\nabla - i\mathbf{a})^2$ when $\mathbf{a} \in L^n(\mathbb{R}^n)^{n+1}$ is $t-$independent and has small $L^n(\mathbb{R}^n)$ norm, we have as a corollary to our Theorem 5.1.3 the first L^2 well-posedness results of the Dirichlet, Neumann and regularity problems in the setting of the half-space. The latter conclusion is novel also for the case of the Schrödinger operator $-\Delta + V$ where $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ is $t-$independent and has small $L^{\frac{n}{2}}$ norm.

The paper is organized as follows.

In Section 2 we review some of the basic definitions and results needed from Harmonic Analysis, such as square and non-tangential maximal functions, Littlewood-Paley families, extrapolation of A_p weights and weighted versions of classical results. We also introduce the notion of off-diagonal $L^r - L^q$ estimates for an operator and show how this relates to weighted bounds. Finally we introduce our main objects the Single and Double Layer Potentials and give some of their properties, as well as some general facts about solutions of our equation.

In Section 3 we develop the necessary extrapolation theorems for both conical and vertical square functions, in the presence of sufficient off-diagonal decay. We first prove these for general operators, then specialize to our objects of interest.

In Section 4 we mention the control on slices obtained from the square function estimates from the previous section.

In Section 5 we prove the non-tangential maximal function estimates for the gradient of the Single Layer and the Double Layer, with the background assumption of good square function bounds.

In section 6 we proceed to 'travel down' on both the square and non-tangential maximal functions, to dispense of the hypothesis of good off-diagonal estimates. For the vertical square function this turns out to be a integration by parts argument, however the conical square function will be tied-up with the non-tangential maximal function in an essential way; we will be forced to work with these two objects simultaneously.

In Section 7 we put all the estimates from the previous sections to use in proving L^2 solvability of the boundary value problems $(D)_2$, $(R)_2$ and $(N)_2$, with representation of solutions via Layer Potentials and with both square and non-tangential maximal function bounds on solutions.

In Section 8, we obtain the existence of solutions to our boundary value problems via the technique of layer potentials, while finally, in Section 9, we prove that the solutions attained in this way are unique.

5.2 Notation and Preliminaries

In this chapter, we heavily borrow the setup from Chapter 4. As such, we direct the reader to Section 4.2 for necessary preliminaries from the previous chapter. Here we recall some main concepts, as well as introduce some new ones.

- We always take $A = A(x)$ to be an $(n+1)\times(n+1)$ matrix of L^∞, $t-$independent complex coefficients satisfying the ellipticity condition (1.1.4), while $B_1, B_2 \in (L^n(\mathbb{R}^n))^{n+1}$ and $V \in L^{n/2}(\mathbb{R}^n)$ are complex-valued, $t-$independent (vector) functions verifying the control (1.2.11), with $\rho \ll 1$. Under these conditions, the term V can be "hidden" into first-order terms $\tilde{B}_1$, $\tilde{B}_2$ which still satisfy our assumptions (see Lemma 4.2.17); therefore, without loss of generality we will omit the zeroth order term V from consideration.
- For a cube $Q \subset \mathbb{R}^n$ we will always denote by R_Q the *Carleson region* above Q; that is, $R_Q := Q \times (0, \ell(Q))$.
- For $R > 0$ we define $I_R := (R, R)^{n+1} \subset \mathbb{R}^{n+1}$, and $I_R^\pm := I_R \cap \mathbb{R}^{n+1}_\pm$.
- We denote by $\mathcal{M}$ the (uncentered) Hardy-Littlewood maximal function in $\mathbb{R}^n$, and more generally for $r > 0$ we define $\mathcal{M}_r(f) := \mathcal{M}(|f|^r)^{1/r}$.
- Given a cube $Q \subset \mathbb{R}^n$ we denote by Q^* a concentric dilate of Q by a factor that depends only on dimension.
- We will often have use for the fractional integral of order 1 in $\mathbb{R}^n$, which we denote by I_1; that is, for nice enough f, we have

$$I_1 f(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\, dy.$$

- We denote by $\mathbb{D}$ the collection of all dyadic cubes in $\mathbb{R}^n$, and for $t > 0$ we define $\mathbb{D}_t$ to be the cubes in $\mathbb{D}$ which satisfy $\ell(Q) < t \leq 2\ell(Q)$. Similarly, for a cube $Q \subset \mathbb{R}^n$ we denote by $\mathbb{D}(Q)$ the collection of dyadic subcubes of Q.
- For $(x,t) \in \mathbb{R}^{n+1}_+$ we define the *Whitney regions*

$$\mathcal{C}_{x,t} := \big\{(y,s) \in \mathbb{R}^{n+1}_+ : |x-y| < t/8, |t-s| < t/8\big\}.$$

  Given $x_0 \in \mathbb{R}^n$, recall that we denote by $\gamma(x_0)$ the non-tangential cone with vertex x_0, given by

$$\gamma(x_0) := \{(x,t) : |x - x_0| < t\}. \tag{5.2.1}$$

- We call a measurable function $\nu : \mathbb{R}^n \to \mathbb{R}$ a *weight* if $\nu > 0$ Lebesgue-a.e. on $\mathbb{R}^n$ and $\nu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that ν is *doubling* if the measure $\nu(x)dx$ is doubling; that is, if (with a slight abuse of notation) $\nu(2Q) \leq C_0\nu(Q)$ for a constant $C_0 > 0$

and all cubes $Q \subset \mathbb{R}^n$.

- For $1 < p < m$, the upper and lower Sobolev exponents of order 1 in m dimensions are
$$p_m^* := \frac{mp}{m-p}, \qquad p_{*,m} := \frac{mp}{m+p},$$
respectively. Sometimes, we will drop m from the subscript when the dimension is clear from the context.

Definition 5.2.2 (Vertical and Conical Square Functions)**.** If $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$, we define the *conical square function* of F as

$$\mathcal{S}F(x) := \Big(\iint_{\gamma(x)} |F(y,t)|^2 \, \frac{dydt}{t^{n+1}} \Big)^{1/2},$$

where $\gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ is the vertical cone with apperture 1 and vertex x. Similarly, we define the *vertical square function* of F as

$$\mathbb{V}F(x) := \Big(\int_0^\infty |F(x,t)|^2 \, \frac{dt}{t} \Big)^{1/2}.$$

Remark 5.2.3. In the definition of $\mathcal{S}$, we could have chosen a different aperture; that is, for $\eta > 0$, we can set

$$\mathcal{S}_\eta F(x) = \Big(\iint_{|x-y|<\eta t} |F(y,t)|^2 \, \frac{dydt}{t^{n+1}} \Big)^{1/2}.$$

It is well-known that different apertures give rise to objects with equivalent L^p norms and even equivalent weighted L^p norms (see for instance [CMS85, Proposition 4] for the unweighted case and [CMP20, Proposition 4.9] for the weighted one).

In contrast with the L^2 case, when $p \neq 2$ the (L^p norms of) conical and vertical square functions are not equivalent.

Proposition 5.2.4 (Comparability of Square Functions [AHM12, Proposition 2.1])**.** *Let $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ be measurable.*

(i) If $0 < p \leq 2$ then $\|\mathbb{V}(F)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)}$.

(ii) If $2 \leq p < \infty$ then $\|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\mathbb{V}(F)\|_{L^p(\mathbb{R}^n)}$.

We will need to use a few different versions of the modified non-tangential maximal function (1.2.12).

Definition 5.2.5 (Non-tangential Maximal Functions)**.** For a function $F : \mathbb{R}^{n+1} \to \mathbb{C}$ and $q > 0$ we define $a_q(F) : \mathbb{R}^{n+1} \to \mathbb{C}$ as

$$a_q(F)(x,t) := \Big(\fint\!\!\!\fint_{\mathcal{C}_{x,t}} |F(y,s)|^q \, dyds\Big)^{1/q}.$$

We define the *non-tangential maximal function* and the *q-modified non-tangential maximal function* of F respectively as

$$\mathcal{N}(F)(x) := \sup_{\gamma(x)} |F|,$$

and

$$\tilde{\mathcal{N}}_q(F)(x) := \mathcal{N}(a_q(F))(x).$$

We also define the *lifted modified non-tangential maximal function*, for $\varepsilon > 0$, as

$$\tilde{\mathcal{N}}_q^{\varepsilon}(F)(x) := \sup_{\substack{|x-y|<t-\varepsilon \\ t>\varepsilon}} a_q(F)(y,t).$$

Similarly, we define a truncated version of the non-tangential maximal function as

$$\tilde{\mathcal{N}}_q^{(\varepsilon)}(F)(x) := \sup_{\substack{|x-y|<t \\ t>\varepsilon}} a_q(F)(y,t).$$

Given a measurable function g on $\mathbb{R}^n \times \{t = 0\}$, we say that $F \longrightarrow g$ *non-tangentially* if for almost every $x \in \mathbb{R}^n$, we have that

$$\lim_{\substack{Y \to x \\ Y \in \gamma(x)}} \tilde{F}(Y) = g(x), \tag{5.2.6}$$

where $\gamma(x)$ is the non-tangential cone defined in (5.2.1), and

$$\tilde{F}(z,t) := \fint\!\!\!\fint_{\mathcal{C}_{x,t}} F(y,s) \, dy \, ds.$$

We now prove a result on the boundary behavior of solutions, under the assumption that we have good control of a modified non-tangential maximal function.

Proposition 5.2.7 (A non-tangential convergence result)**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ be a solu-*

tion to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. Then u converges non-tangentially at every $x \in \mathbb{R}^n$ where $\tilde{\mathcal{N}}_1(\nabla u)(x) < \infty$, in the sense that for any such $x \in \mathbb{R}^n$, the limit in (5.2.6) exists and is finite.

Proof. We will follow the arguments in [KP93b, Theorem 3.1 (a)], with modifications due to the lack of pointwise estimates for u.

Let $x \in \mathbb{R}^n$ be such that $\tilde{\mathcal{N}}_1(\nabla u)(x) < \infty$. Following [KP93b], it is our goal to show that for $Y, Z \in \Gamma(x) \cap B(x, r)$ we have

$$|\tilde{u}(Y) - \tilde{u}(Z)| \leq Cr\tilde{\mathcal{N}}_1(\nabla u)(x), \tag{5.2.8}$$

from which we may easily establish (via the Cauchy criterion) that

$$\lim_{\substack{Y \to x \\ Y \in \Gamma(x)}} \tilde{u}(Y)$$

exists, and consequently define $g_(x)$ to be the limit when it does so. Write $Y = (y, t_1)$ and $Z = (z, t_2)$. Then, to establish (5.2.8), it is clearly enough to establish

$$\max\left\{|\tilde{u}(Y) - \tilde{u}(x, t_1)|\,,\; |\tilde{u}(Z) - \tilde{u}(x, t_2)|\right\} \leq Cr\tilde{\mathcal{N}}_1(\nabla u)(x), \tag{5.2.9}$$

and

$$|\tilde{u}(x, t_2) - \tilde{u}(x, t_1)| \leq Cr\tilde{\mathcal{N}}_1(\nabla u)(x). \tag{5.2.10}$$

To prove (5.2.9) and (5.2.10) we need the following fact.

Claim 5.2.11. For $X \in \mathbb{R}^{n+1}$ and $r > 0$ let $I(X, r) := \{W \in \mathbb{R}^{n+1} : |X - W| < r\}$ be the open cube with center X and side length $2r$. Let $I_i = I(X_i, r_i)$, $i = 1, 2$, $\Omega \subset \mathbb{R}^{n+1}$ open with $I_i \subset \Omega$ $i = 1, 2$ and $\varphi \in W^{1,2}(\Omega)$. If $\alpha \in [0, 2)$ and $|X_1 - X_2| \leq \alpha \min\{r_1, r_2\}$ then

$$\left|\fint_{I_1} \varphi\, dx\, dt - \fint_{I_2} \varphi\, dx\, dt\right| \leq C_\alpha \max\{r_1, r_2\}\left(\max\left\{\tfrac{r_1}{r_2}, \tfrac{r_2}{r_1}\right\}\right)^{\frac{n+1}{2}} \max_{i=1,2}\left(\fint_{I_i} |\nabla\varphi|^2\right)^{1/2},$$

where $C_\alpha = C(n, \alpha)$. In particular, if $r_1 \approx r_2 \approx r$ then

$$\left|\fint_{I_1} \varphi\, dx\, dt - \fint_{I_2} \varphi\, dx\, dt\right| \leq C_\alpha r \max_{i=1,2}\left(\fint_{I_i} |\nabla\varphi|^2\right)^{1/2}, \tag{5.2.12}$$

where C_α depends on the implicit constants in the expression $r_1 \approx r_2 \approx r$.

Proof of Claim 5.2.11. Let $X_3 = \frac{X_1+X_2}{2}$, then

$$|X_3 - X_i| < \frac{\alpha r_i}{2}, \quad i = 1, 2,$$

and hence $I_3 = I(x_3, r) \subset I_i$, $i = 1, 2$ for $r = (1 - \alpha/2)\min\{r_1, r_2\}$. It follows from the triangle inequality and the Poincaré inequality that

$$\begin{aligned}
\Big| \fint_{I_1} \varphi - \fint_{I_2} \varphi \Big| &\le \Big| \fint_{I_1} \varphi - \fint_{I_3} \varphi \Big| + \Big| \fint_{I_1} \varphi - \fint_{I_3} \varphi \Big| \\
&\le 2 \max_{i=1,2} \Big(\fint_{I_3} \Big| \varphi - \fint_{I_i} \varphi \Big|^2 \Big)^{1/2} \\
&\le C_\alpha \big(\max\big\{ \tfrac{r_1}{r_2}, \tfrac{r_2}{r_1} \big\} \big)^{\frac{n+1}{2}} \max_{i=1,2} \Big(\fint_{I_i} \Big| \varphi - \fint_{I_i} \varphi \Big|^2 \Big)^{1/2} \\
&\le C_\alpha \max\{r_1, r_2\} \big(\max\big\{ \tfrac{r_1}{r_2}, \tfrac{r_2}{r_1} \big\} \big)^{\frac{n+1}{2}} \max_{i=1,2} \Big(\fint_{I_i} |\nabla \varphi|^2 \Big)^{1/2},
\end{aligned}$$

as desired. $\square$

Now let us prove (5.2.9) for the term with Y (the proof for the term with Z is identical). Note that $|x - y| \le t_1 < r$ since $Y \in \Gamma(x) \cap B(x, r)$. Let $I_2 = I(z, t_1/2)$, for $z = (x+y)/2$, then $|z - x| = |z - y| \le t_1/2$. This allows us to apply (5.2.12) with $\varphi = u$, $I_1 = I(x, t_1/2)$ and then $I_1 = I(y, t_1/2)$ and use the triangle inequality to obtain (5.2.9).

We turn our attention to (5.2.10) and we assume, without loss of generality, that $t_1 \le t_2$. Let $a = 2/3$, $s_k = a^k t_2$ for $k = 0, 1, \ldots, K$, where $K = \max\{k : a^k t_2 \ge t_1\}$. Notice that for $k = 0, \ldots K - 1$,

$$|s_k - s_{k+1}| = \frac{s_{k+1}}{2} = \min\Big\{ \frac{s_k}{2}, \frac{s_{k+1}}{2} \Big\}.$$

Defining $s_{K+1} = t_1$, we see that the choice of K guarantees the estimate

$$|s_K - s_{K+1}| \le \min\Big\{ \frac{s_K}{2}, \frac{t_1}{2} \Big\}.$$

Set $I_k := I((x, s_k), \frac{s_k}{2})$, $k = 0, \ldots K+1$, then the previous two inequalities allow us to apply (5.2.12) with $\varphi = u$ and the consecutive cubes I_k and I_{k+1}, $k = 0, 1, \ldots, K$ (in place of I_1 and I_2 therein). One then obtains

$$\begin{aligned}|\tilde{u}(x,t_1)-\tilde{u}(x,t_2)| &\leq |\tilde{u}(x,s_{K+1})-\tilde{u}(x,s_K)|+\sum_{k=0}^{K-1}|\tilde{u}(x,s_k)-\tilde{u}(x,s_{k+1})| \\ &\lesssim t_1\tilde{\mathcal{N}}_1(\nabla u)(x)+t_2\sum_{k=0}^{K-1}a^k\tilde{\mathcal{N}}_1(\nabla u)(x) \\ &\lesssim r\tilde{\mathcal{N}}_1(\nabla u)(x),\end{aligned}$$

as desired (since $\sum_{k\geq 0}a^k=3$). □

Definition 5.2.13 (Carleson measures on the half-space)**.** A non-negative measure μ on $\mathbb{R}^{n+1}_+$ is a *Carleson measure* if

$$\|\mu\|_{\mathcal{C}}:=\sup_Q\frac{\mu(R_Q)}{|Q|}<\infty,$$

where the supremum is taken over cubes $Q\subset\mathbb{R}^n$.

Lemma 5.2.14 (John-Nirenberg Lemma for Carleson Measures)**.** *Let μ be a non-negative measure on $\mathbb{R}^{n+1}_+$. Suppose that there exist $\eta\in(0,1)$ and $C_0>0$ such that for every cube $Q\subset\mathbb{R}^n$, there exists a disjoint collection of cubes $(Q_j)_{j\in\mathbb{N}}\subset\mathbb{D}(Q)$ verifying $\sum_{j\geq 1}|Q_j|<\eta|Q|$ and $\mu(R_Q\setminus(\cup_j R_{Q_j}))\leq C_0|Q|$. Then μ is a Carleson measure.*

Remark 5.2.15. We may replace the Lebesgue measure on $\mathbb{R}^n$ by any other Radon measure, the proof is identical. If we assume that the hypotheses only hold for dyadic cubes, then we require the measure to be doubling.

Lemma 5.2.16 (John-Nirenberg Lemma for local Square Functions)**.** *Suppose that $F:\mathbb{R}^{n+1}_+\to\mathbb{R}$, $F\geq 0$ and define the local square function $A_{Q,F}:\mathbb{R}^n\to\mathbb{R}$ by*

$$A_{Q,F}:=\Big(\iint_{|x-y|<t<\ell(Q)}|F(y,t)|^2\,\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}.$$

If there exists $C_0>0$ with the property that for every cube $Q\subset\mathbb{R}^n$, the estimate

$$\int_Q A^2_{Q,F}\,dx\leq C_0|Q|$$

holds, then for every $p>1$, there exists a constant C_1 depending on p,n and C_0 such *that for every cube Q, we have that*

$$\int_Q A_{Q,F}^p \, dx \leq C_1 |Q|.$$

Proof. By Jensen's inequality, the result is trivially true, with $C_1 \leq C_0^{p/2}$, when $p \leq 2$. Therefore we concentrate on the case $p > 2$. For ease of notation, we will write $A_Q = A_{Q,F}$. Moreover, for $\alpha > 0$ we define

$$A_{Q,\alpha}(x) := \Big(\int_0^{\ell(Q)} \int_{|x-y|<\alpha t} |F(y,t)|^2 \, \frac{dydt}{t^{n+1}} \Big)^{1/2}.$$

When $\alpha = 1$, we may omit the subscript α. We also set $K_p := \sup_{Q \subset \mathbb{R}^n} \fint_Q A_Q^p$. Note first that $K_{\alpha,p} \approx_{\alpha,p} K_{1,p} =: K_p$. We defer the proof of this fact to the end, and proceed with the proof of the lemma.

Let us momentarily assume that $K_p < \infty$ *a priori*, and set $\alpha > 0$ and $N \gg 1$, both to be specified later. Consider the open set $\Omega_N := \{x \in Q : A_{Q,\alpha}(x) > N\}$. By the Chebyshev inequality and our assumptions, we see that $|\Omega_N| \lesssim_\alpha C_0 N^{-2} |Q|$. In particular, given $\alpha > 0$, we may chose $N \gtrsim_\alpha \sqrt{C_0}$ so that $\Omega_N \subsetneq Q$. Observe that

$$\int_Q A_Q^p = \int_{\Omega_N} A_Q^p + \int_{Q \setminus \Omega_N} A_Q^p =: I + II.$$

By definition of Ω_N, we have that $II \leq N^p |Q \setminus \Omega_N|$. On the other hand, if $(Q_j)_j$ is a Whitney decomposition of Ω_N, we can write (exploiting the convexity of $s \mapsto s^{p/2}$)

$$I \lesssim_p \sum_{j \geq 1} \int_{Q_j} A_{Q_j}(x)^p \, dx + \sum_{j \geq 1} \int_{Q_j} (A_Q(x)^2 - A_{Q_j}(x)^2)^{p/2} \, dx.$$

For the first term, we easily have that

$$\sum_{j \geq 1} \int_{Q_j} A_{Q_j}(x)^p \, dx \leq \sum_{j \geq 1} K_p |Q_j| = K_p |\Omega_N|.$$

For the second term, by definition of A_Q and A_{Q_j} we see that

$$A_Q(x)^2 - A_{Q_j}(x)^2 = \int_{\ell(Q_j)}^{\ell(Q)} \int_{|x-y|<t} |F(y,t)|^2 \, \frac{dydt}{t^{n+1}}.$$

If $x \in Q_j$, then, by properties of a Whitney decomposition, there exists $x_* \in Q\backslash\Omega_N$ (recall $Q\backslash\Omega_N \neq \emptyset$) such that $|x - x_*| \approx \ell(Q_j)$ with implicit constants depending only on n. In particular, for some $\alpha = \alpha(n) > 0$, we have the inclusion

$$\begin{aligned}\big\{(y,t) \in \mathbb{R}^{n+1}_+ : \, |x-y| < t, \quad \ell(Q_j) < t < \ell(Q)\big\}& \\ \subseteq \big\{(y,t) \in \mathbb{R}^{n+1}_+ : \, |x_*-y| < \alpha t, \quad 0 < t < \ell(Q)\big\}&,\end{aligned}$$

so that $A_Q(x)^2 - A_{Q_j}(x)^2 \leq A_{Q,\alpha}(x_*)^2 \leq N^2$, since $x_* \in Q\backslash\Omega_N$. Accordingly,

$$\sum_{j\geq 1} \int_{Q_j} (A_Q(x)^2 - A_{Q_j}(x)^2)^{p/2} \, dx \lesssim_\alpha N^p |\Omega_N|.$$

Combining these previous estimates, we obtain that $I \lesssim_{p,n} K_p|\Omega_N| + N^p|\Omega_N|$, and so

$$\int_Q A_Q(x)^p \, dx \lesssim_{p,n} K_p|\Omega_N| + N^p|Q| \leq C_0 K_p N^{-2}|Q| + N^p|Q|.$$

Dividing by $|Q|$ and taking supremum over all cubes gives $K_p \lesssim_{p,n} C_0 K_p N^{-2} + N^p$. Choosing $N = M\sqrt{C_0}$ with $M \geq 1$ large enough depending only on p and n, we may hide the first term to the left-hand side, and thus obtain $K_p \lesssim_{p,n} C_0^{p/2}$.

Finally, to do away with the restriction that $K_p < \infty$ a priori, we fix $\eta > 0$ and work with $F_\eta := F\mathbb{1}_{\eta<|F|<1/\eta}\mathbb{1}_{\eta<t<1/\eta}$, for which $K_p < \infty$, and appeal to the monotone convergence theorem in the limit $\eta \to 0^+$.

We now turn to the proof of $K_{\alpha,p} \approx K_p$. Notice that we only used this in the case $p = 2$, so we will only prove this special case. We will also work only with $\alpha > 1$.

By Fubini's theorem, if $\omega_n := |B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^n$,

$$\begin{aligned}\int_Q A_{Q,\alpha}(x)^2 \, dx &= \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbb{1}_Q(x)\mathbb{1}_{|x-y|<\alpha t}|F(y,t)|^2 \, dx dy \frac{dt}{t^{n+1}} \\ &= \alpha^n \omega_n \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \Big(⨍_{|x-y|<\alpha t} \mathbb{1}_Q(x) \, dx\Big)|F(y,t)|^2 \, dy \frac{dt}{t}.\end{aligned}$$

We claim that[5], for some dimensional constants c, c' and every $\beta > 1$,

$$c\beta^{-n}\mathbb{1}_Q(y) \leq \fint_{|x-y|<\beta t} \mathbb{1}_Q(x)\,dx \leq \mathbb{1}_{c'\beta Q}, \qquad \text{whenever } 0 < t < \ell(Q).$$

This claim follows immediately by noting that

$$\fint_{|x-y|<\beta t} \mathbb{1}_Q(x)\,dx = \frac{|Q \cap B(y, \beta t)|}{|B(y, \beta t)|}.$$

Using the second inequality with $\beta = \alpha$ and the first with $\beta = 1$ and $c'\alpha Q$ in place of Q, we arrive at

$$\begin{aligned}
&\int_Q A_{Q,\alpha}(x)^2\,dx \\
&\qquad \lesssim_{\alpha,n} \int_0^{\ell(Q)} \int_{c'\alpha Q} |F(y,t)|^2\,dy\frac{dt}{t} = \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \mathbb{1}_{c'\alpha Q}(y)|F(y,t)|^2\,dy\frac{dt}{t} \\
&\qquad \lesssim_n \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \Big(\fint_{|x-y|<t} \mathbb{1}_{c'\alpha Q}(x)\,dx\Big)|F(y,t)|^2\,dy\frac{dt}{t} \lesssim_{n,\alpha} \int_{c'\alpha Q} A_Q(x)^2\,dx \\
&\qquad\qquad \leq \int_{c'\alpha Q} A_{c'\alpha Q}(x)^2\,dx \lesssim_{n,\alpha} K_p|Q|.
\end{aligned}$$

The result now follows from taking the supremum over all cubes. $\square$

5.2.1 Weights and Extrapolation

Definition 5.2.17 (A_p **weights in** $\mathbb{R}^n$**).** Let $1 < p < \infty$. A weight $\nu \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be an A_p *weight* if there exists a constant $C \geq 1$ such that for every cube $Q \subset \mathbb{R}^n$, the estimate

$$\Big(\fint_Q \nu\Big)\Big(\fint_Q \nu^{-p'/p}\Big)^{\frac{p}{p'}} \leq C$$

holds. The infimum over all these constants is denoted $[\nu]_{A_p}$; we refer to it as the A_p *characteristic* of ν. We say that $\nu \in A_1$ if $(\mathcal{M}\nu)(x) \leq C\nu(x)$ for a.e. $x \in \mathbb{R}^n$. The infimum over such C is denoted by $[\nu]_{A_1}$.

Closely related to A_p weights are the reverse Hölder classes.

[5] We remind the reader that the notation CQ means the concentric dilate of Q by a factor $C > 0$.

Definition 5.2.18 (Reverse Hölder class in $\mathbb{R}^n$)**.** Let $1 < s < \infty$. A weight $\nu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to satisfy a *reverse Hölder inequality with exponent* s, written $\nu \in RH_s$, if there exists a constant $C \geq 1$ such that for every cube $Q \subset \mathbb{R}^n$, we have that

$$\left(\fint_Q \nu^s\right)^{1/s} \leq C \fint_Q \nu.$$

Let us summarize most of the "basic" facts about A_p weights which we will need.

Proposition 5.2.19 ([GR85b, Theorem 1.14, Lemma 2.2, Lemma 2.5, Theorem 2.6])**.** *Let $1 \leq p < q < \infty$. The following statements hold.*

(i) ([GR85b, Ch. IV Theorem 1.14 (a)]) $A_p \subset A_q$.

(ii) A weight ν belongs to A_2 if and only if $\nu^{-1} \in A_2$.

(iii) ([GR85b, Ch. IV Theorem 1.14 (b)]) *If $\nu \in A_p$ then $\nu^\delta \in A_p$ for any* $0 < \delta < 1$.

(iv) ([GR85b, Ch. IV Lemma 2.2]) *If $\nu \in A_p$ then νdx is a doubling measure, and the doubling constant depends on ν only through $[\nu]_{A_p}$ (and p).*

(v) ([GR85b, Ch. IV Lemma 2.5]) *If $\nu \in A_p$ then $\nu \in RH_s$ for some s that depends on the weight only through $[\nu]_{A_p}$ (and p).*

(vi) ([GR85b, Ch. IV Theorem 2.6]) *If $\nu \in A_q$ then $\nu \in A_{q-\varepsilon}$ for some ε depending on ν only through $[\nu]_{A_q}$ (and q).*

(vii) If $\nu \in A_q$ and $s > 1$, then $\nu \in RH_s$ if and only if $\nu^s \in A_{s(q-1)+1}$.

(viii) (Coifman-Rochberg [CR80, Proposition 2], [GR85b, Ch. II Theorem 3.4]) *If $f : \mathbb{R}^n \to \mathbb{C}$ is such that $(\mathcal{M}f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, then for every $0 < \delta < 1$ we have that $\nu_\delta := (\mathcal{M}f)^\delta \in A_1$ and moreover $[\nu_\delta]_{A_1} \leq C_\delta$ depends only on δ.*

(ix) (Muckenhoupt's Theorem [Muc72, Theorem 2], [GR85b, Ch. IV Theorem 2.8]) *For any $1 < p < \infty$, $\nu \in A_p$ and $f \in L^p(\nu)$, $\|\mathcal{M}f\|_{L^p(\nu)} \lesssim_{[\nu]_{A_p}} \|f\|_{L^p(\nu)}$.*

(x) (Coifman-Fefferman [CF74, Theorem III]) *Let T be a "regular" singular integral, as defined in* [CF74]*, and T^* the associated maximal operator. Then, for every $\nu \in A_\infty$ and $f \in C_c^\infty(\mathbb{R}^n)$, we have that $\|T^*\|_{L^p(\nu)} \lesssim_{p,n} \|\mathcal{M}f\|_{L^p(\nu)}$. In particular, by Muckenhoupt's Theorem above, we have that $\|T^* f\|_{L^p(\nu)} \lesssim_{[\nu]_{A_p}} \|f\|_{L^p(\nu)}$.*

The following result was originally proved by Rubio de Francia in [Rub83, Rub84]. We refer to [CMP11, Theorem 1.1] for a simple proof of this fact.

Theorem 5.2.20 (Extrapolation Theorem for A_p weights)**.** *Let $1 < p_0 < \infty$ and let T be an operator satisfying $\|Tf\|_{L^{p_0}(\nu)} \lesssim_{[\nu]_{A_{p_0}}} \|f\|_{L^{p_0}(\nu)}$, for all $\nu \in A_{p_0}$ and all $f \in L^{p_0}(\nu)$.*

Then, for every $p \in (1,\infty)$, $\nu \in A_p$, and $f \in L^p(\nu)$, we have $\|Tf\|_{L^p(\nu)} \lesssim_{[\nu]_{A_p}} \|f\|_{L^p(\nu)}$.

It is important for applications to note that the above theorem does not require any special structure on T; it does not need to be linear or sublinear. In fact, we have

Theorem 5.2.21 (Extrapolation Theorem for A_p Weights Version 2 [CMP11, Theorem 3.9])**.** *Fix $p_0 \in (1,\infty)$ and $\mathcal{F}$ a collection of pairs of non-negative measurable functions (f,g). Suppose that $\|f\|_{L^{p_0}(\nu)} \lesssim_{[\nu]_{A_{p_0}}} \|g\|_{L^{p_0}(\nu)}$ for all $\nu \in A_{p_0}$ and all $(f,g) \in \mathcal{F}$. Then for every $p \in (1,\infty)$, $\nu \in A_p$, and $(f,g) \in \mathcal{F}$, we have $\|f\|_{L^p(\nu)} \lesssim_{[\nu]_{A_p}} \|g\|_{L^p(\nu)}$.*

In practice, the collection $\mathcal{F}$ often takes the form $(|S_1 h|, |S_2 h|)$ for some operators S_i and h in some nice class of functions. A corollary of the previous theorem and this observation is the following.

Corollary 5.2.22 ([CMP11, Corollary 3.14])**.** *Let $r \in (1,2)$, and suppose that T is an operator verifying $\|Tf\|_{L^2(\nu)} \lesssim_{[\nu]_{A_{2/r}}} \|f\|_{L^2(\nu)}$, for each $f \in C_c^\infty(\mathbb{R}^n)$ and all $\nu \in A_{2/r}$. Then $\|Tf\|_{L^q(\mathbb{R}^n)} \lesssim_q \|f\|_{L^q(\mathbb{R}^n)}$ for all $q > r$.*

To prove the corollary, one defines $S_1 f := |Tf|^r$, $S_2 f := |f|^r$. Then, by hypothesis, $\|S_1 f\|_{L^{2/r}(\nu)} \lesssim_{[\nu]_{A_{2/r}}} \|S_2 f\|_{L^{2/r}(\nu)}$, and hence by the previous theorem, $\|S_1 f\|_{L^p(\nu)} \lesssim_{[\nu]_{A_p}} \|S_2 f\|_{L^p(\nu)}$ for $p \in (1,\infty)$. Setting $\nu \equiv 1$ and $p = q/r$ gives the desired result.

Theorem 5.2.23 (Weighted Littlewood-Paley Theorem)**.** *Let $(\mathcal{Q}_s)_s$ be a CLP family (see Definition 4.2.26) and let $\nu \in A_2$. It holds that*

$$\int_{\mathbb{R}^n} \int_0^\infty |(\mathcal{Q}_t f)(x)|^2 \, \frac{dt}{t} \nu(x)\, dx \lesssim_{n,[\nu]_{A_2}} \int_{\mathbb{R}^n} |f(x)|^2 \nu(x)\, dx.$$

Remark 5.2.24. By Theorem 5.2.20, we obtain that the vertical square function associated to $(\mathcal{Q}_s)_s$ is bounded on $L^p(\nu)$ for every $\nu \in A_p$ and $1 < p < \infty$; that is, $\|\mathbb{V}(\mathcal{Q}_s f)\|_{L^p(\nu)} \lesssim \|f\|_{L^p(\nu)}$ for every $\nu \in A_p$.

Proof of Theorem 5.2.23. The idea is to use Rubio de Francia and Duoandikoetxea's method in [DR86, Theorem B], to interpolate a "good" bound with a plain uniform bound in order to obtain another "good" bound in between. We will combine this with interpolation with change of measures as in [SW58, Theorem 2.11], exploiting the self-improvement property of A_p weights. Since this idea will be used quite often throughout the chapter we write out this portion of the the argument in full here, and refer back to it when applicable.

We first claim that it is enough to prove the following estimate:

$$\int_{\mathbb{R}^n} |\mathcal{Q}_s \tilde{\mathcal{Q}}_t^2 f|^2 \nu \lesssim_{[\nu]_{A_2}} \min\left(\frac{t}{s}, \frac{s}{t}\right)^\alpha \int_{\mathbb{R}^n} |\tilde{\mathcal{Q}}_t f|^2 \nu, \qquad \text{for each } s, t > 0, \tag{5.2.25}$$

for some $\alpha > 0$ and some CLP family $(\tilde{\mathcal{Q}}_t)_t$. Indeed, once this is shown, the desired result follows from a familiar quasi-orthogonality argument (see for instance the proof of [Gra14, Theorem 4.6.3]).

To prove (5.2.25), we first claim that it is enough to prove the following estimates.

(i) (Unweighted quasi-orthogonality) There exists $\beta > 0$ such that for any $s, t > 0$, we have the estimate

$$\int_{\mathbb{R}^n} |\mathcal{Q}_s \tilde{\mathcal{Q}}_t^2 f|^2 \leq C_1 \left(\frac{s}{t}, \frac{t}{s}\right)^\beta \int_{\mathbb{R}^n} |\tilde{\mathcal{Q}}_t f|^2.$$

(ii) (Uniform weighted estimate) For any $s > 0$ and $\nu \in A_2$, we have the estimate

$$\int_{\mathbb{R}^n} |\mathcal{Q}_s \tilde{\mathcal{Q}}_t^2 f|^2 \nu \leq C_2([\nu]_{A_2}) \int_{\mathbb{R}^n} |\tilde{\mathcal{Q}}_t f|^2 \nu.$$

Assume that these hold for the moment and fix $\nu \in A_2$. By properties of A_2 weights, there exist $\delta, C > 0$ such that $\nu^{1+\delta} \in A_2$ with $[\nu^{1+\delta}]_{A_2} \leq C$. In particular, the uniform weighted estimate holds with $\nu^{1+\delta}$ in place of ν, with the implicit constants depending only on $[\nu]_{A_2}$. Therefore, if we define the measures $d\mu_\tau := \nu^{(1+\delta)\tau}\, dx$, interpolation with change of measure (see [SW58, Theorem 2.11][6]) gives

$$\int_{\mathbb{R}^n} |\mathcal{Q}_s \tilde{\mathcal{Q}}_t^2 f|^2 \, d\mu_\tau \leq C_1^{1-\tau} C_2^\tau \left(\frac{s}{t}, \frac{t}{s}\right)^{\beta(1-\tau)} \int_{\mathbb{R}^n} |\tilde{\mathcal{Q}}_t f|^2 d\mu_\tau.$$

The desired estimate (5.2.25) is exactly the case $\tau = 1/(1+\delta)$ with $\alpha = \beta\delta/(1+\delta)$. This completes the proof, modulo the above pair of estimates.

The first estimate, the unweighted quasi-orthogonality, is a consequence of classical

[6] Strictly speaking, the statement of [SW58, Theorem 2.11] explicitly excludes the case under consideration (indeed the proof given does not apply in this case); however as is mentioned immediately after the statement of said Theorem, we may run an argument similar to the standard proof of the Riesz-Thorin Theorem, employing instead the three line lemma for sub-harmonic functions as in [CZ56].

Littlewood-Paley theory. On the other hand, the weighted estimate follows from both the fact that $|\mathcal{Q}_s f|, |\tilde{\mathcal{Q}}_t f| \lesssim \mathcal{M} f$ pointwise in $\mathbb{R}^n$ and Muckenhoupt's theorem on the $L^2(\nu)$ boundedness of $\mathcal{M}$ for $\nu \in A_2$ (see Proposition 5.2.19). $\square$

Lemma 5.2.26 (L^p inequalities from weighted L^2 bounds)**.** *Suppose that $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded (not necessarily linear) operator; that is, $\|Tf\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$.*

(i) Suppose that there exists $M > 1$ such that for all $\nu \in A_1$ with the property that $\nu^M \in A_1$ it holds that $\|Tf\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_1}} \|f\|_{L^2(\nu)}$, for every $f \in C_c^\infty(\mathbb{R}^n)$. Then for every $p \in (2, 2 + 1/M)$, it holds that $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$.

(ii) Suppose that there exists $M > 1$ such that for all ν with the property that $\nu^{-M} \in A_1$ it holds that $\|Tf\|_{L^2(\nu)} \lesssim_{[\nu^{-M}]_{A_1}} \|f\|_{L^2(\nu)}$ for every $f \in C_c^\infty(\mathbb{R}^n)$. Then for every $p \in (2 - 1/M, 2)$, we have that $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$.

(iii) Suppose that there exists $M > 1$ such that for all $\nu \in A_2$ with the property that $\nu^M \in A_2$, it holds that $\|Tf\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} \|f\|_{L^2(\nu)}$ for every $f \in C_c^\infty(\mathbb{R}^n)$. Then for every $p \in (2 - 1/M, 2 + 1/M)$, we have $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$.

Proof. This lemma and its proof are contained in [CMP11, Corollary 3.37] for the much more general setting of restricted extrapolation of A_p weights. However, since we will later on need to modify the arguments used in the proof a little to fit our needs, it seems appropriate to write the proof down for future reference. The key fact that we will use is the Coifman-Rochberg theorem (see Proposition 5.2.19).

We start with (i). Fix $p > 2$ such that $M < 1/(p-2)$ and $f \in C_c^\infty(\mathbb{R}^n)$. Note that $\nu := (\mathcal{M}(|Tf|))^{p-2} \in A_1$, and estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf|^p &\leq \int_{\mathbb{R}^n} |Tf|^2 (\mathcal{M}(|Tf|))^{p-2} \lesssim_{[\nu^M]_{A_1}} \int_{\mathbb{R}^n} |f|^2 (\mathcal{M}(|Tf|))^{p-2} \\ &\leq \Big(\int_{\mathbb{R}^n} |f|^p\Big)^{2/p} \Big(\int_{\mathbb{R}^n} (\mathcal{M}(|Tf|))^p\Big)^{(p-2)/p} \lesssim_p \Big(\int_{\mathbb{R}^n} |f|^p\Big)^{2/p} \Big(\int_{\mathbb{R}^n} |Tf|^p\Big)^{1-\frac{2}{p}}.\end{aligned}$$

If we first assume that $\|Tf\|_{L^p(\mathbb{R}^n)} < \infty$, then the result follows. To get rid of this assumption, we instead consider the sequence of operators $S_k f(x) := (Tf)(x) \mathbb{1}_{|Tf| \leq k}(x)$ on $L^2(\mathbb{R}^n)$. Then $\{S_k\}_k$ is uniformly bounded on $L^2(\mathbb{R}^n)$, and they satisfy the same hypotheses as T with constants independent of k. Then, for $f \in C_c^\infty(\mathbb{R}^n)$, we have that $S_k f \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and so by our argument above, $\|S_k f\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$. We now let $k \to \infty$ and use the Monotone Convergence Theorem.

We turn to (ii). Fix $p < 2$ with $\frac{1}{2-p} < M$ and $f \in C_c^\infty(\mathbb{R}^n)$ not identically 0. Note that $\nu := (\mathcal{M}(|Tf| + |f|))^{p-2}$ satisfies $\nu^{-1} \in A_1 \subset A_2$, and hence $\nu \in A_2$. We estimate

$$\int_{\mathbb{R}^n} |Tf|^p \leq \int_{\mathbb{R}^n} (\mathcal{M}(|Tf| + |f|))^p = \int_{\mathbb{R}^n} (\mathcal{M}(|Tf| + |f|))^2 (\mathcal{M}(|Tf| + |f|))^{p-2}$$
$$\lesssim_{[\nu]_{A_2}} \int_{\mathbb{R}^n} (|Tf|^2 + |f|^2)\nu \lesssim_{[\nu^{-M}]_{A_1}} \int_{\mathbb{R}^n} |f|^2 \nu \leq \int_{\mathbb{R}^n} |f|^2 (\mathcal{M}f)^{p-2} \leq \int_{\mathbb{R}^n} (\mathcal{M}f)^p,$$

where we have used Muckenhoupt's theorem, yielding the desired result.

The third statement follows from the first two and Jones's factorization theorem of A_2 weights (see [Jon80]) as quotients of A_1 weights. $\square$

Sometimes we will not be able to conclude boundedness on all weights $\nu^M \in A_2$, but rather only on weights whose characteristic is uniformly bounded by a (large) constant. An inspection of the proof of the above Lemma, together with the Coifman-Rochberg theorem (see Proposition 5.2.19), reveals that this is enough to conclude the unweighted L^p estimates. We record this in the following result.

Corollary 5.2.27. *Let $M \geq 1$, $0 < \delta < 1$ and T be an operator satisfying, for every $\nu \in A_2$ with $[\nu^M] \leq C_\delta$ (where C_δ is as in Proposition 5.2.19), the estimate $\|Tf\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} \|f\|_{L^2(\nu)}$. Then, for every $p \in (2-\delta/M, 2+\delta/M)$, $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$. Analogous statements for the one-sided versions of the estimates also hold.*

Lemma 5.2.28 (Weighted Carleson's Lemma)**.** *Suppose that μ is a measure in $\mathbb{R}^{n+1}_+$ and that $\nu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a doubling weight. Assume further that for every cube $Q \subset \mathbb{R}^n$, it holds that $\mu(R_Q) \lesssim \nu(Q)$. Then, for every measurable function $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ and every $p > 0$, we have that $\iint_{\mathbb{R}^{n+1}_+} |F|^p \, d\mu \lesssim_{n,\mathrm{doub}} \int_{\mathbb{R}^n} (\mathcal{N}F)^p \, \nu$.*

The proof is exactly the same as the usual one when $\nu \equiv 1$, and thus omitted.

We will have need for a version of Carleson's Lemma that introduces the modified non-tangential maximal function $\tilde{\mathcal{N}}$ in place of the usual $\mathcal{N}$.

Lemma 5.2.29 (Modified-Weighted Carleson's Lemma)**.** *Let $d\mu(x,t) = m(x,t)\, dxdt$ be a non-negative measure on $\mathbb{R}^{n+1}_+$ and ν is a doubling weight. For every $(x,t) \in \mathbb{R}^{n+1}_+$, suppose that $d\tilde{\mu}(x,t) := (\sup_{(y,s) \in \mathcal{C}_{x,t}} m(y,s))\, dxdt$ verifies $\tilde{\mu}(R_Q) \leq C_0 \nu(Q)$ for every cube $Q \subset \mathbb{R}^n$. Then, for every $q > 0$, $\iint_{\mathbb{R}^{n+1}_+} |F|^q \, d\mu \lesssim_{\mathrm{doub}} C_0 \int_{\mathbb{R}^n} (\tilde{\mathcal{N}}_q F)^q \, \nu$.*

Proof. Introduce an average as follows:

$$\iint_{\mathbb{R}^{n+1}_+} |F(x,t)|^q \, d\mu(x,t) = \iint_{\mathbb{R}^{n+1}_+} ⨌_{\mathcal{C}_{x,t}} |F(x,t)|^q \, dyds \, m(x,t) \, dxdt$$
$$= \iint_{\mathbb{R}^{n+1}_+} ⨌_{\mathcal{C}_{y,s}} |F(x,t)|^q m(x,t) \, dxdt \, dyds \leq \iint_{\mathbb{R}^{n+1}_+} G(y,s)^q d\tilde{\mu}(y,s),$$

where we used Fubini's Theorem, and $G(y,s) := \Big(⨌_{\mathcal{C}_{y,s}} |F(x,t)|^q \, dxdt\Big)^{1/q}$. The conclusion now follows from Lemma 5.2.28 and the fact that $\mathcal{N}(G) = \tilde{\mathcal{N}}_q(F)$. $\square$

Definition 5.2.30 ($A_{p,q}$ classes)**.** Let $1 < p \leq q < \infty$. We say that a weight $\nu \in A_{p,q} = A_{p,q}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for every cube $Q \subset \mathbb{R}^n$,

$$\big(⨍_Q \nu^q \, dx\big)^{1/q} \big(⨍_Q \nu^{-1/p'} \, dx\big)^{1/p'} \leq C.$$

The infimum over all such C is written $[\nu]_{A_{p,q}}$.

Theorem 5.2.31 ([MW74, Theorem 4])**.** *Let $1 < p < n$ and set $1/q := 1/p - 1/n$. Then $\nu \in A_{p,q}$ if and only if $\|I_1 f\|_{L^q(\nu^q)} \lesssim_{[\nu]_{A_{p,q}}} \|f\|_{L^p(\nu^p)}$.*

Throughout the chapter we will encounter instances where multiplication by an $L^n(\mathbb{R}^n)$ function is acting as, or rather in place of, a (spatial) gradient. The following proposition should be interpreted as stating that, at least in L^p spaces, the two operations are not far from each other. We remind the reader that we assume $n \geq 3$.

Proposition 5.2.32. *Let $B \in L^n(\mathbb{R}^n)$ and $f \in C_c^\infty(\mathbb{R}^n)$. Then, for every $\nu \in A_2$, we have*

$$\|I_1(B \cdot f)\|_{L^2(\nu)} \lesssim_{[\nu]_{A_2}} \|B\|_{L^n(\mathbb{R}^n)} \|f\|_{L^2(\nu)}.$$

In particular, for every $1 < p < \infty$, it holds that $\|I_1(B \cdot f)\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$, where the implicit constants depend on $\|B\|_{L^n(\mathbb{R}^n)}$, p, and n. If in addition we have that $\nu^{2^*/2} \in A_2$ with $2^* = 2^*_n$, then

$$\|B \cdot I_1 f\|_{L^2(\nu)} \lesssim_{[\nu^{2^*}]_{A_2}} \|B\|_{L^n(\mathbb{R}^n)} \|f\|_{L^2(\nu)}.$$

Accordingly, $\|B \cdot I_1 f\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$, for $1 + 2/n < p < 3 - 2/n$.

Proof. Let $\nu \in A_2$ and set $\omega := \nu^{1/2}$. We claim that $\omega \in A_{2_*,2}$. Assuming the claim for a moment, we would have

$$\|I_1(B\cdot f)\|_{L^2(\nu)} = \|I_1(B\cdot f)\|_{L^2(\omega^2)} \lesssim_{[\omega]_{A_{2_*,2}}} \|B\cdot f\|_{L^{2_*}(\omega^{2_*})} \\ \leq \|B\|_{L^n(\mathbb{R}^n)}\|f\|_{L^2(\omega^2)},$$

where we used Hölder's inequality in the last step. This is the result.

To prove the claim, we use Jensen's inequality to see that $\left(\fint_Q \omega^{-2_*}\right)^{2/2_*} \leq \fint_Q \omega^{-2}$. Using this estimate in the definition of $\omega^2 \in A_2$, we deduce that $[\omega]_{A_{2_*,2}} \leq [\nu]_{A_2}^{1/2}$. This completes the proof of the first part.

The second part follows the same lines, using instead that

$$\left(\fint_Q \omega^{2^*}\right)\left(\fint_Q \omega^{-2}\right)^{2^*/2} \leq \left(\fint_Q \omega^{2^*}\right)\left(\fint_Q \omega^{-2^*}\right) \leq [\nu^{2^*/2}]_{A_2},$$

so that $[\omega]_{A_{2,2^*}} \leq [\nu^{2^*/2}]_{A_2}^{1/2^*}$. The L^p estimate finally follows from restricted extrapolation (see Lemma 5.2.26), using the fact that $2/2^* = 1 - 2/n$. $\square$

Proposition 5.2.33. *Let P_t be an approximate identity with smooth, even, compactly supported kernel. Then, for every $\nu \in A_2$ and $f \in C_c^\infty(\mathbb{R}^n)$, it holds that*

$$\|\mathbb{V}(t^{-1}(1-P_t)f)\|_{L^2(\nu)}^2 = \int_0^\infty \int_{\mathbb{R}^n} \left|\frac{I-P_t}{t}f\right|^2 \frac{\nu(x)\,dxdt}{t} \lesssim_{[\nu]_{A_2}} \|\nabla_\| f\|_{L^2(\nu)}^2.$$

Proof. Recall that I_1 denotes the fractional integral of order 1; hence $\nabla_\| I_1 = I_1 \nabla_\| = R$, where R is the vector-valued Riesz-transform (with symbol $\xi/|\xi|$). In particular, $\|Rf\|_{L^2(\nu)} \approx \|f\|_{L^2(\nu)}$ for all $\nu \in A_2$, allowing us to reduce matters to the estimate

$$\int_0^\infty \int_{\mathbb{R}^n} \left|I_1 \frac{I-P_t}{t} f\right|^2 \frac{\nu(x)dt}{t} \lesssim_{[\nu]_{A_2}} \|f\|_{L^2(\nu)}, \qquad f \in C_c^\infty(\mathbb{R}^n).$$

We now use a quasi-orthogonality argument, with a change of measure interpolation (see the proof of Theorem 5.2.23), to reduce matters to the pair of estimates: If we denote $T_t := I_1(1-P_t)/t$, then for some CLP family $(\mathcal{Q}_s)_s$ (see Definition 4.2.26),

$$\|T_t \mathcal{Q}_s^2 f\|_{L^2(\mathbb{R}^n)} \lesssim \left(\frac{s}{t}, \frac{t}{s}\right)^\alpha \|\mathcal{Q}_s f\|_{L^2(\mathbb{R}^n)}, \tag{5.2.34}$$

for some $\alpha > 0$, and

$$\|T_t \mathcal{Q}_s^2 f\|_{L^2(\nu)} \lesssim_{[\nu]_{A_2}} \|\mathcal{Q}_s f\|_{L^2(\nu)}. \tag{5.2.35}$$

Indeed, with (5.2.34) and (5.2.35) in hand, we may follow the proof of Theorem 5.2.23.

For (5.2.34), we compute, via the Fourier transform and Plancherel's theorem, and using φ_t and ψ_s for the kernels of P_t and $\mathcal{Q}_s$ respectively,

$$\|T_t \mathcal{Q}_s h\|_{L^2(\mathbb{R}^n)}^2 = c_n \int_{\mathbb{R}^n} \left| |\xi|^{-1} \frac{1-\hat{\varphi}(t|\xi|)}{t} \hat{\psi}(s|\xi|) \hat{h}(\xi) \right|^2 d\xi,$$

where as usual we have abused notation and written φ, ψ for the one-dimensional functions representing them. Consider first the case $t < s$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| |\xi|^{-1} \frac{1-\hat{\varphi}(t|\xi|)}{t} \hat{\psi}(s|\xi|) \hat{h}(\xi) \right|^2 d\xi \\
&\qquad = \left(\frac{t}{s}\right)^2 \int_{\mathbb{R}^n} \frac{|1-\hat{\varphi}(t|\xi|)|^2}{|t\xi|^4} |s\xi|^2 |\hat{\psi}(s|\xi|)|^2 |\hat{h}(\xi)|^2 \, d\xi \lesssim \left(\frac{t}{s}\right)^2 \|h\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where we used the properties of the CLP family and the fact that $|1-\hat{\varphi}(\tau)| \lesssim \tau^2$ for τ near 0, since φ is even. For the case $s < t$, we use instead the Fundamental Theorem of Calculus to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| |\xi|^{-1} \frac{1-\hat{\varphi}(t|\xi|)}{t} \hat{\psi}(s|\xi|) \hat{h}(\xi) \right|^2 d\xi \\
&\qquad = \left(\frac{s}{t}\right)^2 \int_{\mathbb{R}^n} \left| \int_0^{t|\xi|} \hat{\varphi}'(\tau) \, d\tau \right|^2 \frac{|\hat{\psi}(s|\xi|)|^2}{|s\xi|^2} |\hat{h}(\xi)|^2 \, d\xi \lesssim \left(\frac{s}{t}\right)^2 \|h\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where we used that $\hat{\varphi} \in L^1(0,\infty)$ and $\hat{\psi}(\tau)/\tau \in L^\infty(0,\infty)$. Combining these estimates with $h = \mathcal{Q}_s f$ gives (5.2.34).

The weighted estimate (5.2.35) follows from the pointwise inequality

$$|T_t f(x)| = |t^{-1}(1-P_t) I_1 f(x)| \lesssim \mathcal{M}(Rf)(x),$$

where $R = I_1 \nabla_\|$ is as before. We sketch the argument: Write

$$1 - P_t = (1 - E_t) + (E_t - P_t), \tag{5.2.36}$$

where E_t is the dyadic averaging operator; that is, $E_t f(x) = \fint_{Q_{x,t}} f$, where $Q_{x,t}$ is the unique dyadic cube $Q_{x,t} \in \mathbb{D}_t$ containing x. Writing $g = I_1 f$, we have that

$$|(E_t - P_t)g(x)| \approx \Big|\fint_{Q_{x,t}} \fint_{|x-y|<Ct} \varphi(\tfrac{x-y}{t})(g(y) - g(z))\, dydz\Big|$$
$$\lesssim \fint_{B(x,Ct)} \fint_{B(x,Ct)} |g(y) - g(z)|\, dydz \lesssim t \fint_{B(x,Ct)} |\nabla_\| g(y)|\, dy \leq t\mathcal{M}(\nabla_\| g)(x),$$

where we used Poincare's inequality in the second to last step. Since $I_1 \nabla_\| f = Rf$, we have the right bound for this term.

To handle the first term in (5.2.36), we telescope

$$(1 - E_t)g(x) = \sum_{j=0}^{\infty} (E_{2^{-j-1}t} - E_{2^{-j}t})g(x) =: \sum_{j=0}^{\infty} (E_{t_{j+1}} - E_{t_j})g(x),$$

and we compute that

$$|(E_{t_{j+1}} - E_{t_j})g(x)| = \Big|\fint_{Q_{x,t_{j+1}}} (E_{t_j} g(x) - g(y))\, dy\Big| \lesssim \fint_{Q_{x,t_j}} |E_{t_j} g(x) - g(y)|\, dy$$
$$\lesssim t_j \fint_{Q_{x,t_j}} |\nabla_\| g(y)|\, dy \leq 2^{-j} t\mathcal{M}(\nabla_\| g)(x).$$

The result now follows by summing over j. □

We will have need for the following properties of the heat semigroup associated to the Laplacian Δ in $\mathbb{R}^n$.

Proposition 5.2.37. *Let $P_t := e^{-t^2\Delta}$ and $\mathcal{Q}_t := t\partial_t P_t$. We define the measure*

$$d\mu(x,t) := \frac{|\mathcal{Q}_t \nu(x)|^2}{|P_t \nu(x)|^2} P_t \nu(x)\, \frac{dxdt}{t}.$$

This object satisfies the following properties

(i) For any weight $\nu \in RH_s$ for some $s > 1$ it holds that $|P_t \nu(x)| \lesssim \fint_{|x-y|<t} \nu(y)\, dy$, with constants depending on the RH_s and doubling constants of ν.

(ii) The measure $d\mu$ satisfies the hypotheses of the modified Carleson's Lemma 5.2.29, provided $\nu \in RH_2$.

Proof. The proof of (i) is a simple computation: the kernel of P_t is given by

$$\varphi_t(x-y) = c_n t^{-n} e^{-(|x-y|/2t)^2}, \qquad x, y \in \mathbb{R}^n, \quad t > 0,$$

and we can write

$$P_t\nu(x) = \int_{|x-y|<t} \varphi_t(x-y)\nu(y)\,dy + \sum_{j\geq 0} \int_{2^j t \leq |x-y| < 2^{j+1}t} \varphi_t(x-y)\nu(y)\,dy.$$

Clearly, the first term satisfies the desired estimate; it remains to control the tail. For this, we set $\Delta_j := \{y : 2^j t \leq |x-y| < 2^{j+1}t\}$ and employ Hölder's inequality to obtain

$$\int_{\Delta_j} \varphi_t(x-y)f(y)\,dy \leq \Big(\int_{\Delta_j} \varphi_t(x-y)^{s'}\,dy\Big)^{1/s'} \Big(\int_{\Delta_j} \nu(y)^s\,dy\Big)^{1/s}.$$

Now we see, using that $\nu \in RH_s$,

$$\begin{aligned}\Big(\int_{\Delta_j} \nu^s\Big)^{1/s}& \\ &\lesssim (2^j t)^{n(\frac{1}{s}-1)} \int_{|x-y|<2^{j+1}t} \nu(y)dy \lesssim (C_{doub} 2^{n(1/s-1)})^j t^{\frac{n}{s}} \fint_{|x-y|<t} \nu(y)dy.\end{aligned}$$

On the other hand, for $y \in \Delta_j$ we have that $\varphi_t(x-y) \lesssim t^{-n}\exp(-2^j)$, so that

$$\Big(\int_{\Delta_j} \varphi_t(x-y)^{s'}\,dy\Big)^{1/s'} \lesssim |\Delta_j|^{1/s'} t^{-n}\exp(-2^j) \lesssim t^{-n/s} 2^{jn/s'} \exp(-2^j).$$

Combining these estimates, (i) follows.

The proof of (ii) is somewhat more involved. We ought to show that

$$\sup_{\mathcal{C}_{x,t}} \Big(\frac{|\mathcal{Q}_s\nu(y)|^2}{|P_s\nu(y)|^2} P_s\nu(y)\frac{1}{s}\Big)dxdt \approx \Big(\sup_{\mathcal{C}_{x,t}} \frac{|\mathcal{Q}_s\nu(y)|^2}{|P_s\nu(y)|^2} P_s\nu(y)\Big)\frac{dxdt}{t} =: d\tilde{\mu}(x,t)$$

is a Carleson Measure. For this purpose, first note that, using (i) and the doubling property of ν, it is not hard to show that $P_s\nu(y) \approx P_t\nu(x)$ for all $(y,s) \in \mathcal{C}_{x,t}$. In other words,

$$d\tilde{\mu}(x,y) \approx \Big(\sup_{\mathcal{C}_{x,t}} |\mathcal{Q}_s\nu(y)|^2\Big)\frac{1}{P_t\nu(x)}\frac{dxdt}{t}.$$

Now we let $a > 0$ be small enough so that $s^2 - a^2t^2 \approx t^2$ whenever $|t-s| < t/8$ and write

$$\mathcal{Q}_s\nu(y) = s\partial_s(e^{(a^2t^2-s^2)\Delta}e^{-a^2t^2\Delta}\nu(y)) = -2s^2\Delta e^{(a^2t^2-s^2)\Delta}e^{-a^2t^2\Delta}\nu(y)$$
$$= -\frac{s^2}{a^2t^2}e^{-(s^2-a^2t^2)\Delta}\mathcal{Q}_{at}\nu(y).$$

Therefore, there exists a universal constant $c > 0$ such that $|\mathcal{Q}_s\nu(y)| \lesssim P_{ct}|\mathcal{Q}_{at}\nu|(y)$, for all $|s-t| < t/8$. Setting $g_t(z) := |\mathcal{Q}_{at}\nu(z)|$, we see that

$$P_{ct}g_t(y) = c_n\int_{\mathbb{R}^n}(ct)^{-n}e^{-\frac{|y-z|^2}{4(ct)^2}}g_t(z)\,dz$$
$$= \int_{|x-z|<t}(ct)^{-n}e^{-\frac{|y-z|^2}{4(ct)^2}}g_t(z)\,dz + \sum_{j\geq 0}\int_{\Delta_j}(ct)^{-n}e^{-\frac{|y-z|^2}{4(ct)^2}}g_t(z)\,dz =: I + II.$$

For I, we simply note that

$$I \lesssim \fint_{|x-z|<t} g_t(z)\,dz \lesssim P_t g_t(x).$$

For the tails, we use that $|y-z| \geq \frac{7}{8}|x-z|$ for any $z \in \Delta_j$, to obtain the bound $II \lesssim P_{c't}g_t(x)$. We conclude that $|\mathcal{Q}_s\nu(y)| \lesssim P_{ct}|\mathcal{Q}_{at}\nu|(x)$ for $(y,s) \in \mathcal{C}_{x,t}$.

We have thus reduced matters to proving a (weighted) Carleson Measure estimate for $d\mu'(x,t) := \frac{(P_{ct}|\mathcal{Q}_{at}\nu|(x))^2}{P_t\nu(x)}\frac{dxdt}{t}$; that is, we want to show that $\mu'(R_Q) \lesssim \nu(Q)$, for all $Q \subset \mathbb{R}^n$. So fix a cube $Q \subset \mathbb{R}^n$. We run a stopping time argument to obtain a collection of maximal (dyadic) subcubes $(Q_j)_{j\geq 1}$ of Q with respect to the properties

$$\text{either} \quad \text{(a)} \quad \fint_{Q_j}\nu \geq A\fint_Q \nu, \qquad \text{or} \quad \text{(b)} \quad \fint_{Q_j}\nu \leq A^{-1}\fint_Q \nu,$$

for some $A > 1$ large. We call $\mathcal{F}_1$ the collection of Q_j satisfying the property (a), and $\mathcal{F}_2$ the collection of Q_j verifying (b).

Notice that, by construction, if $Q_j \in \mathcal{F}_1$ we have that $|Q_j| \leq \frac{|Q|}{A\nu(Q)}\int_{Q_j}\nu$, so, after summing over j, $\sum_{Q_j\in\mathcal{F}_1}|Q_j| \leq \frac{|Q|}{A}$. By the A_∞ property of ν, if A is large enough, we may write $\sum_{Q_j\in\mathcal{F}_1}\nu(Q_j) \leq \frac{\nu(Q)}{4}$. On the other hand, if $Q_j \in \mathcal{F}_2$ we obtain directly that $\nu(Q_j) \leq A^{-1}|Q_j|\fint_Q \nu$, so that $\sum_{Q_j\in\mathcal{F}_2}\nu(Q_j) \leq \frac{\nu(Q)}{4}$, if we choose $A > 1/4$.

By the John-Nirenberg Lemma for Carleson Measures (Lemma 5.2.14) it is enough obtain a bound $\mu'(R_Q \backslash (\cup_{\mathcal{F}_1 \cup \mathcal{F}_2} R_{Q_j})) \lesssim \nu(Q)$. Moreover, notice that for $(x,t) \in R_Q \backslash (\cup_{\mathcal{F}_1 \cup \mathcal{F}_2} R_{Q_j})$, we have that $\fint_Q \nu \lesssim P_t \nu(x)$, by construction of the Q_j . Accordingly, it is enough to show the estimate

$$\iint_{R_Q} (P_{ct} |\mathcal{Q}_{at} \nu|)(x)^2 \, \frac{dxdt}{t} \lesssim \frac{\nu(Q)^2}{|Q|}.$$

To do this, we use Minkowski's inequality to write

$$\begin{aligned}\Big(\iint_{R_Q} (P_{ct} |\mathcal{Q}_{at} \nu|)(x)^2 \frac{dxdt}{t} \Big)^{\frac{1}{2}} &\leq \sum_{j=0}^{\infty} \Big(\iint_{R_Q} (P_{ct} |\mathcal{Q}_{at} (\mathbb{1}_{R_j(Q)} \nu)|)(x)^2 \frac{dxdt}{t} \Big)^{\frac{1}{2}} \\ &=: \sum_{j=0}^{\infty} T_j,\end{aligned}$$

where we denote $R_0 = 2Q$ and $R_j = 2^{j+1} Q \backslash 2^j Q$ for $j \geq 1$. For the first term T_0, we employ the fact that P_t is uniformly $L^2(\mathbb{R}^n)-$bounded and that $\mathcal{Q}_t$ satisfies an L^2 square function estimate to obtain

$$\begin{aligned}T_0^2 &\leq \int_0^\infty \int_{\mathbb{R}^n} (P_{ct} |\mathcal{Q}_{at} (\mathbb{1}_{R_0} \nu)|)(x)^2 \, \frac{dxdt}{t} \\ &\lesssim \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{Q}_{at} (\mathbb{1}_{R_0} \nu)(x)|^2 \, \frac{dxdt}{t} \lesssim \int_{2Q} \nu^2.\end{aligned}$$

We now use the Reverse Hölder property of ν to see that $\int_{2Q} \nu^2 \lesssim |Q|^{-1} \nu(Q)^2$, which gives the desired estimate for T_0. For the others, we use the kernel representations; first recall that if φ_t is the kernel for P_t and ψ_t the one for $\mathcal{Q}_t$ then there exist constants such that $|\varphi_t(z)|, |\psi_t(z)| \leq c_1 t^{-n} e^{-c_2 |z|^2/t^2}$. Calling $\nu_j = \mathbb{1}_{R_j} \nu$, we compute

$$\begin{aligned}\int_Q (P_{ct} |\mathcal{Q}_{at} \nu_j|)(x)^2 \, dx &= \int_Q \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi_{ct}(x-y) \psi_{at}(y-z) \nu_j(z) \, dz \Big| \, dy \Big)^2 \, dx \\ &\lesssim \int_Q \Big(\int_{R_j} \nu(z) \int_{\mathbb{R}^n} t^{-2n} e^{-c \frac{|x-y|^2 + |y-z|^2}{t^2}} \, dy \, dz \Big)^2 \, dx.\end{aligned}$$

It is easy to verify that $|x-y|^2+|y-z|^2 \geq (|x-y|^2+|x-z|^2)/4$, and hence

$$\int_Q (P_{ct}|\mathcal{Q}_{at}\nu_j|)(x)^2\,dx \lesssim \int_Q \Big(\int_{R_j} \nu(z)t^{-n}e^{-c\frac{|x-z|^2}{t^2}}\,dz\Big)^2 dx \lesssim e^{-c\frac{(2^j\ell(Q))^2}{2t^2}} \int_Q (\tilde{P}_t\nu)^2,$$

where we define $\tilde{P}_t$ the convolution operator with kernel $t^{-n}e^{-c|z|^2/(2t^2)}$. Now we see, from the proof of part (i), that $\tilde{P}_t\nu(x) \lesssim ⨍_{|x-y|<t} \nu(y)\,dy \lesssim \nu(Q)/t^n$. Therefore,

$$\int_Q (P_{ct}|\mathcal{Q}_{at}\nu_j|)(x)^2\,dx \lesssim e^{-c'\frac{(2^j\ell(Q))^2}{t^2}} \frac{\nu(Q)^2|Q|}{t^{2n}}.$$

The desired estimate for T_j now follows by integrating in t over $(0,\ell(Q))$. $\square$

5.2.2 $L^r - L^q$ Off-diagonal estimates

Throughout this section we denote by T_t, with $t \neq 0$, an operator mapping functions $C_c^\infty(\mathbb{R}^n;\mathbb{C}^{d_1})$ to measurable functions in $\mathbb{R}^n$ with values in $\mathbb{C}^{d_2}$ for some integers d_1, d_2.

Definition 5.2.38 ($L^r \to L^q$ Off-diagonal estimates)**.** Let $1 \leq r \leq q \leq \infty$. We say that a family of operators $(T_t)_{t\neq 0}$ satisfies $L^r - L^q$ *off-diagonal estimates* if there exist $C_0 > 0$ and numbers $\gamma_1 \in \mathbb{R}$, $\gamma_2 > 0$ such that for every cube $Q \subset \mathbb{R}^n$, the following estimates hold with $\gamma := \gamma_1 + \gamma_2$.

(i) If $|t| \approx \ell(Q)$, then

$$\|T_t(f\mathbb{1}_{R_0(Q)})\|_{L^q(Q)} \leq C_0|Q|^{1/q-1/r}\|f\|_{L^r(Q)}.$$

(ii) If $t \in \mathbb{R}$ and we set $R_j(Q) := 2^{j+1}Q\backslash 2^jQ$ for $j \geq 1$, then

$$\|T_t(f\mathbb{1}_{R_j(Q)})\|_{L^q(Q)} \leq C_0 2^{-nj\gamma_1}\Big(\frac{|t|}{2^j\ell(Q)}\Big)^{nj\gamma_2}|Q|^{1/q-1/r}\|f\|_{L^r(R_j(Q))}.$$

(iii) If $t \in \mathbb{R}$ and $\operatorname{supp} f \subset Q$ then

$$\|T_t(f)\|_{L^q(R_j(Q))} \leq C_0 2^{-nj\gamma_1}\Big(\frac{|t|}{2^j\ell(Q)}\Big)^{nj\gamma_2}|Q|^{1/q-1/r}\|f\|_{L^r(Q)}.$$

Proposition 5.2.39 (Weighted estimates from off-diagonal decay)**.** *Suppose* $(T_t)_{t>0}$ *is sublinear and satisfies* $L^r - L^2$ *off diagonal estimates for some* $1 < r < 2$ *and* $\gamma > 1/r$.

Then for all $\nu \in A_{2/r}$, and every $t > 0$,

$$\Big\| \Big(\fint_{|x-y|<t} |T_t f(y)|^2 \, dy \Big)^{1/2} \Big\|_{L^2(\nu dx)} \lesssim_{[\nu]_{A_{2/r}}} \|f\|_{L^2(\nu)}.$$

Proof. This proposition is contained within [GH17], but we provide the proof for completeness.

We first decompose $\mathbb{R}^n$ into cubes in the dyadic grid $\mathbb{D}_t$ of sidelength $\approx t$ to obtain

$$\begin{aligned}
\Big\| \Big(\fint_{|x-y|<t} |T_t f(y)|^2 \, dy \Big)^{1/2} \Big\|_{L^2(\nu dx)} &= \Big(\sum_{Q \in \mathbb{D}_t} \int_Q \fint_{|x-y|<t} |T_t f(y)|^2 \, dy \, \nu(x) dx \Big)^{1/2} \\
&\lesssim \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \int_Q |T_t f(y)|^2 \, dy \, \nu(x) dx \Big)^{1/2} \\
&\lesssim \sum_{j \geq 0} \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \int_Q |T_t(\mathbb{1}_{R_j(Q)} f(y))|^2 \, dy \, \nu(x) dx \Big)^{1/2} =: I,
\end{aligned}$$

where as usual we define $R_0(Q) := 2Q$ and $R_j(Q) := 2^{j+1}Q \backslash 2^j Q$ for $j \geq 1$, and we used Minkowski's inequality in the last line. We now exploit the off-diagonal decay of T_t to get,

$$\Big(\int_Q |T_t(\mathbb{1}_{R_j(Q)} f(y))|^2 \, dy \Big)^{1/2} \lesssim 2^{-jn\gamma} t^{n(1/2-1/r)} \Big(\int_{R_j(Q)} |f(y)|^r \, dy \Big)^{1/r}.$$

Going back to I, we see that

$$\begin{aligned}
I &\lesssim \sum_{j \geq 0} 2^{-jn\gamma} t^{n(1/2-1/r)} \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \Big(\int_{R_j(Q)} |f(y)|^r \, dy \Big)^{2/r} \nu(x) dx \Big)^{1/2} \\
&\lesssim \sum_{j \geq 0} 2^{-jn\gamma} t^{n(1/2-1/r)} (2^j t)^{n/r} \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \Big(\fint_{R_j(Q)} |f(y)|^r \, dy \Big)^{2/r} \nu(x) dx \Big)^{1/2} \\
&\lesssim \sum_{j \geq 0} 2^{-jn(\gamma - 1/r)} t^{n/2} \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \mathcal{M}_r(f)(x)^2 \, \nu(x) dx \Big)^{1/2} \\
&\lesssim \sum_{j \geq 0} 2^{-jn(\gamma - 1/r)} \|\mathcal{M}_r(f)\|_{L^2(\nu)} \lesssim \|\mathcal{M}_r(f)\|_{L^2(\nu)},
\end{aligned}$$

since $\gamma > 1/r$. Since $r < 2$ and $A_{2/r} \subset A_2$, $\mathcal{M}_r$ is bounded on $L^2(\nu)$ and so

$$I \lesssim_{[\nu]_{A_{2/r}}} \|\mathcal{M}_r(f)\|_{L^2(\nu)} \lesssim \|f\|_{L^2(\nu)}.$$

$\square$

5.2.3 Properties of Solutions and Layer Potentials

Properties of weak solutions

We record here the basic properties of solutions to $\mathcal{L}u = 0$ that we will need.

Definition 5.2.40. We define the interval $(2_-, 2_+)$ as the largest open interval, symmetric around 2 with the following two properties:

1. $2n/(n+1) = 2_\# < 2_- < 2 < 2_+ < 2^\# = 2n/(n-1)$.
2. If $p \in (2_-, 2_+)$, then for every weak solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$, the L^p Caccioppoli inequalities (4.3.10) and (4.3.22) hold, with implicit constants depending only on n, p, α, ellipticity of $\mathcal{L}$, and ρ.

Proposition 5.2.41 (Weak Reverse Hölder Inequality for Solutions)**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution to $\mathcal{L}u = \operatorname{div} F$ in $\Omega \subset \mathbb{R}^{n+1}$, with $F \in L^2_{\mathrm{loc}}(\Omega)$. Let B be an $(n+1)$-dimensional ball in $\mathbb{R}^{n+1}$ with $2B \subset \Omega$. Then, for any $q \geq 1$ we have that*

$$\left(\fint\!\!\fint_B |u|^{2^*_{n+1}}\right)^{1/2^*_{n+1}} \lesssim \left(\fint\!\!\fint_{2B} |u|^q\right)^{1/q} + r(B)\left(\fint\!\!\fint_{2B} |F|^2\right)^{1/2}, \tag{5.2.42}$$

with implicit constants depending only on q, n and ellipticity, and where we define

$$\frac{1}{2^*_{n+1}} = \frac{1}{2} - \frac{1}{n+1}, \qquad \frac{1}{2^*_{n+1}} + \frac{1}{2_{*,n+1}} = 1.$$

Proof. We first prove the result for a ball B with $r(B) = 1$. To simplify notation, during this proof we will write $2^* = 2^*_{n+1}$.

Fix $1 \leq t < s \leq 2$, then, from the proof of the Caccioppoli inequality (see Proposition

4.3.1, and note that $f = 0$ for us), we have

$$\begin{aligned}\|\nabla u\|_{L^2(B_t)} &\lesssim \frac{1}{s-t}\|u\|_{L^2(B_s)} + \|F\|_{L^2(B_s)} \\ &\le \frac{1}{s-t}\|u\|_{L^2(B_s)} + \|F\|_{L^2(2B)},\end{aligned}$$

where B_t denotes the concentric dilate of B by a factor of t.

On the other hand, if as usual we denote by u_B the average of u over B, then by the Poincaré-Sobolev inequality we have that

$$\begin{aligned}\|u\|_{L^{2^*}(B_t)} &\lesssim t \cdot t^{(n+1)(1/2^*-1/2)}\|\nabla u\|_{L^2(B_t)} + t^{(n+1)/2^*}|u_{B_t}| \\ &\lesssim \|\nabla u\|_{L^2(B_t)} + \|u\|_{L^1(2B)},\end{aligned}$$

where we use that $t \geq 1/2$ to control the last term in the last line. Combining these two inequalities we obtain

$$\begin{aligned}\|u\|_{L^{2^*}(B_t)} \lesssim \frac{1}{s-t}\|u\|_{L^2(B_s)} &+ \|F\|_{L^2(2B)} \\ &+ \|u\|_{L^1(2B)}.\end{aligned}$$

Note that if here we set $t = 1$ and $s = 2$, the desired estimate (5.2.42) follows for $q \geq 2$. It thus remains to treat the case $q < 2$.

Recall, from interpolation of L^p norms (here we use $q < 2$) and the Cauchy inequality with a parameter,

$$\|u\|_{L^2} \le \|u\|_{L^{2^*}}^{1-\theta}\|u\|_{L^q}^{\theta} \lesssim_\theta \eta^{1/(1-\theta)}\|u\|_{L^{2^*}} + \frac{1}{\eta^{1/\theta}}\|u\|_{L^{2_*}},$$

valid for any $\eta > 0$ and where θ satisfies

$$\frac{1}{2} = \frac{1-\theta}{2^*} + \frac{\theta}{q}.$$

Choosing $\eta^{1/(1-\theta)} \approx t - s$ and setting

$$T := \|F\|_{L^2(2B)} + \|u\|_{L^1(2B)},$$

we arrive at

$$\|u\|_{L^{2^*}(B_t)} \le \frac{1}{2}\|u\|_{L^{2^*}(B_s)} + \frac{C}{(s-t)^{(1-\theta)/\theta}}\|u\|_{L^q(B_s)} + T,$$

the above being valid for any $1 \le t < s \le 2$. We are now in a position to apply the result in [HL97, Lemma 4.3] and conclude

$$\|u\|_{L^{2^*}(B_t)} \lesssim \frac{1}{(s-t)^{(1-\theta)/\theta}}\|u\|_{L^q} + T.$$

Setting now $t = 1$ and $s = 2$ we arrive at

$$\|u\|_{L^{2^*}(B)} \lesssim \|u\|_{L^q(2B)} + \|u\|_{L^1(2B)} + \|F\|_{L^2(2B)}.$$

This is the desired inequality, since $q \ge 1$ and $r(B) = 1$.

To obtain the result for a general B, we simply note that for $r > 0$, $u_r(X) = u(rX)$ solves $\mathcal{L}_r u_r = \operatorname{div} F_r$ in $r\Omega$, where the coefficients of $\mathcal{L}_r$ are given by

$$A_r(X) = A(rX), \quad B_{i,r}(X) = rB_i(rX), \quad F_r(X) = rF(rX).$$

It can be checked that these coefficients satisfy the same conditions as the original coefficients, with the same relevant norms, except for F_r which satisfies $\|F_r\|_{L^2(1/rB)} = r^{(-n+1)/2}\|F\|_{L^2(B)}$. The estimate (5.2.42) follows.

Proposition 5.2.43 (Off-diagonal Estimates. Part 1)**.** *Let $\Theta_{t,m}$ denote either of the following operators:*

$$t^m\partial_t^m\nabla\mathcal{S}_t^{\mathcal{L}}, \qquad t^m\partial_t^{m-1}\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla), \qquad (t^m\partial_t^m\nabla\mathcal{S}_t^{\mathcal{L}}1)\cdot P_t,$$

where P_t is an approximate identity with smooth, even, compactly supported kernel.

Let $2_- < q \le p < 2_+$ and $Q \subset \mathbb{R}^n$ a cube. For every $h \in L^q(\mathbb{R}^n)$, $|t| \approx \ell(Q)$, and $k \ge 0$ it holds that

$$\|\Theta_{t,m}(h\mathbb{1}_Q)\|_{L^p(2Q)} \lesssim_{n,p,q} |Q|^{\frac{1}{p}-\frac{1}{q}}\|h\|_{L^q(Q)}.$$

Moreover, for any $j \geq 1$ and $t \in \mathbb{R}$, if we write $R_j(Q) := 2^{j+1}Q \backslash 2^j Q$, then

$$\|\Theta_{t,m} g\|_{L^p(R_j(Q))} \lesssim_{n,p,q,m} 2^{-n\gamma_1}\Big(\frac{|t|}{2^j \ell(Q)}\Big)^m |Q|^{\frac{1}{p}-\frac{1}{q}}\|g\|_{L^q(Q)}, \tag{5.2.44}$$

and

$$\|\Theta_{t,m} g\|_{L^p(Q)} \lesssim_{n,p,q,m} 2^{-n\gamma_1}\Big(\frac{|t|}{2^j \ell(Q)}\Big)^m |Q|^{\frac{1}{p}-\frac{1}{q}}\|g\|_{L^q(R_j(Q))}, \tag{5.2.45}$$

where $\gamma_1 = 1/2^{\#} - 1/p$ if $\Theta_{t,m} = t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$ (recall that $2^{\#}$ is given in Definition 5.2.40), and $\gamma_1 = 1/2^{\#} - 1/p - 1$ in the case that $\Theta_{t,m} = t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$.

Proof. The proof of this result is essentially contained in Proposition 4.4.28. We sketch some of the modifications needed.

We start with the case $\Theta_{t,m} = t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$. Here, estimate (5.2.44) was obtained in Proposition 4.4.28. By duality, estimate (5.2.45) is equivalent to (5.2.44) for $\Theta_{t,m} = t^m \partial_t^m (\mathcal{S}_t^{\mathcal{L}} \nabla)$; let us thus prove estimate (5.2.44) for $\Theta_{t,m} = t^m \partial_t^m (\mathcal{S}_t^{\mathcal{L}} \nabla)$.

Let $\widetilde{R}_j := (3/2)R_j \times (t - 2^j \ell(Q), t + 2^j \ell(Q))$ and suppose that $g \in C_c^\infty(Q)$. Then $(\mathcal{S}_t^{\mathcal{L}} \nabla_\| g) = \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_\| g$ is a solution in $\widetilde{R}_j$. By a careful application of Caccioppoli's inequality on slices, followed by the standard Caccioppoli inequality in L^p $(m-1)$ times, we obtain

$$\begin{aligned}\|t^m \partial_t^m (\mathcal{S}_t^{\mathcal{L}} \nabla_\|) \cdot g\|_{L^p(R_j(Q))} &= |t|^m \|\partial_t^m \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_\| g\|_{L^p(R_j(Q))} \\ &\lesssim \Big(\frac{|t|}{2^j \ell(Q)}\Big)^m \Big(\frac{1}{2^j \ell(Q)} \iint_{\widetilde{R}_j} |\mathcal{S}_s^{\mathcal{L}} \operatorname{div}_\| g(x)|^p \, dx ds\Big)^{1/p}.\end{aligned}$$

By duality again, it is enough to prove that

$$\|\nabla_\| \mathcal{S}_s^{\mathcal{L}} h\|_{L^{q'}(Q)} \lesssim 2^{-n\gamma_1} |Q|^{1/p - 1/q} \|h\|_{L^{p'}(3/2R_j)},$$

uniformly for $s \in \mathbb{R}$. For this, if we define $\widetilde{Q} := 3/2Q \times (s - \ell(Q), s + \ell(Q))$, by Caccioppoli's inequality we have that

$$\begin{aligned}\|\nabla_\| \mathcal{S}_s^{\mathcal{L}} h\|_{L^{q'}(Q)} &\lesssim \frac{1}{\ell(Q)}\Big(\frac{1}{\ell(Q)} \iint_{\widetilde{Q}} |\mathcal{S}_\tau^{\mathcal{L}} h(x)|^{q'} \, dx d\tau\Big)^{1/q'} \\ &\lesssim |Q|^{1/q' - 1/2^{\#} - 1/n}\Big(\frac{1}{\ell(Q)} \int_{s-\ell(Q)}^{s+\ell(Q)} |R_j|^{q'(1/2_{\#} - 1/p')} \|h\|_{L^{p'}(3/2R_j)}^{q'} \, d\tau\Big)^{1/q'}\end{aligned}$$

$$\lesssim 2^{n(1/p-1/2^{\#})}|Q|^{1/q'-1/p'}\|h\|_{L^{p'}(3/2R_j)},$$

where we used the mapping property $\mathcal{S}_s^{\mathcal{L}} : L^{2_\#}(\mathbb{R}^n) \to L^{2^\#}(\mathbb{R}^n)$ uniformly in $s \in \mathbb{R}$, and Hölder's inequality.

The above proof works the same, with straightforward modifications, in the case $\Theta_{t,m} = t^m \partial_t^{m-1}\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla)$. The case of $\Theta_{t,m} = t^m\partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} 1 \cdot P_t$ is handled with the previous estimates and [AAA+11, Lemma 3.11]. □

The following proposition follows the same lines as the above, the appropriate modifications being outlined in the proof of Proposition 4.4.37.

Proposition 5.2.46 (Off-diagonal Estimates. Part 2)**.** *Let $B \in L^n(\mathbb{R}^n; \mathbb{C}^{n+1})$ and set $\Theta_{t,m}^B := t^m\partial_t^m \mathcal{S}_t^{\mathcal{L}} B\cdot$ acting on functions $C_c^\infty(\mathbb{R}^n;\mathbb{C}^{n+1})$. Then, for $2_- < r < 2 < q < 2_+$ $\Theta_{t,m}^B$ satisfies the $L^r - L^q$ off-diagonal estimates of Definition 5.2.38 with $\gamma = m/n - \alpha$ for some $\alpha > 0$ depending only on dimension, r and q.*

We shall also need the following quasi-orthogonality result.

Proposition 5.2.47 (Quasi-orthogonality)**.** *Let $\Theta_{t,m} := t^m\partial_t^m(\mathcal{S}_t^{\mathcal{L}}\nabla)$, $B \in L^n(\mathbb{R}^n;\mathbb{C}^n)$, and let $\mathcal{Q}_s$ be a standard Littlewood-Paley family. There exists m_0 such that if $m \geq m_0$ and $\nu \in A_{2/r}$ (here r is as in the $L^r - L^2$ off-diagonal estimates for $\Theta_{t,m}$ in Proposition 5.2.43), then the estimate*

$$\Big\|\Big(\fint_{|x-y|<t} |\Theta_{t,m} B I_1 \mathcal{Q}_s^2 g(y)|^2\,dy\Big)^{1/2}\Big\|_{L^2(\nu)} \lesssim_{[\nu]_{A_{2/r}}} \Big(\frac{s}{t}\Big)^\beta \|\mathcal{Q}_s g\|_{L^2(\nu)}, \quad s < t,$$

holds for some $\beta > 0$ (possibly depending on ν only through $[\nu]_{A_{2/r}}$).

Proof. The unweighted case is proved in Lemma 4.4.30. The idea is to use interpolation with change of measure to reduce matters to proving a *uniform* weighted bound of the form

$$\Big\|\Big(\fint_{|x-y|<t} |\Theta_{t,m} B I_1 \mathcal{Q}_s^2 g(y)|^2\,dy\Big)^{1/2}\Big\|_{L^2(\nu)} \lesssim_{[\nu]_{A_{2/r}}} \|g\|_{L^2(\nu)}.$$

This in turn follows from the $L^r - L^2$ off-diagonal estimates of $\Theta_{t,m}$ in Proposition 5.2.43 and Proposition 5.2.39, together with the bounds for $I_1 B$ from Proposition 5.2.32; we omit the details. □

Next, we show the simple computation, first seen in [BES19], that yields the following bound for the vertical maximal function.

Proposition 5.2.48. *Let $m \geq 1$ and $\Theta_{t,m} := t^m \partial_t^m \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$ or $t^m \partial_t^m \nabla \mathcal{D}_t^{\mathcal{L}}$, then for a.e. $x \in \mathbb{R}^n$*

$$\sup_{t>0} |\Theta_{t,m} f(x)| \lesssim_m \mathbb{V}(\Theta_{t,m} f)(x) + \mathbb{V}(\Theta_{t,m+1} f)(x) + |\Theta_{1,m} f(x)|.$$

Proof. First we note that, owing to Lemma 4.2.3, the function $t \mapsto \Theta_{t,m} f(x) =: g_t(x)$ is absolutely continuous for a.e. $x \in \mathbb{R}^n$. Therefore, by the fundamental theorem of calculus, for such an $x \in \mathbb{R}^n$ and every $0 < s < t$,

$$|g_t(x)|^2 = |g_s(x)|^2 + 2 \int_s^t \mathrm{Re}(g_t(x) \overline{\partial_\tau g_\tau(x)})\, d\tau.$$

Notice

$$\Big| \int_s^t \mathrm{Re}(g_t(x) \overline{\partial_\tau g_\tau(x)})\, d\tau \Big| \leq \int_s^t |g_t(x)||\tau \partial_\tau g_\tau(x)| \, \frac{d\tau}{\tau} \leq \mathbb{V}(g_t)(x) \mathbb{V}(\tau \partial_\tau g_\tau)(x),$$

by the Cauchy-Schwarz inequality. The result now follows by setting $s = 1$ and using Cauchy's inequality with a parameter. $\square$

We record here also a weighted version of the Riesz transform estimates for $\mathcal{L}_\|$ and, more importantly for us, estimates for the Hodge decomposition associated to $\mathcal{L}_\|$.

Theorem 5.2.49 ([CMR18, Proposition 9.1])**.** *Let $\mathcal{L}_\| := -\operatorname{div}_\| A_\| \nabla_\|$. Then there exists $M > 0$ (depending only on dimension and the ellipticity of $A_\|$) such that if $\nu \in A_2$ verifies that $\nu^M \in A_2$, then*

$$\max \left\{ \|\nabla_\| \mathcal{L}_\|^{-1/2}\|_{L^2(\nu) \to L^2(\nu)} \, , \, \|\mathcal{L}_\|^{-1/2} \operatorname{div}_\| \|_{L^2(\nu) \to L^2(\nu)} \right\} \lesssim_{[\nu^M]_{A_2}} 1.$$

In particular, if for $f \in L^2(\mathbb{R}^n; \mathbb{C}^n)$ we write Hodge Decomposition $f = A_\| \nabla_\| F + H$ with $F \in \dot{W}^{1,2}(\mathbb{R}^n)$ and $\operatorname{div}_\| H = 0$, then for ν as above, we have that

$$\|\nabla_\| F\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} \|f\|_{L^2(\nu)}.$$

We end this subsection with an identity characterizing the double layer in terms of operators involving only the single layer. This will allow us to focus, as far as the square and non-tangential maximal function estimates are concerned, on operators involving only the single layer.

Lemma 5.2.50 (Double Layer Duality for L^2 functions)**.** *Denote by $\vec{N}$ the outward unit normal vector of the upper-half space. The following formula holds for each $f \in C_c^\infty(\mathbb{R}^n)$:*

$$\mathcal{D}_t^{\mathcal{L},+} f(x) = (\mathcal{S}_t^{\mathcal{L}} \nabla)(A\vec{N} f)(x) + (\mathcal{S}_t^{\mathcal{L}} \overline{B}_2)(\vec{N} f)(x).$$

Proof. We have, by Proposition 4.4.18 (ii), the following formula for $f, g \in C_c^\infty(\mathbb{R}^n)$

$$(\mathcal{D}_t^{\mathcal{L},+} f, g) = (f, \partial_{\nu_{-t}^{\mathcal{L}^*},+} \mathcal{S}^{\mathcal{L}^*} g).$$

On the other hand, since $\mathcal{S}^{\mathcal{L}^*} g$ is in $Y^{1,2}(\mathbb{R}^{n+1})$, we may use the L^2 realization of the conormal (see Lemma 4.4.11 (i)) from which it follows that

$$\begin{aligned}
(f, \partial_{\nu_{-t}^{\mathcal{L}^*},+} \mathcal{S}^{\mathcal{L}^*} g) &= \langle f, \partial_{\nu_{-t}^{\mathcal{L}^*},+} \mathcal{S}^{\mathcal{L}^*} g\rangle_{L^2(\mathbb{R}^n)} \\
&= \langle f, \vec{N} \cdot [A^* \nabla \mathcal{S}_{-t}^{\mathcal{L}^*} g + \overline{B}_2 \mathcal{S}_{-t}^{\mathcal{L}^*} g]\rangle_{L^2(\mathbb{R}^n)} \\
&= \langle \vec{N} f, [A^* \nabla \mathcal{S}_{-t}^{\mathcal{L}^*} g + \overline{B}_2 \mathcal{S}_{-t}^{\mathcal{L}^*} g]\rangle_{L^2(\mathbb{R}^n)^n} \\
&= \langle (\mathcal{S}_t^{\mathcal{L}} \nabla)(A\vec{N} f) + (\mathcal{S}_t^{\mathcal{L}} B_2)(\vec{N} f), g\rangle_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where we used the properties of the operator $(\mathcal{S}_t^{\mathcal{L}} \nabla)$ (see Proposition 4.4.2 (viii)) for the last line. This gives the desired identity for $f \in C_c^\infty(\mathbb{R}^n)$. □

5.2.4 Good Classes of Solutions

Let us now define the function spaces over which we will be able to prove uniqueness.

Definition 5.2.51 (Slice Spaces)**.** For $n \geq 3$, we define

$$D_+^2 := \Big\{ v \in C_0\big((0,\infty); L^2(\mathbb{R}^n)\big) : \|u\|_{D_+^2} < \infty \Big\},$$

with norm given by $\|v\|_{D_+^2} := \sup_{t>0} \|v(t)\|_{L^2(\mathbb{R}^n)}$. We also define

$$S_+^2 := \Big\{ u \in C_0^2\big((0,\infty); Y^{1,2}(\mathbb{R}^n)\big) : u'(t) \in C_0\big((0,\infty); L^2(\mathbb{R}^n)\big),\ \|u\|_{S_+^2} < \infty \Big\},$$

with norm given by

$$\|u\|_{S^2_+} := \sup_{t>0} \|u(t)\|_{Y^{1,2}(\mathbb{R}^n)} + \sup_{t>0} \|u'(t)\|_{L^2(\mathbb{R}^n)} \\ + \sup_{t>0} \|tu'(t)\|_{Y^{1,2}(\mathbb{R}^n)} + \sup_{t>0} \|t^2 u''(t)\|_{Y^{1,2}(\mathbb{R}^n)}.$$

In particular, both D^2_+ and S^2_+ are Banach spaces. Similarly, with obvious modifications, we can define the slice spaces S^2_- and D^2_- in the negative half line $(-\infty, 0)$.

Definition 5.2.52 (Good $\mathcal{D}$ Solutions)**.** We say that $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ is a good $\mathcal{D}$ solution if $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense, $u \in D^2_+$, and $u_\tau := u(\cdot, \cdot + \tau) \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ for any $\tau > 0$.

Definition 5.2.53 (Good $\mathcal{N}/\mathcal{R}$ Solutions)**.** We say that $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ is a good $\mathcal{N}/\mathcal{R}$ solution if $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ in the weak sense, $u \in S^2_+$, and $\partial_t u_\tau \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ for every $\tau > 0$.

As an immediate consequence of Theorems 4.6.12 and 4.6.17 from Chapter 4, we exhibit

Corollary 5.2.54. *Let $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ satisfy $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$.*

(i) If $\||t\nabla\partial_t u\|| < \infty$ and $\lim_{t\to\infty} \nabla u(t) = 0$ in the sense of distributions (see (4.2.2)), then either u is a good $\mathcal{N}/\mathcal{R}$ solution (in the case that $\mathcal{L}1 \neq 0$ in $\mathbb{R}^{n+1}_+$), or $u - c$ is a good $\mathcal{N}/\mathcal{R}$ solution for some constant c (in the case that $\mathcal{L} = 0$ in $\mathbb{R}^{n+1}_+$).

(ii) If $\||t\nabla u\|| < \infty$ and $\lim_{t\to\infty} u(t) = 0$ in the sense of distributions, then u is a good $\mathcal{D}$ solution.

The following result is a companion to Corollary 5.2.54. Together they will imply that our uniqueness statement holds among the two most commonly used classes of solutions (those with either square or non-tangential maximal function estimates).

Lemma 5.2.55. *Let $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ be a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. The following holds*

(i) If $\tilde{N}_2(u) \in L^2(\mathbb{R}^n)$, then u is a good $\mathcal{D}$ solution (see Definition 5.2.52).

(ii) If $\tilde{N}_2(\nabla u) \in L^2(\mathbb{R}^n)$ then either u is a good $\mathcal{N}/\mathcal{R}$ solution (see Definition 5.2.53) if $\mathcal{L}1 \neq 0$, or there exists a constant $c \in \mathbb{C}$ such that $u - c$ is a good $\mathcal{N}/\mathcal{R}$ solution if $\mathcal{L}1 = 0$.

Proof. As will be seen from the proof, (i) will follow the same outline as (ii), and is a bit easier. We first prove that

$$\sup_{t>0} \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_2(\nabla u)\|_{L^2(\mathbb{R}^n)}.$$

Fix $t>0$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be a nonnegative Lipschitz cutoff function such that $\psi(t)=1$, $\psi(3t/4)=0$, and $|\psi'(s)| \leq 4/t$ for each $s \in \mathbb{R}$. We make the computation

$$\begin{aligned}
\|\nabla u(\cdot,t)\|_2^2 &= \int_{\mathbb{R}^n} |\nabla u(\cdot,t)|^2 \psi(t) \\
&= \int_{\mathbb{R}^n} |\nabla u(\cdot,t)|^2 \psi(t) - \int_{\mathbb{R}^n} |\nabla u(\cdot,3t/4)|^2 \psi(3t/4) \\
&= \int_{\mathbb{R}^n} \int_{3t/4}^{t} \partial_s \Big[|\nabla u(x,s)|^2 \psi(s) \Big] \, ds \, dx \\
&\leq \int_{\mathbb{R}^n} \int_{3t/4}^{t} \Big[2|\nabla u(x,s)||\nabla \partial_s u(x,s)|\psi(s) + |\nabla u(x,s)|^2 |\psi'(s)| \Big] \, ds \, dx \\
&\leq 2 \int_{\mathbb{R}^n} \fint_{3t/4}^{t} |\nabla u(x,s)|^2 \, ds \, dx \; + \; \frac{t^2}{16} \int_{\mathbb{R}^n} \fint_{3t/4}^{t} |\nabla \partial_s u(x,s)|^2 \, ds \, dx =: I + II,
\end{aligned}$$

where in the third equality we used the fundamental theorem of calculus and in the last line we used the Cauchy inequality with $\varepsilon > 0$. We now use Fubini's theorem to see that

$$\begin{aligned}
I &= 2 \int_{\mathbb{R}^n} \fint_{|y-x_0|<t} \fint_{3t/4}^{t} |\nabla u(y,s)|^2 \, ds \, dx_0 \, dy \\
&\leq 8 \int_{\mathbb{R}^n} \fint\!\!\fint_{\substack{|x_0 - y| < t \\ |s-t| < t/2}} |\nabla u(y,s)|^2 \, ds \, dy \, dx_0 \\
&\leq 8 \int_{\mathbb{R}^n} \sup_{(x,\tau) \in \gamma(x_0)} \Big(\fint\!\!\fint_{\substack{|x - y| < \tau \\ |s-\tau| < \tau/2}} |\nabla u(y,s)|^2 \, ds \, dy \Big) \, dx_0 = 8 \|\tilde{\mathcal{N}}_2(\nabla u)\|_2^2.
\end{aligned}$$

It remains to control II; for this we will use the Caccioppoli inequality as follows:

$$\begin{aligned}
II \leq \frac{t^2}{16} \int_{\mathbb{R}^n} \fint_{|y-x_0|<t/2} \fint_{3t/4}^{t} |\nabla \partial_s u(y,s)|^2 \, ds \, dx_0 \, dy \lesssim \\
\int_{\mathbb{R}^n} \fint_{|x_0-y|<t} \fint_{t/2}^{5t/4} |\partial_s u(x,s)|^2 \, ds \, dy \, dx_0,
\end{aligned}$$

and thus it is clear that we may handle II as above. We have obtained that

$$\|\nabla u(\cdot,t)\|_2 \lesssim \|\tilde{\mathcal{N}}_2(\nabla u)\|_2, \qquad \text{for each } t>0.$$

Taking supremum over $t>0$ yields the desired result.

We now improve this to

$$\lim_{t\to\infty} \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0,$$

where $\nabla = (\nabla_\|, \partial_t)$ is the full gradient in $n+1$ variables. This follows from the above estimate on slices: Notice that the proof actually gives that

$$\|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_2^{(t)}(\nabla u)\|_{L^2(\mathbb{R}^n)},$$

where we use the truncated non-tangential maximal function (see Definition 5.2.5) on the right hand side.

We claim now that $\tilde{\mathcal{N}}_2^{(t)}(\nabla u)(x) \to 0$ for every $x \in \mathbb{R}^n$ as $t \to \infty$. To see this, assume to the contrary that

$$\limsup_{t\to\infty} \tilde{\mathcal{N}}_2^{(t)}(\nabla u)(x) > \eta > 0,$$

for some $x \in \mathbb{R}^n$. This means there exists a sequence $t_k \to \infty$ and points x_k with $|x - x_k| < t$ such that

$$⨌_{\mathcal{C}_{x_k,t_k}} |\nabla u(y,s)|^2 \, dyds > \eta^2.$$

By the definition of the non-tangential maximal function we then have

$$\tilde{\mathcal{N}}_2(\nabla u)(z)^2 \geq ⨌_{\mathcal{C}_{x_k,t_k}} |\nabla u(y,s)|^2 \, dyds > \eta^2,$$

for every $z \in \mathbb{R}^n$ such that $|z - x_k| < t_k$. Integrating over this set gives

$$\|\tilde{\mathcal{N}}_2(\nabla u)\|_{L^2(\mathbb{R}^n)}^2 \geq \int_{|z-x_k|<t_k} \tilde{\mathcal{N}}_2(\nabla u)(z)^2 \, dz \geq c_n \eta^2 t_k^n.$$

Since $t_k \to \infty$, this contradicts our assumption that $\tilde{\mathcal{N}}_2(\nabla u) \in L^2(\mathbb{R}^n)$.

The claim now proved, and since $\tilde{\mathcal{N}}_2^{(t)}(\nabla u) \leq \tilde{\mathcal{N}}_2(\nabla u)$ by definition, the dominated convergence theorem gives

$$\|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_2^{(t)}(\nabla u)\|_{L^2(\mathbb{R}^n)} \to 0,$$

as $t \to \infty$.

Appealing to Caccioppoli's inequality and the above, together with Proposition 4.6.14, we see that $u \in S_+^2$ when $\mathcal{L}1 \neq 0$. If $\mathcal{L}1 = 0$, we proceed as follows: First, by the sup on slices estimate above and Caccioppoli's inequality on slices we see that $\partial_t u(\cdot,t) \in W^{1,2}(\mathbb{R}^n)$ for every $t > 0$; in particular

$$\int_s^t \partial_\tau u(\cdot,\tau)\, d\tau \in W^{1,2}(\mathbb{R}^n) \subset Y^{1,2}(\mathbb{R}^n), \quad \forall 0 < s < t < \infty.$$

On the other hand, again by the sup on slices and Lemma 4.2.1, we have that for every $t > 0$ there exists a constant $c_t \in \mathbb{C}$ such that $u(\cdot,t) - c_t \in Y^{1,2}(\mathbb{R}^n)$. Therefore, by the fundamental theorem of calculus, for any $0 < s < t < \infty$,

$$\int_s^t \partial_\tau u(\cdot,\tau)\, d\tau - (c_t - c_s) = u(\cdot,t) - c_t - [u(\cdot,s) - c_s] \in Y^{1,2}(\mathbb{R}^n).$$

We conclude $c_t = c_s = c$ as desired, and so $u - c \in S_+^2$.

Finally we show that $\partial_t u_\tau := \partial_t u(\cdot,\cdot+\tau) \in Y^{1,2}(\mathbb{R}_+^{n+1})$ for every $\tau > 0$. For this we simply compute, decomposing $\mathbb{R}^n$ into cubes in $\mathbb{D}_s$ and using Caccioppoli's inequality on slices together with Fubini's Theorem

$$\begin{aligned}\iint_{\mathbb{R}_+^{n+1}} |\nabla\partial_t u(y,s+\tau)|^2\, dyds &= \int_\tau^\infty \int_{\mathbb{R}^n} |\nabla\partial_t u(y,s)|^2\, dyds \\ &\lesssim \sup_{t>0} \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \int_\tau^\infty \frac{ds}{s^2} < \infty.\end{aligned}$$

For (i) we run the same argument with u in place of ∇u. □

5.3 Two General Extrapolation Results

In this section we prove two extrapolation theorems for conical and vertical square functions. The takeaway from these considerations seems to be that conical square functions have good estimates in the range (r, ∞) in the presence of $L^r - L^2$ off-diagonal estimates plus an L^2 square function bound. The vertical square function on the other hand requires (for our argument) that the operator satisfies a reverse Hölder inequality (and in fact, in this case we see that the vertical square function is controlled by the conical on an interval around $p = 2$; this should be compared with Proposition 5.2.4 which is optimal for general functions F, see [AHM12, Proposition 2.1 (c)]).

Lemma 5.3.1 (Extrapolation for Conical Square Functions)**.** *Suppose T_t is an operator satisfying, for $q = 2$ and some $r < 2$, the off-diagonal estimates*[7] *in Definition 5.2.38 with $\gamma > 1/r$. (Notice this allows us to define $T_t 1$ as an element of L^2_{loc}.) Set $R_t := T_t - T_t 1 \cdot P_t$ for a given approximate identity P_t with compactly supported kernel of the form $P_t = \tilde{P}_t \tilde{P}_t$ for another approximate identity $\tilde{P}_t$. Finally assume that for every $f \in L^2(\mathbb{R}^n)$*

$$\|\mathcal{S}(T_t f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

and

$$\|R_t \mathcal{Q}_s^2 f\|_{L^2(\mathbb{R}^n)} \lesssim \left(\frac{s}{t}\right)^{\beta} \|\mathcal{Q}_s f\|_{L^2(\mathbb{R}^n)}, \quad s \leq t \tag{5.3.2}$$

for some (and therefore any) CLP family $\mathcal{Q}_s$ (see Definition 4.2.26) and some $\beta > 0$.

Then

$$\|\mathcal{S}(T_t f)\|_{L^2(\nu)} \lesssim \|f\|_{L^2(\nu)}, \quad \forall \nu \in A_{2/r}. \tag{5.3.3}$$

In particular

$$\|\mathcal{S}(T_t f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall p \in (r, \infty).$$

The above lemma can be thought of as a Calderón-Zygmund-type theorem. In this case the off-diagonal decay plays the role of the usual size condition while the quasi-orthogonality estimate for R_t plays the role of Hölder continuity of the kernel.

Note also that the case $t \leq s$ in the quasi-orthogonality estimate (5.3.2) is a conse-

[7] In fact we will only need the first and second estimates in Definition 5.2.38 for T_t, in the range $|t| \approx \ell(Q)$

quence of the off-diagonal decay of R_t and [AAA+11, Lemma 3.5]. Therefore, with the off-diagonal decay of R_t as a background assumption, (5.3.2) is equivalent to

$$\|R_t \mathcal{Q}_s^2 f\|_{L^2(\mathbb{R}^n)} \lesssim \min\left(\frac{t}{s}, \frac{s}{t}\right)^{\beta} \|\mathcal{Q}_s f\|_{L^2(\mathbb{R}^n)}.$$

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$

We begin by writing

$$T_t f(x) = R_t f(x) + [T_t 1(x)] \cdot P_t f(x), \tag{5.3.4}$$

where R_t and P_t are as in the hypotheses. To handle the first term we use interpolation with change of measure (see the proof of Theorem 5.2.23) to reduce the weighted estimate of $\mathcal{S}(R_t)$ to the pair of estimates

$$\left\|\left(\fint_{|x-y|<t} |R_t \mathcal{Q}_s^2 f(y)|^2\, dy\right)^{1/2}\right\|_{L^2(\mathbb{R}^n)} \lesssim \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\beta} \|\mathcal{Q}_s f\| L^2(\mathbb{R}^n), \tag{5.3.5}$$

for some $\beta > 0$, and

$$\left\|\left(\fint_{|x-y|<t} |R_t \mathcal{Q}_s^2 f(y)|^2\, dy\right)^{1/2}\right\|_{L^2(\nu)} \lesssim_{[\nu]_{A_{2/r}}} \|\mathcal{Q}_s f\|_{L^2(\nu)}, \tag{5.3.6}$$

for r as in the statement of the lemma.

The unweighted quasi-orthogonality estimate (5.3.5) follows from Fubini's Theorem and the good off-diagonal decay.

The uniform weighted estimate follows from Proposition 5.2.39 and the fact that $|\mathcal{Q}_s h(x)| \lesssim \mathcal{M}h(x)$ and $\mathcal{M}$ is bounded on $L^2(\nu)$ (because $A_{2/r} \subset A_2$). This shows the desired weighted estimate, and so by interpolation with change of measure,

$$\left\|\left(\fint_{|x-y|<t} |R_t \mathcal{Q}_s^2 f(y)|^2\, dy\right)^{1/2}\right\|_{L^2(\nu)} \lesssim \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\beta} \|\mathcal{Q}_s f\|_{L^2(\nu)},$$

(for a possibly smaller β than the one for (5.3.5)). The estimate

$$\|\mathcal{S}(R_t f)\|_{L^2(\nu)} \lesssim \|f\|_{L^2(\nu)}$$

now follows from a standard quasi-orthogonality argument, once one realizes that if

$$\tilde{R}_t h(x) := \Big(\fint_{|x-y|<t} |R_t h(y)|^2 \, dy \Big)^{1/2},$$

then $\mathcal{S}(R_t) = \mathbb{V}(\tilde{R}_t)$.

Now it remains to establish the square function bound for $T_t 1(x) \cdot P_t$. For this we first claim that the measure

$$d\mu(x,t) := \Big(\fint_{|x-y|<t} |T_t 1(y)|^2 \, dy \Big) \, d\nu(x) \frac{dxdt}{t},$$

is a ν-Carleson measure, i.e. for every cube Q

$$\mu(R_Q) \lesssim \nu(Q), \qquad R_Q := Q \times (0, \ell(Q)).$$

Assuming the claim we would have, by a weighted version of Carleson's lemma (Lemma 5.2.28), and using the fact that $P_t = \tilde{P}_t \tilde{P}_t$ implies $|P_t f(y)| \lesssim \tilde{P}_t(\mathcal{M} f)(x)$ whenever $|x-y| < t$,

$$\begin{aligned} \int_{\mathbb{R}^n} \iint_{\gamma(x)} |(T_t 1(y))|^2 |P_t f(y)|^2 \, \frac{dydt}{t^{n+1}} \, \nu(x) dx &\lesssim \int_{\mathbb{R}^n} \int_0^\infty |\tilde{P}_t(\mathcal{M} f)(x)|^2 \, d\mu(x,t) \\ &\lesssim \int_{\mathbb{R}^n} \mathcal{N}(\tilde{P}_t(\mathcal{M} f))(x)^2 \, \nu(x) dx \\ &\lesssim \int_{\mathbb{R}^n} \mathcal{M}(\mathcal{M} f)(x)^2 \, \nu(x) dx \\ &\lesssim \int_{\mathbb{R}^n} |f(x)|^2 \, \nu(x) dx, \end{aligned}$$

where we used the fact that $\mathcal{M} : L^2(\nu) \to L^2(\nu)$ since $r > 1$. This accounts for the contribution of the second term in (5.3.4), using Theorem 5.2.23.

To prove the claim we invoke a version of the John-Nirenberg lemma for Carleson measures (see Lemma 5.2.16) and the reverse Hölder inequality for $A_{2/r}$ weights (see Proposition 5.2.19) in the following way: For a fixed cube $Q \subset \mathbb{R}^n$, using Hölder's inequality

$$\begin{aligned}
\mu(R_Q) &= \int_Q \int_0^{\ell(Q)} \fint_{|x-y|<t<\ell(Q)} |T_t 1(y)|^2 \frac{dydt}{t}\,\nu(x)dx \\
&= \int_Q \Big(\iint_{|x-y|<t<\ell(Q)} |T_t 1(y)|^2 \frac{dydt}{t^{n+1}} \Big) \nu(x)\,dx \\
&=: \int_Q A_Q^2(x)\,\nu(x)dx \\
&\le \Big(\int_Q A_Q^{2(1+\delta_1)}\,dx \Big)^{1/(1+\delta_1)} \Big(\int_Q \nu^{1+\delta_2}\,dx \Big)^{1/(1+\delta_2)},
\end{aligned}$$

where $\delta_1 = 1/\delta_2$ and $1+\delta_2$ is the exponent corresponding to the reverse Hölder inequality for ν so that

$$\begin{aligned}
\mu(R_Q) &\lesssim \Big(\int_Q A_Q^{2(1+\delta_1)}\,dx \Big)^{1/(1+\delta_1)} |Q|^{-1+1/(1+\delta_2)} \nu(Q) \\
&\lesssim |Q|^{1/(1+\delta_2)} |Q|^{-1+1/(1+\delta_1)} \nu(Q) \\
&= \nu(Q),
\end{aligned}$$

where we used the John-Nirenberg Lemma for local square functions (see Lemma 5.2.16) in the second to last line.

We should remark here that the implicit constant depends on δ_1 and the constant in the reverse Hölder inequality for ν, but these in turn depend only on $[\nu]_{A_{2/r}}$ (see for instance [Ste93b]).

This finishes the proof of the weighted estimate (5.3.3). The unweighted result now follows from Corollary 5.2.22. □

We now proceed to the extrapolation result for vertical square functions. The idea will be the same: reduce matters to a weighted L^2 estimate, however notice that we used crucially the properties of cones in both the weighted estimates for R_t and the Carleson measure estimate for T_t in Lemma 5.3.1. In order to handle this issue we will transform $\mathbb{V}(T_t)$ into $\mathcal{S}(\tilde{T}_t)$ for an appropriate $\tilde{T}$ involving the weight; this makes the analysis more involved than in Lemma 5.3.1.

Lemma 5.3.7 (A General Extrapolation Result for Vertical Square Functions)**.** *Let T_t be an operator satisfying, for some $r < 2 < q$ and $\delta \in (0,1)$, the $L^r - L^q$ off-diagonal* *estimates in Definition 5.2.38 for some* $\gamma > -1/n + 2/r + \log_2(C_\delta)/n$ *(here* C_δ *is as in (7) of Proposition 5.2.19). We will also require that, for every cube* $Q \subset \mathbb{R}^n$

$$\Big(\fint\!\!\!\fint_{I(Q)} |T_t f(x)|^q \, dxdt\Big)^{1/q} \lesssim \Big(\fint\!\!\!\fint_{I(Q^*)} |S_t f(x)|^2 \, dxdt\Big)^{1/2}, \tag{5.3.8}$$

where as usual $I(Q) = Q \times (\ell(Q)/2, \ell(Q))$ *and* S_t *is an operator satisfying*

$$\|\mathcal{S}(S_t)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} < \infty.$$

We refer to (5.3.8) as the reverse Hölder assumption.

The assumption on γ *allows us to define* $T_t 1$ *as an element of* L^2_{loc} *and we set*

$$R_t f(x) := [T_t - T_t 1(x) \cdot P_t](f)(x),$$

for some approximate identity P_t *with compactly supported kernel. If* R_t *satisfies the quasi-orthogonality estimate*

$$\|R_t \mathcal{Q}_s^2 f\|_{L^2(\mathbb{R}^n)} \lesssim \Big(\frac{s}{t}\Big)^{\beta} \|f\|_{L^2(\mathbb{R}^n)}, \quad s < t,$$

for all $f \in L^2(\mathbb{R}^n)$ *and some* $\beta > 0$*; and if* T_t *satisfies the* L^2 *square function estimate*

$$\|\mathbb{V}(T_t f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

Then if $\nu \in RH_M \cap A_1$ *for* $M > \max(2r/(r-2), (q/2)')$ *and* $[\nu]_{A_1} \leq C_\delta$,

$$\|\mathbb{V}(T_t f)\|_{L^2(\nu)} \lesssim \|f\|_{L^2(\nu)}.$$

In particular, for any $p \in (2 - \delta/M, 2 + \delta/M)$ *it holds*

$$\|\mathbb{V}(T_t f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

If $T_t = 0$*, i.e.* $T_t = R_t$*, then we can dispense of the reverse Hölder assumption.*

Proof. We first note that, by Proposition 5.2.4, in the range $p < 2$ we have

$$\|\mathbb{V}(T_t f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{S}(T_t f)\|_{L^p(\mathbb{R}^n)},$$

and for $r < p < 2$, by Lemma 5.3.1 (recall that vertical and conical square functions coincide on L^2) and Corollary 5.2.22, we have

$$\|\mathcal{S}(T_t f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Therefore, it is enough to consider the case $p > 2$.

We now proceed to write our vertical square function into a conical square function by introducing an average adapted to ν. For this purpose we set, for $x \in \mathbb{R}^n$ and $t > 0$ fixed,

$$\nu_{x,t} := \fint_{|x-y|<t} \nu(y)\, dy.$$

We now write, using Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^n} |\mathbb{V}(T_t f)(x)|^2\, \nu(x)dx &= \int_{\mathbb{R}^n} \int_0^\infty |T_t f(x)|^2\, \frac{dt}{t}\, \nu(x)dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty \fint_{|x-y|<t} \frac{\nu(y)}{\nu_{x,t}} |T_t f(x)|^2\, dy \frac{dt}{t^{n+1}}\, \nu(x)dx \\
&= \int_{\mathbb{R}^n} \iint_{|x-y|<t} \frac{\nu(x)}{\nu_{x,t}} |T_t f(x)|^2\, \frac{dxdt}{t^{n+1}}\, \nu(y)dy \\
&=: \int_{\mathbb{R}^n} \iint_{|x-y|<t} |\tilde{T}_t f(x)|^2\, \frac{dxdt}{t^{n+1}}\, \nu(y)dy \\
&= \|\mathcal{S}(\tilde{T}_t f)\|_{L^2(\nu)}.
\end{aligned}$$

We are now in a position to try and mimic the proof of Lemma 5.3.1. Unfortunately the process is quite a bit more involved and, rather than proving a full weighted estimate, we will use the specific form of our weight ν. To simplify notation we introduce the operators:

$$\tilde{R}_t f(x) = \sqrt{\frac{\nu(x)}{\nu_{x,t}}} R_t,$$

where R_t is as in the statement of the Lemma. With this in hand we write

$$\tilde{T}_t f(x) = \tilde{R}_t f(x) + \tilde{T}_t 1(x) \cdot P_t f(x).$$

As was done in the case of the conical square function, to handle the second term it is enough to show the ν-Carleson measure estimate (see Lemma 5.2.28)

$$\mu(R_Q) \lesssim \nu(Q),$$

for every cube $Q \subset \mathbb{R}^n$, where $R_Q := Q \times (0, \ell(Q))$ and the measure μ is defined as

$$d\mu(x,t) := \Big(\fint_{|x-y|<t} |\tilde{T}_t 1(y)|^2 \, dx \Big) \, \nu(x) \frac{dxdt}{t}.$$

For this we first reduce matters to an unweighted estimate via the John-Nirenberg lemma for local square functions (see Lemma 5.2.16) as follows: Notice that, for any $\bar{q} > 1$,

$$\begin{aligned}
\mu(R_Q) &= \int_Q \Big(\iint_{|x-y|<t<\ell(Q)} |\tilde{T}_t 1(x)|^2 \, \frac{dxdt}{t^{n+1}} \Big) \, \nu(y) \, dy \\
&=: \int_Q A_Q^2(y) \, \nu(y) \, dy \\
&\leq \Big(\int_Q A_Q^{2\bar{q}} \, dy \Big)^{1/\bar{q}} \Big(\int_Q \nu(y)^{\bar{q}'} \, dy \Big)^{1/\bar{q}'} \\
&\lesssim_{[\nu]_{RH_{\bar{q}'}}} \Big(\int_Q A_Q^{2\bar{q}} \Big)^{1/\bar{q}} |Q|^{-1/\bar{q}} \int_Q \nu(y) \, dy \\
&= \Big(\fint_Q A_Q^{2\bar{q}} \, dy \Big)^{1/\bar{q}} \nu(Q),
\end{aligned}$$

where as before the quantity $[\nu]_{RH_{\bar{q}'}}$ is admissible if say $M > \bar{q}'$(see Proposition 5.2.19). Therefore it is enough to show that

$$\Big(\fint_Q A_Q^{2\bar{q}} \, dy \Big)^{1/\bar{q}} \lesssim 1.$$

By the John-Nirenberg lemma for local square functions (see Lemma 5.2.16), it is enough to show

$$\fint_Q A_Q^2 \, dy \lesssim 1.$$

Using Fubini's theorem, we see that this last estimate is equivalent to the unweighted

Carleson measure estimate

$$\int_0^{\ell(Q^*)} \int_{Q^*} |\tilde{T}_t 1(x)|^2 \, \frac{dxdt}{t} \lesssim |Q^*|, \tag{5.3.9}$$

where as before $Q^* = c_n Q$ is a dilate of Q. Since the above has to hold for every cube, we write Q in place of Q^* in what follows. Moreover, since the quantity $\nu/\nu_{x,t}$ is invariant under scalar multiplication of ν by a positive constant, for a fixed cube Q we may assume that $\nu(Q)/|Q| = 1$.

First we use a stopping time argument to deal with $\nu_{x,t}$: For a fixed constant $A^{-1} < 1/4$, to be selected later, we let $\{Q_j\}_{j\in\mathbb{N}}$ be the collection of maximal dyadic sub-cubes of Q with respect to the conditions

$$\fint_{Q_j} \nu(x)\, dx > A, \quad \text{or} \quad \fint_{Q_j} \nu(x)\, dx < A^{-1}.$$

We say $j \in I_1$ if the first condition holds, and $j \in I_2$ if the second does.

By the first condition we have

$$\sum_{j\in I_1} |Q_j| < \sum_{j\in I_1} A^{-1} \int_{Q_j} \nu(x)\, dx \le A^{-1} \int_Q \nu(x)\, dx = A^{-1}|Q|,$$

by our condition $\nu(Q)/|Q| = 1$. On the other hand if $j \in I_2$

$$\int_{Q_j} \nu(x)\, dx < A^{-1}|Q_j|, \quad \text{and} \quad \int_{Q_j^*} \nu(x)\, dx \ge A^{-1}|Q_j^*|,$$

where Q_j^* is the dyadic parent of Q_j. Therefore

$$\sum_{j\in I_2} \nu(Q_j) \le \sum_{j\in I_2} A^{-1}|Q_j| \le A^{-1}|Q| = A^{-1}\nu(Q).$$

By the A_∞ property of ν we can choose A, depending only on the A_1 characteristic of ν, small enough such that the above inequality implies

$$\Big| \cup_{j\in I_2} Q_j \Big| < \frac{1}{2}|Q|.$$

Combining this with the corresponding estimate for I_1, and using the fact that the cubes

Q_j are pairwise disjoint,

$$\sum_{j\geq 0}|Q_j| < B|Q|,$$

for $B = 1/2 + A^{-1} < 1$. By the John-Nirenberg lemma for Carleson measures (see Lemma 5.2.14), the above implies that it is enough to show the estimate

$$\iint_{E_Q} \frac{\nu(x)}{\nu_{x,t}}|T_t 1(x)|^2\, dx\frac{dxdt}{t} \leq C_0|Q|, \tag{5.3.10}$$

where we define the sawtooth region

$$E_Q := R_Q \backslash \Big(\cup_{j\geq 0} R_{Q_j}\Big).$$

To handle (5.3.10) we first claim the following:

$$\nu_{x,t} \gtrsim 1, \qquad \forall (x,t) \in E_Q, \tag{5.3.11}$$

with implicit constants depending only on the doubling constant of ν. To see this fix $(x,t) \in E_Q$ and consider first the case $x \notin \cup_{j\geq 0} Q_j$ so that

$$A^{-1} \leq \fint_{Q'} \nu(y)\, dy \leq A,$$

for any dyadic subcube $Q' \in \mathbb{D}(Q)$ containing x. In particular, choosing $Q'_t \in \mathbb{D}_t(Q)$ with this property, and using the doubling property of ν we see

$$A^{-1} \leq \fint_{Q'_t} \nu(y)\, dy \approx \fint_{|x-y|<t} \nu(y)\, dy = \nu_{x,t}.$$

On the other hand if $x \in Q_j$ for some $j \geq 0$ we proceed as follows: If $t > 4\ell(Q_j)$, it means that, if as before $Q'_t \in \mathbb{D}_t(Q)$ is the unique dyadic subcube of Q containing x, then Q'_t is not in the collection $\{Q_j\}_j$ so by definition

$$A^{-1} \leq \fint_{Q'_t} \nu(y)\, dy \leq A,$$

and we conclude as before since this average is comparable, by doubling of ν, to $\nu_{x,t}$. If $\ell(Q_j) \leq t \leq 4\ell(Q_j)$ (the first inequality owing to the definition of E_Q) then by definition the dyadic parent $\tilde{Q}_j$ of Q_j satisfies

$$A^{-1} \leq \fint_{\tilde{Q}_j} \nu(y)\, dy \leq A,$$

so that, again by doubling of ν, the claim follows.

We conclude, using (5.3.10) and (5.3.11), that it is enough to establish (recall $\nu(Q) = |Q|$)

$$\iint_{E_Q} |T_t 1(x)|^2 \nu(x)\, \frac{dxdt}{t} \leq C_0 |Q|. \tag{5.3.12}$$

To show this we first fix $\psi = \psi_Q \in C_c^\infty(4Q)$ with the property that $\psi \equiv 1$ in $2Q$ and $0 \leq \psi \leq 1$, so that

$$\begin{aligned}
\iint_{E_Q} |T_t 1(x)|^2 \nu(x)\, \frac{dxdt}{t} &\lesssim \iint_{E_Q} |T_t \psi(x)|^2 \nu(x)\, \frac{dxdt}{t} \\
&\quad + \iint_{E_Q} |T_t (1-\psi)(x)|^2 \nu(x)\, \frac{dxdt}{t} \\
&=: II + III.
\end{aligned}$$

We first handle III: Using Hölder's inequality, with $q/2 > 1$ as in the hypotheses, we recall that we have chosen $M > (q/2)'$ so that $\nu \in RH_{(q/2)'}$ (see Proposition 5.2.19),

$$\begin{aligned}
III &\leq \int_0^{\ell(Q)} \int_Q |T_t(1-\psi)|^2 \nu(x)\, \frac{dxdt}{t} \\
&\leq \int_0^{\ell(Q)} \Big(\int_Q \nu^{(q/2)'}(x)\, dx \Big)^{1/(q/2)'} \Big(\int_Q |T_t(1-\psi)|^q\, dx \Big)^{2/q} \frac{dt}{t} \\
&\lesssim \int_0^{\ell(Q)} |Q|^{2/q} \nu(Q) \Big(\fint_Q |T_t(1-\psi)|^q\, dx \Big)^{2/q} \frac{dt}{t} \\
&= \int_0^{\ell(Q)} |Q|^{1-2/q} \Big(\int_Q |T_t(1-\psi)|^q\, dx \Big)^{2/q} \frac{dt}{t},
\end{aligned}$$

where we used the normalization $\nu(Q)/|Q| = 1$ in the last line. Now, since T_t satisfies $L^2 - L^q$ off-diagonal estimates (see Definition 5.2.38), using as usual $R_j = R_j(Q) = 2^{j+1}Q \backslash 2^j Q$ for $j \geq 1$ and recalling that $1 - \psi \equiv 1$ outside of $4Q$,

$$\Big(\int_Q |T_t(1-\psi)|^q\,dx\Big)^{1/q} \le \sum_{j\ge 1}\Big(\int_Q |T_t[(1-\psi)\mathbb{1}_{R_j}]|^q\,dx\Big)^{1/q}$$

$$\le \sum_{j\ge 1} 2^{-nj\gamma_1}\Big(\frac{t}{(2^j\ell(Q))}\Big)^{n\gamma_2}\ell(Q)^{n(1/q-1/2)}\Big(\int_{R_j}|1-\psi|^2\,dx\Big)^{1/2}$$

$$\lesssim \sum_{j\ge 1} 2^{-nj\gamma}\Big(\frac{t}{\ell(Q)}\Big)^{n\gamma_2}\ell(Q)^{n(1/q-1/2)}|R_j|^{1/2}$$

$$\lesssim \sum_{j\ge 1} 2^{-nj(\gamma-1/2)}\Big(\frac{t}{\ell(Q)}\Big)^{n\gamma_2}|Q|^{1/q}$$

$$\lesssim \Big(\frac{t}{\ell(Q)}\Big)^{n\gamma_2}|Q|^{1/q}$$

since $\gamma > 1/2$. Plugging this into the estimate for III above we see, since $\gamma_2 > 0$,

$$III \lesssim \int_0^{\ell(Q)} |Q|\Big(\frac{t}{\ell(Q)}\Big)^{2n\gamma_2}\,\frac{dt}{t} \lesssim |Q|.$$

This is the desired estimate for III.

To handle II we first define, for $Q' \in \mathbb{D}(Q)$,

$$I(Q') := \{(x,t) \in R_Q : x \in Q',\ \ell(Q')/2 < t \le \ell(Q')\},$$

the Whitney region in $\mathbb{R}^{n+1}_+$ associated to Q'. We see that

$$II = \sum_{\substack{Q'\in\mathbb{D}(Q)\\ Q'\cap Q_j=\emptyset,\,\forall j}} \iint_{I(Q')} |T_t\psi(x)|^2\nu(x)\,\frac{dxdt}{t}$$

We now use Hölder's Inequality with $q > 2$ so that the L^q reverse Hölder inequality for T_t holds, again noting that we have chosen M large enough to guarantee $\nu \in RH_{(q/2)'}$, to conclude

$$II \le \sum_{\substack{Q'\in\mathbb{D}(Q)\\ Q'\not\subset Q_j,\,\forall j}} \Big(\iint_{I(Q')} |T_t\psi|^q\,\frac{dxdt}{t}\Big)^{2/q}\Big(\iint_{I(Q')} \nu^{(q/2)'}\,\frac{dxdt}{t}\Big)^{1/(q/2)'}$$

$$\lesssim \sum_{\substack{Q'\in\mathbb{D}(Q)\\ Q'\not\subset Q_j,\,\forall j}} |Q'|^{1/(q/2)'}\Big(\iint_{I(Q')} |T_t\psi|^q\,\frac{dxdt}{t}\Big)^{2/q},$$

where we used that for Q' satisfying $Q' \not\subset Q_j$ for all j (i.e. for Q' not contained in any of the Q_j) we have, by construction of the Q_j,

$$\fint_{Q'} \nu(x)\, dx \approx 1.$$

We now use reverse Hölder assumption on T_t to obtain

$$\Big(\fint\!\!\fint_{I(Q')} |T_t \psi|^q \, \frac{dxdt}{t} \Big)^{2/q} \lesssim \fint\!\!\fint_{I([Q']^*)} |S_t \psi|^2 \, \frac{dxdt}{t},$$

Therefore, using this in the estimate for II,

$$\begin{aligned} II &\lesssim \sum_{\substack{Q' \in \mathbb{D}(Q) \\ Q' \not\subset Q_j, \forall j}} |Q'|^{1/p'} |Q'|^{1/p} \fint\!\!\fint_{I([Q']^*)} |S_t \psi|^2 \, dxdt \\ &\lesssim \sum_{\substack{Q' \in \mathbb{D}(Q) \\ Q' \not\subset Q_j, \forall j}} \iint_{I([Q']^*)} |S_t \psi|^2 \, \frac{dxdt}{t} \\ &\lesssim \iint_{R_{Q^*}} |S_t \psi|^2 \, \frac{dxdt}{t}. \end{aligned}$$

We conclude from the fact that T_t satisfies an $L^2(\mathbb{R}^n)$ square function estimate and $\|\psi\|_{L^2(\mathbb{R}^n)} \lesssim |Q|^{1/2}$ by construction.

Combining the estimates for II and III, (5.3.12) follows and thus, by our previous reductions, we have shown

$$\|\mathcal{S}(T_t 1 \cdot P_t f)\|_{L^2(\nu)} \lesssim \|f\|_{L^2(\nu)}.$$

It remains to handle the contribution of $\tilde{R}_t$. Notice that so far, we've only required that $\nu \in A_{2/r}$ and $\gamma > 1/r$. The extra assumptions will be needed in order to handle $\tilde{R}_t$.

Again as in the proof of Lemma 5.3.1 we will appeal to interpolation with change of measure (see Theorem 5.2.23). For this it is enough to prove the following pair of estimates:

$$\Big\| \Big(\fint_{|x-y|<t} |\tilde{R}_t \mathcal{Q}_s^2 f(y)|^2 \, dy \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^n)} \lesssim_{[\nu^M]_{A_1}} \min\Big(\frac{t}{s}, \frac{s}{t}\Big)^{\beta} \|\mathcal{Q}_s f\|_{L^2(\mathbb{R}^n)}, \quad (5.3.13)$$

valid for some (and therefore all) Littlewood-Paley family $(\mathcal{Q}_s)_s$ and some $\beta > 0$; and

$$\Big\| \Big(\fint_{|x-y|<t} |\tilde{R}_t \mathcal{Q}_s^2 f(y)|^2 \, dy \Big)^{1/2} \Big\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_1}} \|\mathcal{Q}_s f\|_{L^2(\nu)}. \tag{5.3.14}$$

We remark that in the first quasi-orthogonality estimate (5.3.13), even though the estimate itself is unweighted, $\tilde{R}_t$ still has a dependence on ν.

The uniform $L^2(\nu)$ estimate is handled the same way it was done for the conical; setting $h := \mathcal{Q}_s^2 f$ we see

$$\begin{aligned} &\Big(\int_{\mathbb{R}^n} \fint_{|x-y|<t} |\tilde{R}_t h(y)|^2 \, dy \nu(x) dx \Big)^{1/2} \\ &\qquad \lesssim \Big(\sum_{Q \in \mathbb{D}_t} \int_Q \int_{|x-y|<t} |\tilde{R}_t h(y)|^2 \, dy \, \nu(x) dx \Big)^{1/2} \\ &\qquad\qquad \lesssim \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \int_Q \frac{\nu(y)}{\nu_{y,t}} |R_t h(y)|^2 \, dy \, \nu(x) dx \Big)^{1/2}. \end{aligned}$$

By Hölder's Inequality with exponent $q/2 > 1$ and $M > (2/q)'$, we see

$$\begin{aligned} \int_Q \frac{\nu(y)}{\nu_{y,t}} |R_t h(y)|^2 \, dy \, \nu(x) dx &\leq \Big(\int_Q \Big| \frac{\nu(y)}{\nu_{y,t}} \Big|^{(q/2)'} dy \Big)^{1/(q/2)'} \Big(\int_Q |R_t h(y)|^q \, dy \Big)^{2/q} \\ &\lesssim_{[\nu^M]_{A_1}} |Q|^{1-2/q} \Big(\int_Q |R_t h(y)|^q \, dy \Big)^{2/q}. \end{aligned}$$

Plugging this into the first estimate, we can now proceed as in the conical case (see Lemma 5.3.1), exploiting the $L^r - L^q$ off-diagonal decay in place of the $L^r - L^2$.

For the quasi-orthogonality estimate we proceed as follows: For this we will exploit the off-diagonal decay that $\tilde{R}_t$ inherits from R_t. More explicitly we have, for fixed $t, s > 0$

using Fubini's Theorem and duality

$$\begin{aligned}
\int_{\mathbb{R}^n} \fint_{|x-y|<t} |\tilde{R}_t \mathcal{Q}_s^2 f(x)|^2 \, dx \, dy &= \int_{\mathbb{R}^n} \fint_{|x-y|<t} \frac{\nu(x)}{\nu_{x,t}} |R_t \mathcal{Q}_s^2 f(x)|^2 \, dx \, dy \\
&= \int_{\mathbb{R}^n} \frac{\nu(x)}{\nu_{x,t}} |R_t \mathcal{Q}_s^2 f(x)|^2 \, dx \\
&= \int_{\mathbb{R}^n} \frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s^2 f(x) \cdot \overline{R_t \mathcal{Q}_s^2 f(x)} \, dx \, dy \\
&= \int_{\mathbb{R}^n} R_t^* \Big(\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s^2 f \Big)(x) \cdot \overline{\mathcal{Q}_s^2 f(x)} \, dx \\
&\le \| R_t^* ((\nu(x)/\nu_{x,t} R_t Q_s^2 f) \|_{L^2(\mathbb{R}^n)} \| Q_s^2 f \|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where R_t^* is the adjoint of R_t, for fixed $t > 0$, in $L^2(\mathbb{R}^n)$. Since $\|\mathcal{Q}_s^2 f\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{Q}_s f\|_{L^2(\mathbb{R}^n)}$, we have reduced matters to showing

$$\int_{\mathbb{R}^n} \Big| R_t^* \Big(\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s^2 f \Big)(x) \Big|^2 dx \lesssim \min\Big(\frac{s}{t}, \frac{t}{s} \Big)^\alpha \int_{\mathbb{R}^n} |\mathcal{Q}_s f(x)|^2 \, dx, \tag{5.3.15}$$

for some $\alpha > 0$. To save space we denote by I the left-hand-side of this last inequality. Recall that we denote by $\mathbb{D}_t$ the collection of dyadic cubes of scale 2^{-k} where $t/2 < 2^{-k} \le t$. We compute, denoting by $Q^* = c_n Q$ for any cube $Q \subset \mathbb{R}^n$ where c_n is a dimensional constant,

$$\begin{aligned}
I^{1/2} &= \Big(\sum_{Q \in \mathbb{D}_t} \int_Q \Big| R_t^* \Big(\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s f \Big) \Big|^2 dx \Big)^{1/2} \\
&= \Big(\sum_{Q \in \mathbb{D}_t} \int_Q \fint_{|x-y|<t} \Big| R_t^* \Big(\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s f \Big) \Big|^2 dy \, dx \Big)^{1/2} \\
&\lesssim \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \int_Q \Big| R_t^* \Big(\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s f \Big) \Big|^2 dx \, dy \Big)^{1/2} \\
&\lesssim \sum_{j \ge 0} \Big(\sum_{Q \in \mathbb{D}_t} \fint_{Q^*} \int_Q \Big| R_t^* \Big(\mathbb{1}_{R_j(Q)} \frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s f \Big) \Big|^2 dx \, dy \Big)^{1/2},
\end{aligned} \tag{5.3.16}$$

where we define $R_0(Q) := 2Q$ and for $j \ge 1$, $R_j(Q) := 2^{j+1}Q \backslash 2^j Q$ and we used the triangle inequality in the last line, together with the $L^2(\mathbb{R}^n)$-boundedness of R_t^*. We now use the off-diagonal decay for R_t^* to write,

$$\int_Q \Big| R_t^*\Big(\mathbb{1}_{R_j(Q)} \frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s f\Big)\Big|^2\,dx$$
$$\leq 2^{-nj\gamma}|Q|^{1-2/r} \times \Big(\int_{R_j(Q)} \Big|\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s^2 f(x)\Big|^r\,dx\Big)^{2/r}$$
$$\approx 2^{-nj(\gamma-2/r)}|Q|\Big(\fint_{R_j(Q)} \Big|\frac{\nu(x)}{\nu_{x,t}} R_t \mathcal{Q}_s^2 f(x)\Big|^r\,dx\Big)^{2/r}$$
$$\leq 2^{-nj(\gamma-2/r)}|Q|\Big(\fint_{R_j(Q)} |R_t \mathcal{Q}_s^2 f(x)|^2\,dx\Big) \times \Big(\fint_{R_j(Q)} \Big|\frac{\nu(x)}{\nu_{x,t}}\Big|^{\tilde r}\,dx\Big)^{2/\tilde r},$$

where $\tilde r^{-1} = 1/r - 1/2$ by Hölder's inequality. Plugging this estimate into (5.3.16), we see that

$$I^{1/2} \lesssim \sum_{j\geq 0}\Big(\sum_{Q\in\mathbb{D}_t}\fint_{Q^*} C_j^2|Q|\Big(\fint_{R_j(Q)}|R_t\mathcal{Q}_s^2 f(x)|^2\,dx\Big)\Big(\fint_{R_j(Q)}\Big|\frac{\nu(x)}{\nu_{x,t}}\Big|^{\tilde r}\,dx\Big)^{2/\tilde r}\,dy\Big)^{1/2}, \tag{5.3.17}$$

where we have defined $C_j := 2^{-nj(\gamma-2/r)}$. Since $M > \tilde r$, so that $\nu \in RH_{\tilde r}$ (see Proposition 5.2.19). Moreover, using the doubling property of ν and denoting C_{doub} to be the doubling constant, we have

$$\nu_{x,t} \geq 2^j C_{doub}^{-j}\nu_{x,2^j t} \approx C_{doub}^{-j} 2^j \nu(Q_j'), \tag{5.3.18}$$

where Q' is any cube with $\ell(Q_j') \approx 2^j t$ containing x. Therefore decomposing $R_j(Q)$ into $N = N(n)$ cubes Q' of sidelenght $2^j t$ we compute

$$\Big(\fint_{R_j(Q)}\Big|\frac{\nu(x)}{\nu_{x,t}}\Big|^{\tilde r}\,dx\Big)^{1/\tilde r} \lesssim \Big(\sum_{Q'}\fint_{Q'}\Big|\frac{\nu(x)}{\nu_{x,t}}\Big|^{\tilde r}\,dx\Big)^{1/\tilde r}$$
$$\lesssim_{[\nu^M]_{A_2}} \Big(\sum_{Q'}\Big(\fint_{Q'}\frac{\nu(x)}{\nu_{x,t}}\,dx\Big)^{\tilde r}\Big)^{1/\tilde r}$$
$$\lesssim C_{doub}^j 2^{-j}\Big(\sum_{Q'}\Big(\frac{\nu(Q')|Q'|}{\nu(Q')|Q'|}\Big)^{\tilde r}\Big)^{1/\tilde r}$$
$$\lesssim C_{doub}^j 2^{-j},$$

where we used (5.3.18) in the second to last line. In what follows, we absorb this constant into C_j, now writing $\widetilde{C}_j := 2^{-nj(\gamma+1/n-2/r-\log_2(C_{doub})/n)}$.

Plugging this into the estimate for I, appearing in (5.3.17), and using Fubini's Theorem,

$$
\begin{aligned}
I^{1/2} &\lesssim \sum_{j\geq 0}\Big(\sum_{Q\in\mathbb{D}_t}\fint_{Q^*}\widetilde{C}_j^2|Q|\fint_{R_j(Q)}|R_t\mathcal{Q}_s^2 f(x)|^2\,dx\,dy\Big)^{1/2}\\
&\approx \sum_{j\geq 0}\Big(\sum_{Q\in\mathbb{D}_t}\int_{Q^*}\widetilde{C}_j^2\fint_{|x-y|<2^{k+1}t}|R_t\mathcal{Q}_s^2 f(x)|^2\,dx\,dy\Big)^{1/2}\\
&\approx \sum_{j\geq 0}\Big(\widetilde{C}_j^2\int_{\mathbb{R}^n}\fint_{|x-y|<2^{(k+1)}t}|R_t\mathcal{Q}_s^2 f(x)|^2\,dx\,dy\Big)^{1/2}\\
&\lesssim \Big(\int_{\mathbb{R}^n}|R_t\mathcal{Q}_s^2 f(x)|^2\,dx\Big)^{1/2}\Big(\sum_{j\geq 0}\widetilde{C}_j^2\Big)^{1/2}\\
&\lesssim \Big(\int_{\mathbb{R}^n}|R_t\mathcal{Q}_s^2 f(x)|^2\,dx\Big)^{1/2},
\end{aligned}
$$

where in the last step we used that $C_{doub}\lesssim_n [\nu]_{A_1}\leq C_\delta$. This gives the desired estimate (5.3.15), since we have good quasi-orthogonality estimates for this object (see the proof of Theorem 5.4.4).

This concludes the proof.

□

Remark 5.3.19. As seen from the proof above, we can weaken the Reverse Hölder condition on $\Theta_{t,m}$ to

$$
\Big(\fint\!\!\fint_{I(Q)}|T_t f(x)|^{\bar{q}}\,dxdt\Big)^{1/\bar{q}}\lesssim\Big(\fint\!\!\fint_{I(Q^*)}|S_t f(x)|^{\bar{q}}\,dxdt\Big)^{1/\bar{q}},
$$

for every $r\leq\bar{q}\leq q$, and where the operator S_t satisfies $\|\mathcal{S}(S_t f)\|_{L^2(\mathbb{R}^n)}\lesssim\|f\|_{L^2(\mathbb{R}^n)}$, and a reverse Hölder inequality (what one should keep in mind here is that, in our intended application, where $T_t=t^m\partial_t^m\nabla\mathcal{S}_t^{\mathcal{L}}$, we do not have a reverse Hölder inequality for gradients of solutions, but we do have them for solutions, say $S_t=t^{m-1}\partial_t^m\mathcal{S}_t^{\mathcal{L}}$).

5.4 Extrapolation of Square Function Estimates

In this section, we obtain certain weighted and L^p estimates for operators of the form $t^m\partial_t^m\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla)$, for an $m\in\mathbb{N}$ large. The main ingredients for these estimates are the L^r-L^q off-diagonal diagonal decay for our operators (see Propositions 5.2.43 and 5.2.46) for $r < 2 < q$, used implicitly through the extrapolation results of the previous sections.

At this stage we also mention the work [Pri19], where the vertical and conical square functions for objects associated to the heat and Poisson semigroups of $\mathcal{L}$ (without lower order terms) are considered. We remark that our objects are a bit more technically involved to handle, in part due to the mild off-diagonal decay that they enjoy. Nevertheless, the basic idea of extrapolation and control of the vertical square function by a conical is the same.

In order to simplify the statement of our results, we make the following definition to condense the assumption that $\mathcal{L}$ satisfies the basic square function, Caccioppoli estimates and their consequences.

Definition 5.4.1 (Hypothesis A)**.** We say that $\mathcal{L}$ satisfies hypothesis A if the following conditions holds.

1. $\mathcal{L}$ has the form
$$\mathcal{L} = -\operatorname{div}(A\nabla + B_1) + B_2 \cdot \nabla,$$
for some $t-$independent $B_i \in L^n(\mathbb{R}^n; \mathbb{C}^{n+1})$ and a complex, t-independent matrix A verifying the uniform ellipticity condition (1.1.4).
2. With $\tilde{\rho}_1 > 0$ as in Theorem 5.1.1, we have
$$\max\{\|B_1\|_{L^n(\mathbb{R}^n)}, \|B_2\|_{L^n(\mathbb{R}^n)}\} < \tilde{\rho}_1.$$

We will say that a quantity depends on ellipticity if it depends only on C_A and $\tilde{\rho}_1$.

We summarize the main results, as far as applications to later sections are involved, of this section in the following (see also Theorem 5.4.15 for bounds on the Double Layer)

Theorem 5.4.2. *Suppose that $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). There exist $\varepsilon_0 > 0$, $m_0 \in \mathbb{N}$, and $\rho_0 > 0$ depending on dimension and ellipticity, such that if $\Theta_{t,m}$ is one of the following operators*

$$t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla), \quad t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} B_i), \quad t^m \partial_t^{m-1} B_i(\mathcal{S}_t^{\mathcal{L}} \nabla), \qquad i = 1, 2,$$

and if the coefficients of $\mathcal{L} = -\operatorname{div}(A\nabla + B_1) + B_2 \cdot \nabla$ satisfy

$$\max\{\|B_1\|_{L^n(\mathbb{R}^n)}, \|B_2\|_{L^n(\mathbb{R}^n)}\} < \rho_0,$$

then for every $m \geq m_0$ and $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ we have the estimates

$$\|\mathcal{S}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}.$$

For quick referencing, we mention that the proof of this Theorem is contained in the following results below:

- The conical estimates for $t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$ are obtained in Theorem 5.4.12, while the vertical ones in Theorem 5.4.13.
- The estimates for $t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} B)$ are contained in Corollary 5.4.14.
- The results for $t^m \partial_t^{m-1} B(\mathcal{S}_t^{\mathcal{L}} \nabla)$ are obtained in Lemma 5.4.7. There the results are obtained for the operator with the gradient replaced by a t derivative. A careful inspection of the proof though shows that, as long as we have good estimates for the operator $t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$, the same argument applies.
- Estimates for the Double Layer are obtained in Theorem 5.4.15.

5.4.1 Estimates for $\nabla \mathcal{S}_t^{\mathcal{L}}$

In this subsection we prove the relevant estimates for operators of the form $t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$. These will follow immediately from the extrapolation results from the previous section; together with the off-diagonal estimates obtained Propositions 5.2.43 and 5.2.46.

Remark 5.4.3. We would like to be able to apply Lemmas 5.3.1 and 5.3.7 to $\Theta_{t,m} = t^m \partial_t^m (\mathcal{S}_t^{\mathcal{L}} \nabla_{\|})$ to handle the Double Layer; it is not a simple matter however to obtain the necessary quasi-orthogonality condition in those (one reason is that in the regime $s < t$ we need to “add" derivatives to $\Theta_{t,m}$, while taking them away from $\mathcal{Q}_s$; however adding derivatives to $\Theta_{t,m}$ is tricky since we already have a $\nabla_{\|}$ in front. We will have to use the equation to circumvent this issue). We will treat this operator separately.

Theorem 5.4.4. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\Theta_{t,m} = t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$, then there exists $\varepsilon_1 > 0$ and $m_1 \in \mathbb{N}$, depending on dimension and ellipticity, such that if $m \geq m_1$ and $2 - \varepsilon_1 < p < \infty$ then*

$$\|\mathcal{S}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)} \lesssim_{p,m} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. This follows immediately from Lemma 5.3.1. More precisely, the off-diagonal decay is contained in Proposition 5.2.43, while the quasi-orthogonality estimate (5.3.2) for R_t is obtained in the proof of the L^2 square function bound for $\Theta_{t,m}$ (see Theorem 4.5.1) we sketch it here for completeness. Fix $0 < s < t$ and we choose $\mathcal{Q}_s = s \operatorname{div}_\| \widetilde{\mathcal{Q}}_s$, so that

$$\Theta_{t,m}\mathcal{Q}_s h(x) = t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}(s \operatorname{div}_\| \widetilde{\mathcal{Q}}_s h)(x) = \frac{s}{t} t^{m+1}\partial_t^m \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla_\|)(\widetilde{\mathcal{Q}}_s h),$$

and we appeal to Lemma 4.6.2, which shows that the operator $t^{m+1}\partial_t^m \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla_\|)$ is uniformly bounded in $L^2(\mathbb{R}^n)$, moreover so is $\widetilde{\mathcal{Q}}_s$. This takes care of the contribution of $\Theta_{t,m}$ to R_t. To handle the other term we further choose $P_t = \widetilde{P}_t \widetilde{P}_t$ for an approximate identity $\widetilde{P}_t$ and note that $|\Theta_{t,m}1(x)|\widetilde{P}_t$ is uniformly bounded in $L^2(\mathbb{R}^n)$ while $\widetilde{P}_t \mathcal{Q}_s$ satisfies good quasi-orthogonality estimates when $s < t$. Finally, the L^2 square function bound is obtained in Theorem 4.5.1 and Lemma 4.5.2. $\square$

We now turn to the appropriate vertical square function bounds.

Theorem 5.4.5 (L^p Bounds for Vertical Square Function)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\Theta_{t,m} := t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$. There exists $\varepsilon_2 > 0$ and $m_2 \in \mathbb{N}$, depending on dimension and ellipticity, such that if $p \in (2 - \varepsilon_2, 2 + \varepsilon_2)$ and $m \geq m_2$ then*

$$\|\mathbb{V}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We use Remark 5.3.19, with $T_t = t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$ and $S_t = t^{m-1}\partial_t^m \mathcal{S}_t^{\mathcal{L}}$. Then the square function bound for T_t follow from Theorem 4.5.1 and Lemma 4.5.2, while the square function bound S_t follows from Theorem 4.5.1. The comparability of T and S, as in Remark 5.3.19, follows from Caccioppoli's inequality (Proposition 4.3.9), and the Reverse Hölder inequality for S_t is contained in Proposition 5.2.41, recalling that $S_t f(x)$ is a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$ (see for instance Proposition 4.3.16) The necessary off-diagonal decay for both S and T is in Proposition 5.2.43, choosing m large enough. The conclusion now follows from Lemma 5.3.7. $\square$

While the extrapolation result in Lemma 5.3.7 is interesting on its own, it turns out that in our context, exploiting Caccioppoli's inequality, it is easy to get a much stronger bound (in fact the moral of the proof seems to be that, if T_t enjoys a reverse Hölder inequality on slices, then we can always control the vertical square function by the conical in an interval *around* $p = 2$). We state this in the following

Theorem 5.4.6 (Weighted Bounds for Vertical Square Function)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\Theta_{t,m} := t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}}$. There exist $m_2' \in \mathbb{N}$ and $M_2 \geq 1$, depending on dimension and ellipticity, such that for every $m \geq m_2'$, $M \geq M_2$ and every $\nu \in A_2$ with the property $\nu^M \in A_2$ it holds*

$$\|\mathbb{V}(\Theta_{t,m})\|_{L^2(\nu)} \approx \|\mathcal{S}(\tilde{\Theta}_{t,m} f)\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} \|f\|_{L^2(\nu)},$$

where we define

$$\tilde{\Theta}_{t,m} f(x) := \sqrt{\frac{\nu(x)}{\fint_{|x-y|<t} \nu(y) dy}}\, \Theta_{t,m} f(x) = \sqrt{\frac{\nu(x)}{\nu_{x,t}}}\, \Theta_{t,m} f(x).$$

Proof. We note, from the beginning of the proof of Lemma 5.3.7, the comparability

$$\|\mathbb{V}(\Theta_{t,m} f)(x)\|_{L^2(\nu)} \approx \|\mathcal{S}(\tilde{\Theta}_{t,m} f)\|_{L^2(\nu)},$$

holds for any weight $0 < \nu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore it remains to estimate the conical square function associated to $\tilde{\Theta}_{t,m}$. First writing

$$\|\mathcal{S}(\tilde{\Theta}_{t,m} f)\|^2_{L^2(\nu)} = \int_{\mathbb{R}^n} \int_0^\infty \fint_{|x-y|<t} \frac{\nu(y)}{\nu_{y,t}} |\Theta_{t,m} f(y)|^2 \, dy \frac{dt}{t}\, \nu(x) dx,$$

and using, by Hölder's and Caccioppoli's Inequalities,

$$\begin{aligned}
&\fint_{|x-y|<t} \frac{\nu(y)}{\nu_{y,t}} |\Theta_{t,m} f(y)|^2 \, dy \\
&\qquad \leq \Big(\fint_{|x-y|<t} \Big| \frac{\nu(y)}{\nu_{y,t}} \Big|^{q'} dy \Big)^{1/q'} \Big(\fint_{|x-y|<t} |\Theta_{t,m} f(y)|^{2q} \, dy \Big)^{1/q} \\
&\qquad\qquad \lesssim_{[\nu^M]_{A_2}} \Big(\fint_{|x-y|<t} |\Theta_{t,m} f(y)|^{2q} \, dy \Big)^{1/q} \\
&\qquad\qquad\qquad \lesssim \Big(\fint_{t/2}^{3t/2} \fint_{|x-y|<2t} |\theta_{s,m-1} f(y)|^{2q} \, dy ds \Big)^{1/q}
\end{aligned}$$

where we have defined $\theta_{s,m-1} := t^{m-1} \partial_t^m \mathcal{S}_t^{\mathcal{L}}$, and chosen $q \in (1,2)$ such that our operators satisfy a $2q$ Caccioppoli Inequality on slices (see Lemma 4.3.20) and then chosen $M > q'$. Now since $\theta_{s,m-1}$ satisfies a Reverse Hölder Inequality (see Proposition

5.2.41) we see that

$$\Big(\fint_{t/2}^{3t/2}\fint_{|x-y|<2t}|\theta_{s,m-1}f(y)|^{2q}\,dyds\Big)^{1/q}\lesssim\fint_{t/4}^{7t/4}\fint_{|x-y|<3t}|\theta_{s,m-1}f(y)|^{2}\,dyds$$
$$\lesssim\fint_{t/4}^{7t/4}\fint_{|x-y|<4s}|\theta_{s,m-1}f(y)|^{2}\,dyds.$$

The desired result follows now from Fubini's theorem and the fact that conical square functions with different cone appertures are comparable (see the comments after Definition 5.2.2). ☐

In what follows we will need square function estimates for the operators $t^m\partial_t^m B\mathcal{S}_t^{\mathcal{L}}$, where $B\in L^n(\mathbb{R}^n)$ is independent of the transversal variable. The L^2 case follows from the bounds for $t^m\partial_t^m\nabla\mathcal{S}_t^{\mathcal{L}}$ and Sobolev's inequality, the case $p\neq 2$ requires a bit of an argument both in the case of the vertical and conical square functions.

Lemma 5.4.7. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). For a function $B\in L^n(\mathbb{R}^n)$, independent of the t variable and $m\in\mathbb{N}$ consider the operators*

$$\Theta^B_{t,m}f(x):=t^m\partial_t^m B\mathcal{S}_t^{\mathcal{L}}f(x),\qquad \Theta_{t,m}f(x):=t^m\partial_t^m\nabla_{\|}\mathcal{S}_t^{\mathcal{L}}f(x).$$

For every $1<p<n$ and $f\in C_c^\infty(\mathbb{R}^n)$ it holds

$$\|\mathbb{V}(\Theta^B_{t,m}f)\|_{L^p(\mathbb{R}^n)}\lesssim\|\mathbb{V}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)}.$$

Moreover, if

$$\theta_{t,m}f:=t^m\partial_t^{m+1}\mathcal{S}_t^{\mathcal{L}}f,$$

then for any $1<p<\infty$

$$\|\mathcal{S}(\Theta^B_{t,m}f)\|_{L^p(\mathbb{R}^n)}\lesssim\|\mathcal{S}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)}+\|\mathcal{S}(\theta_{t,m-1}f)\|_{L^p(\mathbb{R}^n)}$$
$$\lesssim\|\mathcal{S}(\theta_{t,m-1}f)\|_{L^p(\mathbb{R}^n)}.$$

Proof. We begin with the bound for the conical versions. First note that the second inequality follows from the fact that conical square functions, in our setting, always "travel up" by the L^2 Caccioppoli inequality. To handle the first inequality we note that for fixed $x \in \mathbb{R}^n$ and $t > 0$ we have, by Hölder's and Poincaré-Sobolev Inequalities,

$$\Big(\fint_{|x-y|<t} |\Theta_{t,m}^B f(y)|^2 \, dy\Big)^{1/2} \lesssim \frac{\|B\|_{L^n(\mathbb{R}^n)}}{t} \Big(\fint_{|x-y|<t} |t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} f(y)|^{2^*} \, dy\Big)^{1/2^*}$$
$$\lesssim \|B\|_{L^n(\mathbb{R}^n)} \Big[\Big(\fint_{|x-y|<t} |\Theta_{t,m} f(y)|^2 \, dy\Big)^{1/2} + |(\theta_{t,m-1} f)_{x,t}|\Big],$$

where $(\theta_{t,m-1} f)_{x,t}$ denotes the average of $\theta_{t,m-1} f$ on the n-ball $|x - y| < t$. The result now follows from Jensen's inequality and the definition of $\mathcal{S}$.

The vertical square function is a bit more involved. The idea is to write

$$\Theta_{t,m}^B f(x) = B I_1 R \nabla_{\|} t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} f(x) = B I_1 R \Theta_{t,m} f(x),$$

where I_1 is the fractional integral of order 1 and R is a vector valued Riesz Transform (note that the above makes sense in $L^2(\mathbb{R}^n)$ owing to the slices estimates of Theorem 4.1.2 and the mapping properties of I_1 and R). Therefore, by Hölder's Inequality

$$\|\mathbb{V}(\Theta_{t,m}^B f)\|_{L^p(\mathbb{R}^n)} \leq \|B\|_{L^n(\mathbb{R}^n)} \|\mathbb{V}(I_1 R \Theta_{t,m} f)\|_{L^{p^*}(\mathbb{R}^n)},$$

where $1/p^* = 1/p - 1/n$ is the Sobolev exponent in dimension n.

The desired result follows from the following estimate: Let $F : \mathbb{R}_+^{n+1} \to \mathbb{C}$, then for every $1 < p < n$

$$\|\mathbb{V}(I_1 R F)\|_{L^{p^*}(\mathbb{R}^n)} \lesssim \|\mathbb{V}(F)\|_{L^p(\mathbb{R}^n)}.$$

To show this first note that for every $1 < p < \infty$ we have

$$\|\mathbb{V}(RF)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbb{V}(F)\|_{L^p(\mathbb{R}^n)},$$

This is a consequence of the weighted estimate

$$\int_{\mathbb{R}^n} \int_0^\infty |RF(x,t)|^2 \, \frac{dt}{t} \nu(x) dx \lesssim \int_{\mathbb{R}^n} \int_0^\infty |F(x,t)|^2 \, \frac{dt}{t} \nu(x) dx, \quad \nu \in A_2,$$

and the extrapolation theorem for A_p weights (see Theorem 5.2.20). Therefore it is enough to prove the estimate for I_1 alone. For this we will need an off-diagonal extrapolation result (see [CMP11, Theorem 3.23]) to reduce matters to proving

$$\Big(\int_{\mathbb{R}^n}\int_0^\infty |I_1F(x,t)|^2\,\frac{dt}{t}\,\nu^2(x)dx\Big)^{1/2}$$
$$\lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty |I_1F(x,t)|^2\,\frac{dt}{t}\Big)^{2_*/2}\,\nu(x)^{2_*}dx\Big)^{1/2_*},$$

where $1/2_* = 1/2+1/n$, and the above holds for every $\nu \in A_{2_*,2}$ (see Definition 5.2.30). To prove the above inequality we appeal to Theorem 5.2.31 to obtain, for a weight ν as above,

$$\|I_1 g\|_{L^2(\nu^2)} \lesssim \|g\|_{L^{2_*}(\nu^{2_*})}.$$

Therefore

$$\Big(\int_{\mathbb{R}^n}\int_0^\infty |I_1F(x,t)|^2\,\frac{dt}{t}\,\nu^2(x)dx\Big)^{1/2}$$
$$\lesssim \Big(\int_0^\infty\Big(\int_{\mathbb{R}^n} |F(x,t)|^{2_*}\,\nu(x)^{2_*}dx\Big)^{2/2_*}\frac{dt}{t}\Big)^{1/2}.$$

The desired bound now follows from Minkowski's inequality (in $L^{2/2_*}$). □

Remark 5.4.8. More generally, the proof above gives weighted inequalities and, in fact, shows the following: Weighted bounds $T : L^2(\nu) \to L^2(\nu)$ imply that $\|\mathbb{V}(TF)\|_{L^2(\nu)} \lesssim \|\mathbb{V}(F)\|_{L^2(\nu)}$. The same is true for the conical square function if T in addition has good local estimates, we refer to [AP17].

5.4.2 Estimates for $(\mathcal{S}_t^{\mathcal{L}}\nabla)$

We will need the analogue of Lemma 5.4.7 for the dual (in $L^2(\mathbb{R}^n)$) operator.

Lemma 5.4.9. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $B \in L^n(\mathbb{R}^n;\mathbb{C}^n)$ and set $\Theta_{t,m} := t^m\partial_t^m(\mathcal{S}_t^{\mathcal{L}}\nabla_\|)$ and $\Theta_{t,m}^B := t^m\partial_t^m\mathcal{S}_t^{\mathcal{L}}B$. then for any weight $\nu \in A_2$ we have*

$$\|\mathcal{S}(\Theta_{t,m}^B f)\|_{L^2(\nu)} \lesssim_{[\nu]_{A_2}} \|B\|_{L^n(\mathbb{R}^n)}\|\mathcal{S}(\Theta_{t,m})\|_{L^2(\nu)\to L^2(\nu)}\|f\|_{L^2(\nu)}.$$

In fact the constant can be shown to be at most a dimensional constant times $[\nu]_{A_2}^{1+\alpha}$ for some $\alpha < 1$ (see for instance [Pet08] and [LMPT10])

Proof. We begin by writing, for $f \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^n)$, $B\cdot f = \mathrm{div}_\| I_1I_1\nabla_\|(B\cdot f) =$

$\operatorname{div}_{\|} I_1 R(B \cdot f)$, where I_1 is the fractional integral of order 1. Therefore

$$\Theta_{t,m}^B f = \Theta_{t,m}(I_1 R(B \cdot f)),$$

and so

$$\|\mathcal{S}(\Theta_{t,m}^B f)\|_{L^2(\nu)} \lesssim \|\mathcal{S}(\Theta_{t,m})\|_{L^2(\nu)\to L^2(\nu)} \|RI_1(B \cdot f)\|_{L^2(\nu)}.$$

Since $R : L^2(\nu) \to L^2(\nu)$ by the Coifman-Fefferman maximal inequality (see Proposition 5.2.19), the result follows from Proposition 5.2.32. □

Theorem 5.4.10 (Square Function bounds for $(\mathcal{S}_t^{\mathcal{L}}\nabla)$)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\Theta_{t,m} := t^m \partial_t^m(\mathcal{S}_t^{\mathcal{L}}\nabla)$ and $\Theta_{t,m}^{\|} := t^m \partial_t^m(\mathcal{S}_t^{\mathcal{L}}\nabla_{\|})$ and $\delta \in (0,1)$. Then there exist $M > 0$, $m_3 \in \mathbb{N}$ (depending only on dimension, ellipticity, and for m_3 also δ) such that for every $m \geq m_3$ and if $\nu^M \in A_2$ is such that $[\nu^M]_{A_2} \leq C_\delta$ (with C_δ as in 7 of Proposition 5.2.19) then*

$$\|\mathcal{S}(\Theta_{t,m}^{\|} f)\|_{L^2(\nu)} \lesssim_{C_\delta, m} \|f\|_{L^2(\nu)},$$

provided[8] $\|B_{2\|}\|_{L^n(\mathbb{R}^n)} \leq \rho_3$*, for some ρ_3 depending only on dimension, ellipticity of $\mathcal{L}_{\|}$*[9] *and C_δ.*

In particular, for $p \in (2 - 1/2M, 2 + 1/2M)$, it holds

$$\|\mathcal{S}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We will follow the same outline as in the proof the corresponding unweighted L^2 bound for this object (see Lemma 4.5.26, which in turn is based on the method in [HMM15b]).

Throughout we will fix $\mathcal{Q}_s$ a CLP family (see Definition 4.2.26) with smooth compactly supported kernel, and set

$$P_t := -\int_t^\infty \mathcal{Q}_s^2 \, \frac{ds}{s}.$$

By the Hodge decomposition and the weighted estimates in Theorem 5.2.49, we see

[8]This is one of the few places we may require additional smallness in addition to that imposed in Chapter 4, prior to discussing existence and uniqueness for boundary value problems.

[9]More specifically on the constants appearing in Theorem 5.2.49.

that it is enough to show

$$\|\mathcal{S}(\Theta_{t,m} A_{\|} \nabla_{\|} F)\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} \|\nabla_{\|} F\|_{L^2(\nu)}.$$

We start by writing, via the Coifman-Meyer technique [CM86],

$$\begin{aligned}
&\Theta^{\|}_{t,m} A_{\|} \nabla_{\|} F(x) \\
&\quad = (\Theta^{\|}_{t,m}(A_{\|} \nabla_{\|} F)(x) - [\Theta^{\|}_{t,m} A_{\|}(x)] \cdot P_t \nabla_{\|} F(x)) + [\Theta^{\|}_{t,m} A_{\|}(x)] \cdot P_t \nabla_{\|} F(x) \\
&\qquad\qquad\qquad\qquad\qquad =: R_t(\nabla_{\|} F)(x) + [\Theta^{\|}_{t,m} A_{\|}(x)] \cdot P_t \nabla_{\|} F(x).
\end{aligned}$$

Since these objects already satisfy good (unweighted) L^2 estimates, the difficulties now shift to the “error" term R_t; indeed, using the weighted version of Carleson's lemma (see Lemma 5.2.28) to handle the second term it is enough to show that

$$\mu(R_Q) \lesssim \nu(Q), \qquad \forall Q \subset \mathbb{R}^n, \tag{5.4.11}$$

where we have defined the measure μ as (recall that we are trying to control a *conical* square function)

$$d\mu(x,t) := \Big(\fint_{|x-y|<t} |\Theta^{\|}_{t,m} A_{\|}(y)|^2 \, dy \Big) \frac{\nu(x) dx dt}{t}.$$

To obtain (5.4.11), owing to the off-diagonal decay of $\Theta^{\|}_{t,m}$ in Proposition 5.2.43 and the fact that $d\mu$ is a Carelson measure when $\nu = 1$ by Lemma 4.5.26, we can mimic the argument used in the proof of the extrapolation theorem for conical square functions (Theorem 5.3.1) involving the John-Nirenberg lemma for local square functions (Lemma 5.2.16); we omit the details.

It now remains to show that R_t has good square function bounds. This is the main part of the proof; we will follow almost verbatim the proof of Lemma 4.5.26, replacing weighted bounds where appropriate.

We start by rewriting R_t in the following way:

$$\begin{aligned}
R_t &= \Theta^{\|}_{t,m} A_{\|} - [\Theta^{\|}_{t,m} A_{\|}] P_t \\
&= \Big(\Theta^{\|}_{t,m} A_{\|} P_t - [\Theta^{\|}_{t,m} A_{\|}] P_t \Big) + \Theta^{\|}_{t,m} A_{\|} (I - P_t) =: R^{[1]}_t + R^{[2]}_t.
\end{aligned}$$

Since $\Theta^{\|}_{t,m}$ has good off-diagonal decay by Proposition 5.2.43 (see also [AAA+11, Lemma 3.3]), so does R_t and satisfies the quasi orthogonality estimate (5.3.2), thanks to the presence of the P_t term. We can then apply the extrapolation lemma for conical square functions (Lemma 5.3.1) to conclude that

$$\|\mathcal{S}(R_t^{[1]}\nabla_{\|}F)\|_{L^2(\nu)} \lesssim \|\nabla_{\|}F\|_{L^2(\nu)}, \quad \nu \in A_{2/r},$$

for some $1 < r < 2$.

For the term $R_t^{[2]}$ we'll use the equation in the form of the identities on slices (see Proposition 4.3.19). For notational convenience, we will denote $Z_t := (1 - P_t)$, and also

$$\vec{b} := (A_{n+1,1}, \ldots, A_{n+1,n}), \quad \vec{a} := (A_{1,n+1}, \ldots, A_{n+1,n+1}).$$

We write

$$\begin{aligned}
R_t^{[2]}(\nabla_{\|}F) &= \Theta^{\|}_{t,m}A_{\|}Z_t\nabla_{\|}F = \Theta^{\|}_{t,m}A_{\|}\nabla_{\|}Z_tF \\
&= \partial_t\Theta_{t,m}\vec{a}Z_tF - \theta_{t,m}(\vec{b}\nabla Z_tF) + \Theta_{t,m}B_1Z_tF \\
&\quad - t\theta_{t,m-1}(B_{2\|}\nabla_{\|}Z_tF) + \theta_{t,m}(B_{2\perp}Z_tF) \\
&=: J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned}$$

where as usual we have defined

$$\theta_{t,m} := t^m\partial_t^{m+1}\mathcal{S}_t^{\mathcal{L}}.$$

To handle J_1 we note that, owing to the $L^r - L^2$ off-diagonal decay of $\Theta_{t,m}$ (Proposition 5.2.43) and the average weighted estimates on slices of Proposition 5.2.39, we see that

$$\begin{aligned}
\|\mathcal{S}(J_1)\|^2_{L^2(\nu)} &= \int_0^\infty \int_{\mathbb{R}^n} \fint_{|x-y|<t} |\Theta_{t,m+1}\vec{a}t^{-1}Z_tF(y)|^2\, dy\nu(x)dx\,\frac{dt}{t} \\
&\lesssim_{[\nu]_{A_{2/r}}} \int_0^\infty \int_{\mathbb{R}^n} |\vec{a}t^{-1}Z_tF(x)|^2\,\nu(x)dx\frac{dt}{t} \\
&\lesssim \|\nabla_{\|}F\|_{L^2(\nu)},
\end{aligned}$$

where we have used Proposition 5.2.33 to handle the square function associated to Z_t in the last line.

J_2 we rewrite as follows:

$$\begin{aligned}J_2 &= \theta_{t,m}(\vec{b}\cdot\nabla_{\|}F) + \Big(\theta_{t,m}(\vec{b}\cdot P_t\nabla_{\|}F) - [\theta_{t,m}\vec{b}]\cdot P_t\nabla_{\|}F\Big) + [\theta_{t,m}\vec{b}]\cdot P_t\nabla_{\|}F\\ &=: J_{2,1} + J_{2,2} + J_{2,3}.\end{aligned}$$

Again appealing to the John-Nirenberg lemma for local square functions (Lemma 5.2.16, see also the proof of Lemma 5.3.1) we see that the contribution of $J_{2,3}$ is under control by the weighted version of Carleson's lemma (Lemma 5.2.28). The term $J_{2,2}$ we can handle the same way we did $R_t^{[1]}$; we omit the details. Finally, by Theorem 5.4.6, we have good weighted conical square function bounds for $\theta_{t,m}$ and $\vec{b}\in L^\infty(\mathbb{R}^n)$, so the contribution of $J_{2,1}$ is also under control.

For J_3 we appeal to Proposition 5.2.47, which, for $s<t$, gives the bound (for $I_1 g = F$)

$$\Big\|\Big(\fint_{|x-y|<t}|\Theta_{t,m}B_1 I_1 \mathcal{Q}_s^2 g(y)|^2\,dy\Big)^{1/2}\Big\|_{L^2(\nu)} \lesssim \Big(\frac{s}{t}\Big)^{\beta}\|\mathcal{Q}_s g\|_{L^2(\nu)}.$$

Therefore

$$\begin{aligned}\|\mathcal{S}(J_3)\|_{L^2(\nu)} &= \int_0^\infty\int_{\mathbb{R}^n}\fint_{|x-y|<t}|\Theta_{t,m}B_1(1-P_t)F(y)|^2\,dy\,\nu(x)dx\,\frac{dt}{t}\\ &= \int_0^\infty\int_{\mathbb{R}^n}\fint_{|x-y<t}\Big|\Theta_{t,m}B_1I_1\int_0^t \mathcal{Q}_s^2 g(y)\,\frac{ds}{s}\Big|^2\,dy\,\nu(x)dx\,\frac{dt}{t}\\ &\lesssim_\beta \int_0^\infty\int_{\mathbb{R}^n}\fint_{|x-y|<t}\int_0^t\Big(\frac{t}{s}\Big)^{\beta/2}|\Theta_{t,m}B_1I_1\mathcal{Q}_s^2 g(y)|^2\,\frac{ds}{s}\,dy\,\nu(x)dx\,\frac{dt}{t}\\ &\lesssim \int_0^\infty\int_s^\infty\Big(\frac{s}{t}\Big)^{\beta/2}\|\mathcal{Q}_s g\|^2_{L^2(\nu)}\,\frac{st}{t}\,\frac{ds}{s}\\ &\lesssim \|\mathbb{V}(\mathcal{Q}_s g)\|_{L^2(\nu)}\\ &\lesssim \|g\|_{L^2(\nu)},\end{aligned}$$

where we invoked Theorem 5.2.23 in the last line. To conclude we note that $I_1 g = F$ and so $Rg = \nabla_{\|}F$, where R is the vector-valued Riesz transform (with symbol $\xi/|\xi|$) and we know $\|Rg\|_{L^2(\nu)}\approx\|g\|_{L^2(\nu)}$ for every $\nu\in A_2$; the desired bound follows from this since $A_{2/r}\subset A_2$.

To handle J_4 we write it as

$$J_4 = -t\theta_{t,m-1} B_{2\|} \nabla_\| F + t\theta_{t,m-1} B_{2\|} \nabla_\| P_t F =: J_{4,1} + J_{4,2}.$$

For $J_{4,1}$ we appeal to Lemma 5.4.9 to bound

$$\|\mathcal{S}(J_{4,1})\|_{L^2(\nu)} \lesssim_{[\nu]_{A_2}} \|B_{2\|}\|_{L^n(\mathbb{R}^n)} \|\nabla_\| F\|_{L^2(\nu)} \|\Theta^{\|}_{t,m}\|_{L^2(\nu)\to L^2(\nu)}.$$

Therefore, if $\|B_2\|_{L^n(\mathbb{R}^n)}$ is small enough we may hide this term on the left hand side.

We rewrite $J_{4,2}$ in the following way:

$$\begin{aligned} J_{4,2} &= \Big(t\theta_{t,m-1}(B_{2\|} \cdot P_t \nabla_\| F) - [t\theta_{t,m-1} B_{2\|}] \cdot P_t \nabla_\| F\Big) + [t\theta_{t,m-1} B_{2\|}] \cdot P_t \nabla_\| F \\ &=: R_t^{[3]} + [t\theta_{t,m-1} B_{2\|}] \cdot P_t \nabla_\| F. \end{aligned}$$

$R_t^{[3]}$ may be handled the same way as $R_t^{[1]}$, using Proposition 5.2.46 to obtain the right $L^r - L^2$ off-diagonal estimates. It remains to show, by an application of the weighted version of Carelson's lemma, a ν-Carleson measure estimate for

$$d\mu(x,t) := \left(\fint_{|x-y|<t} |t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} B_{2\|}(y)|^2 \, dy\right) \nu(x) \frac{dxdt}{t}.$$

This follows, once again, by an application of the John-Nirenberg lemma for local square functions (Lemma 5.2.16, see also the proof of Lemma 5.3.1)[10].

Finally, to handle J_5, we appeal again to the $L^r - L^2$ off-diagonal estimates of $t^{m+1}\partial_t^{m+1} \mathcal{S}_t^{\mathcal{L}} B_{2\perp}$ (Proposition 5.2.46) and Proposition 5.2.39 (which give that the term $t^{m+1}\partial_t^{m+1} \mathcal{S}_t^{\mathcal{L}} B_{2\perp}$ satisfies good averaged weighted bounds on slices) to conclude

$$\|\mathcal{S}(J_5)\|_{L^2(\nu)} \lesssim \int_0^\infty \int_{\mathbb{R}^n} \left|\frac{Z_t}{t} F\right|^2 \nu(x) \frac{dt}{t}.$$

We conclude by the square function estimates of Proposition 5.2.33, the same as we did for J_1.

[10]Notice that, since we already have good unweighted L^2 square function estimates for $\Theta_{t,m}$, the John-Nirenberg lemma gives us that this object is under control; as opposed to the unweighted case, where we were forced to hide this term.

Combining all the above, we see that we have shown:

$$\|\mathcal{S}(\Theta_{t,m}^{\|})\|_{L^2(\nu)\to L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} 1 + \|B_{2\|}\|_{L^n(\mathbb{R}^n)}\|\mathcal{S}(\Theta_{t,m}^{\|})\|_{L^2(\nu)\to L^2(\nu)},$$

this gives the desired bound if the left hand side is finite and $\|B\|_{L^n(\mathbb{R}^n)}$ is small enough. To achieve the former we may work with the truncated square functions given by

$$\mathcal{S}_\eta(\Theta_{t,m}^{\|} f)(x) := \int_\eta^{1/\eta} \fint_{|x-y|<t} |\Theta_{t,m}^{\|} f(y)|^2 \, dy \frac{dt}{t},$$

which satisfy $\|\mathcal{S}_\eta(\Theta_{t,m}^{\|})\|_{L^2(\nu)\to L^2(\nu)} < \infty$, owing to the estimates on slices of Proposition 5.2.39. Fix now C_δ as in the assumptions, i.e. $[\nu^M] \leq C_\delta$, then our estimates read (see also Theorem 5.2.49)

$$\|\mathcal{S}(\Theta_{t,m}^{\|})\|_{L^2(\nu)\to L^2(\nu)} \lesssim_{C_0} 1 + \|B_{2\|}\|_{L^n(\mathbb{R}^n)}\|\mathcal{S}(\Theta_{t,m}^{\|})\|_{L^2(\nu)\to L^2(\nu)}.$$

Thus, choosing $\|B_{2\|}\|_{L^n(\mathbb{R}^n)} < \rho = \rho(C_\delta)$ we can hide the second term on the right hand side and conclude the result.

The L^p bounds are a consequence of this and Corollary 5.2.27 if we choose $\delta = 1/2$, where recall $C_{1/2}$ is defined as in 7 of Proposition 5.2.19. $\square$

Theorem 5.4.12. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\Theta_{t,m} := t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$ and $\delta \in (0,1)$. There exist $m_4 \in \mathbb{N}$ and $M > 0$ (depending only on dimension and ellipticity, and for m_4 also on δ) such that if $m \geq m_4$ and $\nu \in A_2$ is such that $[\nu^M]_{A_2} \leq C_\delta$, then*

$$\|\mathcal{S}(\Theta_{t,m} f)\|_{L^2(\nu)} \lesssim_{[\nu^M]_{A_2}} \|f\|_{L^2(\nu)},$$

provided $\|B_{2\|}\|_{L^n(\mathbb{R}^n)} < \rho_4$, for some ρ_4 depending on dimension, ellipticity of $\mathcal{L}_\|$, and C_δ only.

In particular there exists $\varepsilon_4 > 0$ (depending on dimension and ellipticity) such that if $p \in (2-\varepsilon_4, 2+\varepsilon_4)$ and $m \geq m_4$ then

$$\|\mathcal{S}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Notice that it is enough to consider $\nabla_\|$ instead of ∇ in both instances; otherwise we are in the situation of Theorem 5.4.4 or Theorem 5.4.10. Therefore, without loss of generality, $\Theta_{t,m} = t^m \partial_t^{m-1} \nabla_\| (\mathcal{S}_t^{\mathcal{L}} \nabla_\|)$. In this case, for $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^n)$ we can write

$$\Theta_{t,m} f(x) = t^m \partial_t^{m-1} \nabla_\| \mathcal{S}_t^{\mathcal{L}} (\operatorname{div}_\| f)(x) =: t^m \partial_t^{m-1} \nabla_\| \mathcal{S}_t^{\mathcal{L}} g(x).$$

By the Caccioppoli inequality on slices (Lemma 4.3.20) we see that, for fixed $x \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned}
&\int_0^\infty \fint_{|x-y|<t} |t^m \partial_t^{m-1} \nabla_\| \mathcal{S}_t^{\mathcal{L}} f(y)|^2 \, dy \frac{dt}{t} \\
&\qquad \lesssim \int_0^\infty \fint_{t/4}^{5t/4} \fint_{|x-y|<2t} |t^{m-1} \partial_t^{m-1} \mathcal{S}_t^{\mathcal{L}} g(y)|^2 \, dy \frac{dt}{t} \\
&\qquad = \int_0^\infty \fint_{t/4}^{5t/4} \fint_{|x-y|<2t} |s^{m-1} \partial_s^{m-1} (\mathcal{S}_s^{\mathcal{L}} \nabla_\|) f(y)|^2 \, dy ds \frac{dt}{t} \\
&\qquad \lesssim \int_0^\infty \fint_{|x-y|<2t} |s^{m-1} \partial_s^{m-1} (\mathcal{S}_s^{\mathcal{L}} \nabla_\|) f(y)|^2 \, dy \frac{ds}{s}.
\end{aligned}$$

The result now follows from Theorem 5.4.10 and the change of angle formula for square functions (see the comments after Definition 5.2.2). $\square$

The following is the weighted version of Theorem 5.4.2 for the vertical square function.

Theorem 5.4.13. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\Theta_{t,m} := t^m \partial_t^{m-1} \nabla (\mathcal{S}_t^{\mathcal{L}} \nabla)$ and $\delta \in (0,1)$. There exists $m_5 \in \mathbb{N}$ and $M > 0$ (depending only on dimension and ellipticity, and in the case of m_5 also on δ) such that if $m \geq m_5$ and $\nu \in A_2$ satisfies $[\nu^M]_{A_2} \leq C_\delta$, then*

$$\|\mathbb{V}(\Theta_{t,m} f)\|_{L^2(\nu)} \lesssim_{C_0} \|f\|_{L^2(\nu)},$$

provided $\|B_{2\|}\| < \rho_5$, for some $\rho_5 > 0$ depending only on dimension, ellipticity of $\mathcal{L}_\|$, and C_δ.

In particular there exists $\varepsilon_5 > 0$ (depending on dimension and ellipticity) such that for $p \in (2 - \varepsilon_5, 2 + \varepsilon_5)$

$$\|\mathbb{V}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. By Theorem 5.4.6 and the t-independence, it is enough to consider $\Theta_{t,m} :=$

$t^m\partial_t^{m-1}\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla_\|)$. Now the idea is to repeat the proof of the weighted bound for $\mathbb{V}(t^m\partial_t^m\nabla\mathcal{S}_t^{\mathcal{L}}g)$, with $g = \operatorname{div}_\| f$, in Theorem 5.4.6, using now instead Theorem 5.4.12; we omit the details. $\square$

The following Corollary will be useful when dealing with square functions involving the double layer potential.

Corollary 5.4.14. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\varepsilon_0 > 0$ and $M_0 > 0$ be as in Theorem 5.4.2. Let $\Theta_{t,m}^B := t^m\partial_t^{m-1}\nabla(\mathcal{S}_t^{\mathcal{L}}B)$ for some $B \in L^n(\mathbb{R}^n;\mathbb{C}^{n+1})$. Then for every $f \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^{n+1})$*

$$\|\mathcal{S}(\Theta_{t,m}^B f)\|_{L^2(\nu)}, \|\mathbb{V}(\Theta_{t,m}^B f)\|_{L^2(\nu)} \lesssim_{[\nu^m]_{A_2}} \|f\|_{L^2(\nu)}.$$

Proof. Write $B_\| \cdot f_\| = \operatorname{div}_\| \nabla_\| I_1 I_1(B_\| \cdot f_\|) = \operatorname{div}_\| R I_1(B_\| \cdot f_\|) = \operatorname{div}_\| g_\|$. Notice that, by the proof of Proposition 5.2.32, we know

$$\|g_\|\|_{L^p(\mathbb{R}^n)} \lesssim \|f_\|\|_{L^p(\mathbb{R}^n)}$$

for every $1 < p < \infty$. On the other hand we can also write $B_\perp f_\perp = \operatorname{div}_\| RI_1(B_\perp f_\perp) = \operatorname{div}_\| g_\perp$ where

$$\|g_\perp\|_{L^p(\mathbb{R}^n)} \lesssim \|f_\perp\|_{L^p(\mathbb{R}^n)}.$$

Combining these two observations the result follows from either Theorem 5.4.12 for the conical version or Theorem 5.4.13 for the vertical. $\square$

Recall that for $\vec{N} = -e_{n+1}$ the exterior normal to $\partial\mathbb{R}_+^{n+1}$ we have the following representation formula for the double layer:

$$\mathcal{D}_t^{\mathcal{L}} f = (\mathcal{S}_t^{\mathcal{L}}\nabla)\cdot A\vec{N}f + (\mathcal{S}_t^{\mathcal{L}}\overline{B_2})\cdot\vec{N}f,$$

for $f \in C_c^\infty(\mathbb{R}^n)$. As an immediate consequence of this and the previous results we have

Theorem 5.4.15 (Square Function Bounds for $\mathcal{D}_t^{\mathcal{L}}$. Part 1)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $\varepsilon_0 > 0$, $m_0 \in \mathbb{N}$, and $\rho_0 > 0$ as in Theorem 5.4.2. Suppose $\Theta_{t,m} = t^m\partial_t^m\nabla\mathcal{D}_t^{\mathcal{L}}$. Then for $m \geq m_0$, $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ and $f \in C_c^\infty(\mathbb{R}^n)$*

$$\|\mathcal{S}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)}, \|\mathbb{V}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

5.5 Control on Slices

Theorem 5.5.1 (Estimates on Slices for $\nabla \mathcal{S}_t^{\mathcal{L}}$)**.** *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. If $\varepsilon_0 > 0$ is as in Theorem 5.4.2 and $p \in (2-\varepsilon_0, 2+\varepsilon_0)$, then for $f \in C_c^\infty(\mathbb{R}^n)$*

$$\|\mathcal{S}_t^{\mathcal{L}} f\|_{L^{p^*}(\mathbb{R}^n)} + \|\nabla \mathcal{S}_t^{\mathcal{L}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{m,p} \|f\|_{L^p(\mathbb{R}^n)}, \qquad t > 0.$$

Proof. The proof of this result is essentially contained in Theorem 4.1.2, using the L^p square function estimates from the previous section instead. We omit the details. □

As a consequence of this result we obtain the necessary boundedness on slices of our operators.

Corollary 5.5.2. *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. If ε_0 is as in Theorem 5.4.2, then for every $m \geq 1$, $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ and $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$*

$$\|t^m \partial_t^{m-1} (\mathcal{S}_t^{\mathcal{L}} \nabla) \cdot f\|_{L^{p^*}(\mathbb{R}^n)} + \|t^m \partial_t^{m-1} \nabla (\mathcal{S}_t^{\mathcal{L}} \nabla) \cdot f\|_{L^p(\mathbb{R}^n)} \lesssim_{m,p} \|f\|_{L^p(\mathbb{R}^n)}, \qquad t > 0.$$

If either of the gradients is replaced by ∂_t, then the above remains true for $m = 0$.

Proof. It is enough to treat $\nabla(\mathcal{S}_t^{\mathcal{L}} \nabla_\|)$ by the t-independence of the coefficients. The idea is to write $\nabla(\mathcal{S}_t^{\mathcal{L}} \nabla_\| f) = \nabla \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_\| f$ and apply the L^p Caccioppoli's inequality on slices (Lemma 4.3.20) once, and then use induction (recall that the off-diagonal decay already gives uniform L^p bounds for m large enough). Details can be found in [AAA+11, Lemma 2.11]. □

Similarly we have the result for B_i in place of the gradient, the proof is a simple application of Sobolev's inequality for $m = 0$, Caccioppoli's inequality on slices, and duality.

Corollary 5.5.3. *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. If $\varepsilon_0 > 0$ is as in Theorem 5.4.2, $m \geq 0$, and $p \in (2-\varepsilon_0, 2+\varepsilon_0)$, then for $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^{n+1})$*

$$\|t^m \partial_t^m B_i \mathcal{S}_t^{\mathcal{L}} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

and

$$\|t^m \partial_t^m (\mathcal{S}_t^{\mathcal{L}} B_i) \cdot g\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}.$$

The following result is really a Corollary of the above estimates; we state it on its own for future reference.

Theorem 5.5.4 (Estimates on Slices for $\mathcal{D}^{\mathcal{L}}$)**.** *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. If ε_0 is as in Theorem 5.4.2, then for $m \geq 0$ and $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ and $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\|t^m \partial_t^m \mathcal{D}_t^{\mathcal{L}} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \qquad t > 0$$

Proof. Again by Caccioppoli's inequality on slices it is enough to treat the case $m = 0$.

The result is an immediate consequence of the following representation formula for the double layer given earlier: For $f \in C_c^\infty(\mathbb{R}^n)$ we have

$$\mathcal{D}_t^{\mathcal{L}} f = (\mathcal{S}_t^{\mathcal{L}} \nabla) \cdot (A \vec{N} f) + (\mathcal{S}_t^{\mathcal{L}} \overline{B}_2) \cdot \vec{N} f,$$

where $\vec{N} = -e_{n+1}$ is the exterior normal to $\partial \mathbb{R}^{n+1}_+$. $\square$

The following result will be used in the proof of the non-tangential maximal function estimate.

Lemma 5.5.5. *Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ is a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$, given by either $u = \mathcal{S}_t^{\mathcal{L}} f$ or $u = \mathcal{D}^{\mathcal{L}} g$ for some $f, g \in C_c^\infty$. For any positive Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \leq 1$, if we define the function*

$$u_\varphi(x,t) := u(x, t + \varphi(x)), \qquad (x,t) \in \mathbb{R}^{n+1}_+,$$

then we have

$$\sup_{t>0} \|u_\varphi(\cdot, t)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbb{V}(t \nabla u)\|_{L^p(\mathbb{R}^n)},$$

for $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ as in Theorem 5.7.1 (one has to keep in mind here that the vertical square function "travels down" as long as we have good estimates on slices).

Proof. We sketch the proof: The function u_φ solves an elliptic equation $\mathcal{L}_\varphi u_\varphi = 0$ of the same type as $\mathcal{L}$, with the corresponding norms of the operator $\mathcal{L}_\varphi$ controlled in terms of those for $\mathcal{L}$ and $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}$. Next, we apply Theorem 4.6.12 (or, more precisely, its proof of the uniform $Y^{1,2}(\mathbb{R}^n)$ estimate). $\square$

5.6 Nontangential Maximal Function Estimates

Proposition 5.6.1. *Let $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ be a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. For all $q \geq 1$ and $\varepsilon > 0$ it holds* [11]

$$\tilde{\mathcal{N}}_2^{(\varepsilon)}(\nabla u) \lesssim \mathcal{M}(\tilde{\mathcal{N}}_q(\partial_t u)) + \mathcal{M}(\nabla_\| u(\cdot,\varepsilon)) + \mathcal{M}(u(\cdot,\varepsilon)) \cdot \mathcal{M}_2(B_1),$$

with implicit constants depending on dimension, ellipticity and q. Here we recall that we have defined, for any $r > 0$ and $g \in L^r_{\text{loc}}(\mathbb{R}^n)$,

$$\mathcal{M}_r(g) := \mathcal{M}(|g|^r)^{1/r}.$$

In particular, if $1 < p < n$ and $u(\cdot,\varepsilon) \in Y^{1,p}(\mathbb{R}^n)$ for every $\varepsilon > 0$,

$$\|\tilde{\mathcal{N}}_2(\nabla u)\|_{L^p(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_1(\partial_t u)\|_{L^p(\mathbb{R}^n)} + \sup_{\varepsilon>0} \|\nabla_\| u(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^n)},$$

whenever the right hand side is finite.

Moreover for u we have the estimate

$$\|\tilde{\mathcal{N}}_2(u)\|_{L^p(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_1(u)\|_{L^p(\mathbb{R}^n)},$$

for any $p > 1$.

Proof. The statement for u follows from the reverse Hölder inequality for solutions (Proposition 5.2.41) and the comparability of $\tilde{\mathcal{N}}$ defined with different parameters for $\mathcal{C}_{x,t}$.

Fix $\varepsilon > 0$ and set $u_\varepsilon(x) = u(x,\varepsilon)$. Fix $z \in \mathbb{R}^n$ and $(x,t) \in \gamma(z)$. Recall that we defined the cylinders

$$\mathcal{C}_{x,t} := \{(y,s) \in \mathbb{R}^{n+1}_+ : |x-y| < t/8,\ |t-s| < t/8\},$$

[11] See Definition 5.2.5 for the truncated maximal function $\tilde{\mathcal{N}}_2^{(\varepsilon)}$.

and we set, for $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$,

$$\tilde{\mathcal{N}}_2(F)(z) := \sup_{(x,t)\in\gamma(z)} a_2(u)(x,t) := \sup_{(x,t)\in\gamma(z)} \left(\fint\!\!\fint_{\mathcal{C}_{x,t}} |F(w,\tau)|^2 \, dw d\tau \right)^{1/2}.$$

We denote by $\mathcal{C}^*_{x,t}$ the concentric dilate $2\mathcal{C}_{x,t}$ and a_1^* the corresponding L^1 average built with $\mathcal{C}^*_{x,t}$ instead of $\mathcal{C}_{x,t}$. By the reverse Hölder inequality in Proposition 5.2.41, for any $c \in \mathbb{C}$

$$a_2(\nabla u)(x,t) \lesssim \frac{1}{t} a_1^*(u-c)(x,t) + |c| a_2^*(B_1)(x,t) =: I + II.$$

If we choose

$$c := \fint_{\Delta_{x,t}} u_\varepsilon \, dz,$$

then we immediately see, exploiting the t-independence of B_1,

$$II \le \mathcal{M}(u_\varepsilon)(z) \cdot a_2^*(B_1)(x,t) \lesssim \mathcal{M}(u_\varepsilon)(z) \cdot \mathcal{M}_2(B_1)(z).$$

It remains to estimate I. For this purpose we compute

$$a_1^*(u-c)(x,t) \le a_1^*(u-u_\varepsilon)(x,t) + a_1^*(u_\varepsilon - c)(x,t).$$

From the definition of c we see that

$$\begin{aligned} a_1^*(u_\varepsilon - c)(x,t) &= \fint\!\!\fint_{\mathcal{C}^*_{x,t}} \left| u_\varepsilon(y) - \fint_{\Delta_{x,t}} u_\varepsilon(w)\, dw \right| dy ds \\ &\lesssim t \fint_{\Delta_{x,t}} |\nabla_\| u_\varepsilon(y)| \, dy \\ &\le t \mathcal{M}(\nabla_\| u_\varepsilon)(z), \end{aligned}$$

where we used the Poincaré inequality in L^1 for the second line.

On the other hand, by the fundamental theorem of calculus, and introducing an average in space we have

$$
\begin{aligned}
a_1^*(u-u_\varepsilon)(x,t) =& \fint\!\!\fint_{\mathcal{C}^*_{x,t}} \Big|u(y,s)-u(y,\varepsilon)\Big|\,dyds \\
\leq& \fint\!\!\fint_{\mathcal{C}^*_{x,t}} \int_\varepsilon^s |\partial_\tau u(y,\tau)|\,d\tau\,dyds \\
\leq& \fint\!\!\fint_{\mathcal{C}^*_{x,t}} \int_0^s |\partial_\tau u(y,\tau)|\,d\tau\,dyds \\
=& \fint_{|t-s|<t/8} \int_0^s \fint_{|x-y|<t/8} \fint_{|y-w|<\tau/8} |\partial_\tau u(y,\tau)|\,dwdy\,d\tau ds \\
\lesssim& \fint_{|t-s|<t/8} \fint_{|x-w|<t/2} \int_0^s \fint_{|w-y|<\tau/8} |\partial_\tau u(y,\tau)|\,dyd\tau\,dwds.
\end{aligned}
$$

Now we notice, for fixed $s>0$,

$$
\begin{aligned}
\int_0^s \fint_{|w-y|<\tau/8} |\partial_\tau u(y,\tau)|\,dyd\tau &\lesssim \sum_{k\geq 0} 2^{-k} s \fint_{2^{-k-1}s}^{2^{-k}s} \fint_{|w-y|<2^{-k-3}s} |\partial_\tau u(y,\tau)|\,dyd\tau \\
&\lesssim \sum_{k\geq 0} 2^{-k} s a_1^*(\partial_\tau u)(y, 2^{-k-1/2}s) \\
&\lesssim \sum_{k\geq 0} 2^{-k} s \tilde{\mathcal{N}}_1(\partial_\tau u)(y) \\
&\lesssim s\tilde{\mathcal{N}}_1(\partial_\tau u)(y).
\end{aligned}
$$

Plugging this estimate into the inequality preceding it we arrive at

$$
a_1^*(u-u_\varepsilon)(x,t) \lesssim \fint_{|t-s|<t/8} \fint_{|x-w|<t/2} s\tilde{\mathcal{N}}_1(\partial_\tau u)(y)\,dyds \lesssim t\mathcal{M}(\tilde{\mathcal{N}}_1(\partial_\tau u))(z),
$$

where we used the fact that $t\approx s$ for the last inequality. We conclude

$$
I \lesssim \mathcal{M}(\tilde{\mathcal{N}}_1(\partial_\tau u))(z),
$$

which combined with the estimate for II yields the desired result in the case $q=1$. $\square$

Lemma 5.6.2. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Let $u(x,t) = \partial_t \mathcal{S}_t^{\mathcal{L}} f(x)$ for some $f \in C_c^\infty(\mathbb{R}^n)$ or $u(x,t) = \mathcal{D}_t^{\mathcal{L}} g$ for some $g \in C_c^\infty(\mathbb{R}^n)$. There exists*

$m_6 \in \mathbb{N}$ and $\varepsilon_6 > 0$ such that if $m \geq m_6$ and $p \in (2-\varepsilon_6, 2+\varepsilon_6)$ then for every $q < p$

$$\|\tilde{\mathcal{N}}_q(\theta_{t,m} f)\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|\mathcal{S}(\Theta_{t,m+1} f)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,1} f)\|_{L^p(\mathbb{R}^n)},$$

with implicit constants depending on p, m, n and ellipticity; and where we have defined, in the case of $u(x,t) = \partial_t \mathcal{S}_t^{\mathcal{L}} f(x)$,

$$\Theta_{t,m} f := t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} f = t^m \partial_t^{m-1} \nabla u,$$

and

$$\theta_{t,m} f := t^m \partial_t^m \partial_t \mathcal{S}_t^{\mathcal{L}} f = t^m \partial_t^m u$$

and in the case of $u(x,t) = \mathcal{D}_t^{\mathcal{L}} g$

$$\Theta_{t,m} f = t^m \partial_t^{m_1} \nabla \mathcal{D}^{\mathcal{L},+} f$$

and

$$\theta_{t,m} f = t^m \partial_t^m \mathcal{D}^{\mathcal{L},+} f.$$

Therefore the conclusion can be rewritten, in terms of u, as

$$\|\tilde{\mathcal{N}}_q(t^m \partial_t^m u)\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|\mathcal{S}(t^{m+1} \partial_t^m \nabla u)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(t \nabla u)\|_{L^p(\mathbb{R}^n)}.$$

Proof. For $m \geq 0$ let us define a modified version of the averages a_q given by

$$a_{q,m}(u)(x,t) := \Big(\fint\!\!\fint_{\mathcal{C}_{x,t}} |t^m \partial_s^m u(y,s)|^q \, dyds \Big)^{1/q} \approx \Big(\fint\!\!\fint_{\mathcal{C}_{x,t}} |s^m \partial_s^m u(y,s)|^q \, dyds \Big)^{1/q}.$$

In particular, for $z \in \mathbb{R}^n$,

$$\tilde{\mathcal{N}}_q(\theta_{t,m})(z) \approx \sup_{(x,t) \in \gamma(z)} a_{q,m}(u)(x,t).$$

Writing for $\lambda > 0$,

$$\begin{aligned} &|\{z \in \mathbb{R}^n : \tilde{\mathcal{N}}_q(\theta_{t,m} f)(z) > \lambda\}| \\ &\qquad \leq |\{z \in \mathbb{R}^n : \tilde{\mathcal{N}}_q(\theta_{t,m} f)(z) > \lambda, \, \mathcal{S}(\Theta_{t,m+1} f)(z) \leq \gamma\lambda\}| \end{aligned}$$

$$+ |\{z \in \mathbb{R}^n : \mathcal{S}(\Theta_{t,m+1} f)(z) > \gamma\lambda\}|,$$

we see that it is enough to prove that, for $\gamma > 0$,

$$|E_{\lambda,\varepsilon}| := |\{z \in \mathbb{R}^n : \tilde{\mathcal{N}}_q^{\varepsilon}(\theta_{t,m} f)(z) > \lambda,\, \mathcal{S}(\Theta_{t,m+1} f)(z) \leq \gamma\lambda\}| \\ \lesssim \lambda^{-p} \|\mathbb{V}(\Theta_{t,1} f)\|_{L^p(\mathbb{R}^n)},$$

with constants uniform in ε and λ. Define the function φ_ε as follows:

$$\varphi_\varepsilon(x) := \inf \Big\{ t > \varepsilon : \sup_{(y,s) \in (x,t) + \gamma(0)} a_{q,m}(u)(y,s) \leq \lambda \Big\}.$$

Recall that we have, from the control on slices in Corollary 5.5.2, if $p \in (2 - \varepsilon_0, 2 + \varepsilon_0)$ (here ε_0 is as in Theorem 5.4.2 and Corollary 5.5.2)

$$\sup_{t>0} \|t^m \partial_t^m u(\cdot, t)\|_{L^p(\mathbb{R}^n)} < \infty, \tag{5.6.3}$$

so in particular

$$a_{q,m}(u)(y,s) \lesssim s^{-n/q} \sup_{t>0} \|t^m \partial_t^m u\|_{L^p(\mathbb{R}^n)}. \tag{5.6.4}$$

We conclude from this

$$\lim_{s \to \infty} \sup_{y \in \mathbb{R}^n} a_{q,m}(u)(y,s) = 0,$$

and so $\varepsilon \leq \varphi_\varepsilon(x) < \infty$ for every $x \in \mathbb{R}^n$. Moreover φ_ε is a Lipschitz function with constant 1, since it satisfies the appropriate uniform cone condition. We set

$$\gamma_{\varphi_\varepsilon} := \{(x,t) \in \mathbb{R}^{n+1}_+ : t = \varphi_\varepsilon(x)\},$$

the graph of φ_ε. Finally recall that we denote $(u)_A$ the average of u over a set $A \subset \mathbb{R}^{n+1}_+$ of finite measure.

We first claim that, for every $z \in E_{\lambda,\varepsilon}$ and if we denote $Z_\varepsilon := (z, \varphi_\varepsilon(z))$,

$$\lambda \lesssim \mathcal{M}_{\gamma_{\varphi_\varepsilon}}(a_{q,m}(u))(Z_\varepsilon),$$

for some implicit constant independent of λ and ε, and where $\mathcal{M}_{\gamma_{\varphi_\varepsilon}}$ denotes the maximal function on $\gamma_{\varphi_\varepsilon}$ with its natural surface measure, which we denote by σ.

To see this fix $z \in E_{\lambda,\varepsilon}$ and $(x,t) \in Z_\varepsilon + \gamma(0)$ and note that, owing to (5.6.4) there exists $R > 0$ such that $a_{q,m}(u)(x,t) \leq \lambda/2$ if $t > R$. Moreover using that $\tilde{\mathcal{N}}_q^\varepsilon(\theta_{t,m}f)(z) > \lambda$, so that $\varphi_\varepsilon(z) > \varepsilon$, there exists $(y,s) \in \overline{Z_\varepsilon + \gamma(0)}$ satisfying $a_{q,m}(u)(x,t) > \lambda$. By continuity of $a_{q,m}$ in $\mathbb{R}^{n+1}_+$ we conclude from the above that there exists a point $(x,t) \in \overline{Z_\varepsilon + \gamma(0)}$ such that $a_{q,m}(u)(x,t) = \lambda$, and $(x,t) \in \overline{\{a_{q,m}(u) > \lambda\}}$.

Note also that the above implies $(x,t) \in \gamma_{\varphi_\varepsilon}$, i.e. $t = \varphi_\varepsilon(x)$: For every $\delta > 0$ there exists $(y,s) \in B((x,t),\delta)$ such that $a_{q,m}(y,s) > \lambda$, and so, since $B((x,t),\delta) \subset (x, t - \sqrt{2}\delta) + \gamma(0)$, we have $\varphi_\varepsilon(x) > t - \sqrt{2}\delta$. Since δ was arbitrary we conclude $\varphi_\varepsilon(x) \geq t$. On the other hand, by the Lipschitz condition on φ_ε it can't happen that $\gamma_{\varphi_\varepsilon}$ intersects the interior of the cone $Z_\varepsilon + \gamma(0)$, therefore $\varphi_\varepsilon(x) \leq t$. Notice that, in fact, the above shows that $(x,t) = (x, \varphi_\varepsilon(x)) \in \partial(Z_\varepsilon + \gamma(0))$.

Given such a point $(x,t) = (x, \varphi_\varepsilon(x)) := X_\varepsilon$, and for any $(y,s) \in B(X_\varepsilon, t/100)$ we have, by the Poincaré-Sobolev inequality and writing $v_m(w,\tau) := \partial_\tau^m u(w,\tau)$,

$$
\begin{aligned}
&\lambda = a_{q,m}(u)(x,t) \\
&\leq \Big(\fint\!\!\!\fint_{\mathcal{C}_{x,t}} t^{2m} |v_m - (v_m)_{\mathcal{C}_{x,t}}|^q \, dw d\tau \Big)^{1/q} + t^m |(v_m)_{\mathcal{C}_{x,t}} - (v_m)_{\mathcal{C}_{y,s}}| + t^m |(v_m)_{\mathcal{C}_{y,s}}| \\
&\lesssim \Big(\iint_{\mathcal{C}_{x,t}} |\tau^m \nabla v_m|^2 \tau^{1-n} \, dw d\tau \Big)^{1/2} + t^m |(v_m)_{\mathcal{C}_{y,s}}| \lesssim \mathcal{S}(\Theta_{t,m+1} f)(x) + t^m |(v_m)_{\mathcal{C}_{y,s}}| \\
&\qquad \leq \gamma\lambda + t^m |(v_m)_{\mathcal{C}_{y,s}}| \leq \gamma\lambda + a_{q,m}(u)(y,s),
\end{aligned}
$$

where we also used the fact that $x \in E_{\lambda,\varepsilon}$ and that $\tau \approx s \approx t$. Choosing $\gamma < 1$ small enough above we can write

$$
\lambda \lesssim a_{q,m}(u)(y,s), \quad \forall (y,s) \in B(X_\varepsilon, t/100).
$$

From here (5.6.4) follows easily: Integrating the above inequality on $\gamma_{\varphi_\varepsilon}$ we have

$$
\lambda \lesssim \fint_{B(X_\varepsilon, t/100) \cap \gamma_{\varphi_\varepsilon}} a_{q,m}(u)(W) \, d\sigma(W) \lesssim \fint_{B(X_\varepsilon, t) \cap \gamma_{\varphi_\varepsilon}} a_{q,m}(u)(W) \, d\sigma(W),
$$

where we have used that $\sigma(B) \approx r(B)$ for any ball centered on $\gamma_{\varphi_\varepsilon}$. Moreover, since $X_\varepsilon \in \partial(Z_\varepsilon + \gamma(0))$ we have

$$
|z - x| = |t - \varphi_\varepsilon(z)| = t - \varphi_\varepsilon(x) < t,
$$

and so we conclude $Z_\varepsilon \in B(X_\varepsilon, t)$ and (5.6.4) follows.

Since (5.6.4) holds for any $z \in E_{\lambda,\varepsilon}$ we see that

$$|E_{\lambda,\varepsilon}| \lesssim |\{W \in \gamma_{\varphi_\varepsilon} : \mathcal{M}_{\gamma_{\varphi_\varepsilon}}(a_{q,m}(u))(W)\}| \lesssim \lambda^{-p} \int_{\gamma_{\varphi_\varepsilon}} a_{q,m}(u)^p(W)\, d\sigma(W).$$

Therefore, it is enough to prove

$$\int_{\gamma_{\varphi_\varepsilon}} a_{q,m}(u)^p(W)\, d\sigma(W) \lesssim \|\mathbb{V}(t\nabla u)\|^p_{L^p(\mathbb{R}^n)}. \tag{5.6.5}$$

We make a further reduction as follows: Note that, by the L^q Caccioppoli's inequality applied m times, for any $(x,t) \in \mathbb{R}^{n+1}_+$

$$\begin{aligned} a_{q,m}(u)(x,t) &= \Big(\fint\!\!\fint_{\mathcal{C}_{x,t}} |t^m \partial_t^m u(y,s)|^q \, dyds \Big)^{1/q} \\ &\lesssim \Big(\fint\!\!\fint_{\mathcal{C}^*_{x,t}} |u(y,s)|^q \, dyds \Big)^{1/q} \\ &\approx a_q^*(u)(x,t). \end{aligned}$$

Therefore the result would follow from

$$\int_{\mathbb{R}^n} |a_q^*(u)(x,\varphi_\varepsilon(x))|^p \, dy \approx \int_{\gamma_{\varphi_\varepsilon}} |a_q^*(W)|^p \, d\sigma(W) \lesssim \|\mathbb{V}(t\nabla u)\|_{L^p(\mathbb{R}^n)}.$$

To simplify notation in what follows we set

$$g(x) := a_q^*(u)(x,\varphi_\varepsilon(x)), \quad \nu(x) = \mathcal{M}(g)(x)^{p-q}.$$

We then have

$$\int_{\mathbb{R}^n} |a_q^*(u)(x,\varphi_\varepsilon(x))|^p \, dy = \int_{\mathbb{R}^n} g^p(x)\, dx \leq \int_{\mathbb{R}^n} g^q(x)\nu(x)\, dx.$$

Note that

$$\frac{4}{5}\varphi_\varepsilon(y) \leq \varphi_\varepsilon(x) \leq \frac{4}{3}\varphi_\varepsilon(y), \quad \text{whenever } |x-y| < \frac{\varphi_\varepsilon(x)}{4},$$

owing to the fact φ_ε is Lipschitz with constant one.

We now go back to the definition of g and a_q^* to compute,

$$
\begin{aligned}
\int_{\mathbb{R}^n} g^q(x)\nu(x)\,dx &\approx \int_{\mathbb{R}^n} \fint_{|x-y|<\varphi_\varepsilon/4} \fint_{3\varphi_\varepsilon(x)/4}^{5\varphi_\varepsilon(x)/4} |u(y,s)|^q\,dsdy\,\nu(x)dx \\
&\lesssim \int_{4/5}^{5/3} \int_{\mathbb{R}^n} \fint_{|x-y|<\varphi_\varepsilon(x)/8} |u(y,\tau\varphi_\varepsilon(y))|^q\,dy\,\nu(x)dx\,d\tau \\
&\lesssim \int_{4/5}^{5/3} \int_{\mathbb{R}^n} \mathcal{M}\Big(|u(\cdot,\tau\varphi_\varepsilon(\cdot))|^q\Big)(x)\,\nu(x)dx\,d\tau.
\end{aligned}
$$

By Hölder's Inequality with exponents p/q and $p/(p-q)$ we see

$$
\int_{\mathbb{R}^n} \mathcal{M}\Big(|u(\cdot,\tau\varphi_\varepsilon(\cdot))|^q\Big)(x)\,\nu(x)dx \le \|\mathcal{M}_q(u(\cdot,\tau\varphi_\varepsilon(\cdot)))\|_{L^p(\mathbb{R}^n)}^q \|\nu\|_{L^{p/(p-q)}(\mathbb{R}^n)},
$$

and, since $q<p$, by the boundedness of $\mathcal{M}_q$ in L^p

$$
\|\mathcal{M}_q(u(\cdot,\tau\varphi_\varepsilon(\cdot)))\|_{L^p(\mathbb{R}^n)}^q \lesssim \Big(\int_{\mathbb{R}^n} |u(x,\tau\varphi_\varepsilon(x))|^p\,dx\Big)^{q/p}.
$$

Similarly,

$$
\|\nu\|_{L^{p/(p-q)}(\mathbb{R}^n)} = \|\mathcal{M}(g)\|_{L^p(\mathbb{R}^n)}^{p-q} \lesssim \Big(\int_{\mathbb{R}^n} |g(x)|^p\,dx\Big)^{(p-q)/p}
$$

Combining the above estimates, we obtain

$$
\int_{\mathbb{R}^n} g^p(x)\,dx \lesssim \int_{4/5}^{5/3} \Big(\int_{\mathbb{R}^n} |u(x,\tau\varphi_\varepsilon(x))|^p\,dx\Big)^{q/p} \Big(\int_{\mathbb{R}^n} g^p(x)\,dx\Big)^{(p-q)/p}\,d\tau.
$$

Using Hölder's inequality again (perhaps working with $g(x)\mathbb{1}_{g<M}$ if necessary in order to divide by $\|g\|_{L^p(\mathbb{R}^n)}$) we arrive at

$$
\int_{\mathbb{R}^n} g^p(x)\,dx \lesssim \int_{4/5}^{5/3} \int_{\mathbb{R}^n} |u(x,\tau\varphi_\varepsilon(x))|^p\,dxd\tau.
$$

We now note that $\tau\varphi_\varepsilon$ is a Lipschitz function with constant τ. Therefore the function $v(x,t) = u(x,t+\varphi_\varepsilon(x))$ solves $\mathcal{L}_{\tau\varphi_\varepsilon} v = 0$ in $\mathbb{R}^{n+1}_+$, where the operator $\mathcal{L}_{\tau\varphi_\varepsilon}$ is of the same type as $\mathcal{L}$ and moreover its coefficients are controlled (in the appropriate norms) by those of $\mathcal{L}$. Therefore, by the control on slices by the vertical square function in Theorem

5.5.1 (see also Lemma 5.5.5)

$$\int_{\mathbb{R}^n} |u(x,\tau\varphi_\varepsilon(x))|^p\,dx = \int_{\mathbb{R}^n} |v(x,0)|^p\,dx \lesssim \int_{\mathbb{R}^n} \Big(\int_0^\infty |t\nabla v(x,t)|^2\,\frac{dt}{t}\Big)^{p/2}\,dx$$
$$\lesssim \int_{\mathbb{R}^n} \Big(\int_{\tau\varphi_\varepsilon(x)}^\infty |\nabla u(x,t)|^2\,tdt\Big)^{p/2}\,dx$$
$$\le \|\mathbb{V}(t\nabla u)\|^p_{L^p(\mathbb{R}^n)}.$$

This yields (5.6.5) and the result is proved. □

Remark 5.6.6. Notice that in the above lemma, the only things required for its proof were:

1. $\theta_{t,m}$ satisfies good estimates on slices (as in (5.6.3)).
2. We already have, *for all* operators of the form $\mathcal{L} = -\operatorname{div}(A\nabla + B_1) + B_2\cdot\nabla$ (with sufficient smallness of the first order coefficients), the control on slices by the vertical square function

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbb{V}(t\nabla u)\|_{L^p(\mathbb{R}^n)}.$$

The following result uses the fact, proved in the next section, that $\mathbb{V}(t^m\partial_t^{m-1}\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla))$ satisfies L^p bounds for all $m \ge 1$.

Corollary 5.6.7 (L^p estimate for non-tangential maximal functions of layer potentials)**.** *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. If $\varepsilon_0 > 0$ and m_0 are as in Theorem 5.4.2, and if $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ and $m \ge m_0 \ge 1$ then*

$$\|\tilde{\mathcal{N}}_2(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

where $\Theta_{t,m}$ is either $t^m\partial_t^m\nabla(\mathcal{S}_t^{\mathcal{L}}\nabla)$ or $t^m\partial_t^{m-1}\nabla\mathcal{D}_t^{\mathcal{L}}$.

Proof. We treat only the single layer. The double layer argument is identical. Also, notice by t independence it is enough to treat the operator with the inside gradient replaced by $\nabla_\|$.

First, from the pointwise inequality in Proposition 5.6.1 and the dominated convergence theorem, we see that for any $q \ge 1$ and p in the above range, and setting

$\theta_{t,m} = t^m \partial_t^m (\mathcal{S}_t^{\mathcal{L}} \nabla_{\|}) = -t^m \partial_t^m \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_{\|}$

$$\|\tilde{\mathcal{N}}_2(\Theta_{t,m} f)\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_q(\theta_{t,m} f)\|_{L^{p,\infty}(\mathbb{R}^n)} + \sup_{t>0} \|\Theta_{t,m} f\|_{L^p(\mathbb{R}^n)}.$$

In particular by the slices estimates in Corollary 5.5.2 and Theorem 5.5.4, and choosing q as in the above Lemma,

$$\|\tilde{\mathcal{N}}_2(\Theta_{t,m} f)\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|\mathcal{S}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,1} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

where we have used Theorem 5.7.1 for the last step; ensuring that $\mathbb{V}(\Theta_{t,1} f)$ is under control. The result now follows from real interpolation. □

5.7 Traveling Down

We first establish the vertical square function estimates, since there is little difficulty there. The discrepancy between these so-called traveling down procedures for the vertical and conical square functions should be contrasted with the situation in the extrapolation arguments.

Theorem 5.7.1. *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. Let ε_0 be as in Theorem 5.4.2, then for $p \in (2-\varepsilon_0, 2+\varepsilon_0)$ and every $m \geq 1$ it holds*

$$\|\mathbb{V}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim_m \|f\|_{L^p(\mathbb{R}^n)},$$

where $\Theta_{t,m}$ is either $t^m \partial_t^{m-1} \nabla (\mathcal{S}_t^{\mathcal{L}} \nabla)$ or $t^m \partial_t^{m-1} \nabla \mathcal{D}_t^{\mathcal{L}}$.

Proof of Theorem 5.7.1. We employ the same idea as in the L^2 case from [BHL+b]; integrating by parts in t to control the square function of $\Theta_{t,m}$ in terms of $\Theta_{t,m+1}$ plus terms that are bounded in L^p. Notice that for $m \geq m_0$ large enough the desired bound is a consequence of Theorem 5.4.2 and Lemma 5.4.9 for the case of the double layer.

We start by defining, for $\eta > 0$,

$$\int_{\mathbb{R}^n} \Big(\int_0^\infty |\Theta_{t,m} f(x)|^2 \, \frac{dt}{t} \Big)^{p/2} dx = \lim_{\eta \to 0^+} \int_{\mathbb{R}^n} \Big(\int_\eta^{1/\eta} |\Theta_{t,m} f(x)|^2 \, \frac{dt}{t} \Big)^{p/2} dx =: I_\eta.$$

In particular, owing to the estimates on slices from Corollary 5.5.2, we see that $I_\eta < \infty$

for all such η.

We now carry out the integration by parts

$$
\begin{aligned}
I_\eta = \int_{\mathbb{R}^n} \Big(\int_\eta^{1/\eta} t^{2m-1} |\partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_\| f(x)|^2 \, dt \Big)^{p/2} dx \\
\leq \int_{\mathbb{R}^n} \Big(|t^m \partial_t^m \nabla \mathcal{S}_t^{\mathcal{L}} \operatorname{div}_\| f(x)|^2 \Big|_{t=\eta}^{1/\eta} \Big)^{p/2} dx \\
+ \int_{\mathbb{R}^n} \Big(\int_\eta^{1/\eta} |\Theta_{t,m} f(x)| |\Theta_{t,m+1} f(x)| \frac{dt}{t} \Big)^{p/2} dx \\
\leq C_m \|f\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \Big(\int_\eta^{1/\eta} |\Theta_{t,m} f(x)| |\Theta_{t,m+1} f(x)| \frac{dt}{t} \Big)^{p/2} dx,
\end{aligned}
$$

where we used the estimates on slices in Corollary 5.5.2 for the single layer and Theorem 5.5.4 for the double layer for the last line. Finally we use Cauchy's inequality with a parameter to obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} \Big(\int_\eta^{1/\eta} |\Theta_{t,m} f(x)|^2 \frac{dt}{t} \Big)^{p/2} dx \\
\lesssim \|f\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \Big(\int_\eta^{1/\eta} |\Theta_{t,m+1} f(x)|^2 \frac{dt}{t} \Big)^{p/2} dx.
\end{aligned}
$$

Letting $\eta \to 0$ we can write

$$
\|\mathbb{V}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,m+1} f)\|_{L^p(\mathbb{R}^n)}.
$$

The result now follows by induction and Theorem 5.4.2. □

We now turn to the much harder task of traveling down with the conical square function. Here, although the idea is the same basic integration by parts technique, the arguments become much more elaborate due to the 'space averaging' happening alongside the integration over the transversal variable. To handle this we will make use of the non-tangential maximal function estimates from the previous section (see Lemma 5.6.2) through a modified version of the classical Carleson embedding lemma (see Lemma 5.2.29). However, the use of the non-tangential maximal function makes the traveling down procedure for either this object or the conical square function a bit subtle. It is our hope that the following lemma (which should be read with the results of the previous section in mind) and Theorem 5.7.11 clarifies the intertwining of these two procedures.

We also note that for the range $p > 2$ we already have the conical square function bounded by the vertical (see Proposition 5.2.4) by general facts about square functions, so it is only the case $p < 2$ that is of interest.

Lemma 5.7.2. *Let $\Theta_{t,m}$ be either $t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$ or $t^m \partial_t^{m-1} \nabla \mathcal{D}_t^{\mathcal{L}}$. Suppose that $m \geq 1$ is given such that*

$$\mathcal{S}(\Theta_{t,m} f), \mathcal{S}(\Theta_{t,m+1} f), \mathbb{V}(\Theta_{t,m+1} f), \mathbb{V}(\Theta_{t,m} f), \tilde{\mathcal{N}}(\Theta_{t,m} f) \in L^2(\mathbb{R}^n),$$

and

$$\sup_{t>0} \|\Theta_{t,m} f\|_{L^2(\mathbb{R}^n)} < \infty, \quad \lim_{t\to\infty} \|\Theta_{t,m} f\|_{L^2(\mathbb{R}^n)} = 0$$

for every $f \in C_c^\infty(\mathbb{R}^n)$. Then, for every $1 < p < 2$

$$\begin{aligned} \|\mathcal{S}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} &\lesssim \|\mathcal{S}(\Theta_{t,m+1} f)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,m+1} f)\|_{L^p(\mathbb{R}^n)} \\ &\quad + \|\mathbb{V}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} + \|\tilde{\mathcal{N}}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} + \sup_{t>0} \|\Theta_{t,m} f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

for every $f \in C_c^\infty(\mathbb{R}^n)$.

Proof. We fix $m \geq 1$ as in the hypotheses and define $g_m := \sup_{t>0} |\Theta_{t,m} f|$, $h_m = \tilde{\mathcal{N}}(\Theta_{t,m})$, and $H_m := \mathcal{S}(\Theta_{t,m+1} f) + g_m + h_m$. Recall from Proposition 5.2.48

$$\|g_m\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathbb{V}(\Theta_{t,m} f)\|_{L^2(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,m+1} f)\|_{L^2(\mathbb{R}^n)} + \sup_{t>0} \|\Theta_{t,m} f\|_{L^2(\mathbb{R}^n)}.$$

Therefore $H_m \in L^2(\mathbb{R}^n)$ and so, by the Coifman-Rochberg theorem, we see that if we define $\nu(x) := \mathcal{M}(\mathcal{S}(\Theta_{t,m} f) + H_m)(x)^{p-2}$ for some $1 < p < 2$, then $\nu^M \in A_2$ for any $M > 1$ satisfying $M(2-p) < 1$ and moreover $[\nu^M]_{A_2}$ depends only on the quantity $M(2-p)$.

We now mimic the proof of the extrapolation lemma 5.2.26 and write, for fixed $1 < p < 2$,

$$\begin{aligned} \int_{\mathbb{R}^n} |\mathcal{S}(\Theta_{t,m} f)(x)|^p \, dx &\leq \int_{\mathbb{R}^n} \mathcal{M}(\mathcal{S}(\Theta_{t,m} f) + H_m)(x)^p \, dx \\ &= \int_{\mathbb{R}^n} \mathcal{M}(\mathcal{S}(\Theta_{t,m} f) + H_m)(x)^2 \nu(x) \, dx \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} \mathcal{M}(\mathcal{S}(\Theta_{t,m}f))(x)^2\nu(x)\,dx + \int_{\mathbb{R}^n} \mathcal{M}(H_m)(x)^2\nu(x)\,dx \\
&\quad \lesssim_{[\nu]_{A_2}} \int_{\mathbb{R}^n} \mathcal{S}(\Theta_{t,m}f)(x)^2\nu(x)\,dx + \int_{\mathbb{R}^n} \mathcal{M}(H_m)(x)^p\,dx \\
&\qquad \lesssim \int_{\mathbb{R}^n} \mathcal{S}(\Theta_{t,m}f)(x)^2\nu(x)\,dx + \int_{\mathbb{R}^n} H_m(x)^p\,dx, \quad (5.7.3)
\end{aligned}
$$

where we used the boundedness of $\mathcal{M}$ in $L^2(\nu)$, by the above discussion, and in $L^p(\mathbb{R}^n)$. By definition of H_m, and Proposition 5.2.48, we have

$$
\begin{aligned}
\|H_m\|_{L^p(\mathbb{R}^n)} \lesssim_p \|\mathbb{V}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,m+1}f)\|_{L^p(\mathbb{R}^n)} + \sup_{t>0}\|\Theta_{t,m}f\|_{L^p(\mathbb{R}^n)} \\
+ \|\mathcal{S}(\Theta_{t,m+1}f)\|_{L^p(\mathbb{R}^n)} + \|\tilde{\mathcal{N}}(\Theta_{t,m}f)\|_{L^p(\mathbb{R}^n)}. \quad (5.7.4)
\end{aligned}
$$

It thus remains to estimate the first term in (5.7.3). For this we will try to emulate the procedure for the vertical square function, introducing an approximate identity P_t to smooth-out the averaging implicit in the definition of $\mathcal{S}$. We will first fix the approximate identity: For $t > 0$ we define

$$
P_t := e^{t^2\Delta}, \quad Q_t := t\partial_t e^{t^2\Delta} = t\partial_t P_t.
$$

We will also need to truncate our weight to formally justify our computations so we define, for $N > 0$,

$$
\nu_N := \min(\nu, N).
$$

We first compute, using Fubini's theorem

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{S}(\Theta_{t,m}f)(x)^2\,\nu(x)dx &= \int_{\mathbb{R}^n}\int_0^\infty \fint_{|x-y|<t} |\Theta_{t,m}f(y)|^2\,dy\frac{dt}{t}\nu(x)\,dx \\
&= \int_{\mathbb{R}^n}\int_0^\infty |\Theta_{t,m}f(y)|^2 \fint_{|x-y|<t}\nu(x)\,dx\frac{dt}{t}\,dy \\
&\lesssim \int_{\mathbb{R}^n}\int_0^\infty |\Theta_{t,m}f(y)|^2 P_t\nu(y)\,\frac{dydt}{t} =: I
\end{aligned}
$$

Now, by the Monotone convergence theorem,

$$
I \quad = \quad \lim_{N\to\infty}\lim_{\varepsilon\to 0} I_{\varepsilon,N} \quad := \quad \lim_{N\to\infty}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_\varepsilon^{1/\varepsilon} |\Theta_{t,m}f(y)|^2 P_t\nu_N(y)\,\frac{dydt}{t}.
$$

Recalling the definition of $\Theta_{t,m}$ and integrating by parts in t we obtain, recalling that $Q_t := t\partial_t P_t$,

$$
\begin{aligned}
&I_{\varepsilon,N}\\
&= -\frac{1}{2m}\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon} \partial_t\Big(|\partial_t^m \mathcal{S}_t^{\mathcal{L}} f|^2 P_t\nu_N\Big) t^{2m}\,dydt + \frac{1}{2m}\int_{\mathbb{R}^n} |\Theta_{t,m}f|^2 P_t\nu_N\,dy\Big|_{t=\varepsilon}^{1/\varepsilon}\\
&\leq \frac{1}{m}\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m}f||\Theta_{t,m+1}f|P_t\nu_N\,\frac{dydt}{t} + \frac{1}{2m}\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m}f|^2|Q_t\nu_N|\,\frac{dydt}{t}\\
&\qquad + \frac{1}{2m}\int_{\mathbb{R}^n} |\Theta_{1/\varepsilon,m}f|^2 P_{1/\varepsilon}\nu_N\,dy - \frac{1}{2m}\int_{\mathbb{R}^n} |\Theta_{\varepsilon,m}f|^2 P_\varepsilon\nu_N\,dy\\
&\qquad =: II_{\varepsilon,N} + III_{\varepsilon,N} + IV_{\varepsilon,N} + V_{\varepsilon,N}. \qquad (5.7.5)
\end{aligned}
$$

We handle the boundary terms first. To start we note that by Theorem 1.4 in [BHL$^+$b] we have that $\lim_{t\to\infty}\|\Theta_{t,m}f\|_{L^2(\mathbb{R}^n)} \to 0$, so that, using $|P_t\nu_N| \lesssim N$ pointwise in $\mathbb{R}^n$, we obtain

$$IV_{\varepsilon,N} \lesssim N\int_{\mathbb{R}^n} |\Theta_{1/\varepsilon,m}f|^2\,dy \to 0 \qquad \text{as} \quad \varepsilon\to 0.$$

Therefore

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0} IV_{\varepsilon,N} = 0. \qquad (5.7.6)$$

On the other hand, as $\varepsilon\to 0$, we have $P_\varepsilon\nu_N \to \nu_N$ pointwise a.e. and $|\Theta_{\varepsilon,m}f|^2 \leq g_m^2$, by definition of g_m. By the Dominated Convergence Theorem (recall $g_m \in L^2(\mathbb{R}^n)$ and $P_\varepsilon\nu_N \lesssim N$) we get that

$$V_{\varepsilon,N} \lesssim \int_{\mathbb{R}^n} g_m^2 P_\varepsilon\nu_N\,dy \overset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^n} g_m^2\nu_N\,dy \leq \int_{\mathbb{R}^n} g_m^2\nu\,dy \lesssim \int_{\mathbb{R}^n} g_m^p\,dy,$$

where we used the definition of ν in the last line to conclude $\nu(y) \leq \mathcal{M}(g_m)(y)^{p-2}$ and the Hardy-Littlewood maximal theorem. We conclude

$$\limsup_{N\to\infty}\limsup_{\varepsilon\to 0} V_{\varepsilon,N} \lesssim \int_{\mathbb{R}^n} g_m^p\,dx. \qquad (5.7.7)$$

The first term $II_{\varepsilon,N}$ we can treat as usual; using Cauchy's inequality with a parameter we see

$$II_{\varepsilon,N} \lesssim \delta\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m}f|^2 P_t\nu_N\,\frac{dydt}{t} + C(\delta)\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m+1}f|^2 P_t\nu_N\,\frac{dydt}{t}$$

$$= \delta I_{\varepsilon,N} + C(\delta) \int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m+1} f|^2 P_t \nu_N \, \frac{dydt}{t} \quad (5.7.8)$$

Choosing δ small enough we can hide the first term on the right hand side of (5.7.5).

Finally, we rewrite $III_{\varepsilon,N}$ in the following way, using the Cauchy-Schwarz inequality,

$$III_{\varepsilon,N} = \int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m} f| \frac{|Q_t \nu_N|}{|P_t \nu_N|} \sqrt{P_t \nu_N} |\Theta_{t,m} f| \sqrt{P_t \nu_N} \, \frac{dydt}{t}$$
$$\leq I_{\varepsilon,N} \int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m} f|^2 d\mu_N(y,t),$$

where we have defined

$$d\mu_N(x,t) := \frac{|Q_t \nu_N(x)|^2}{|P_t \nu_N(x)|^2} P_t \nu_N(x) \, \frac{dxdt}{t}.$$

By Proposition 5.2.37 and the modified Carleson's lemma (Lemma 5.2.29) we obtain

$$\int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m} f|^2 d\mu_N(y,t) \lesssim \int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\Theta_{t,m})^2 \, \nu_N \, dy = \int_{\mathbb{R}^n} h_m \nu_N \, dy \lesssim \int_{\mathbb{R}^n} h_m^p \, dy.$$

Therefore, applying once again Cauchy's inequality with a parameter, we see that

$$III_{\varepsilon,N} \lesssim \delta I_{\varepsilon,N} + C(\delta) \int_{\mathbb{R}^n} h_m^p \, dy. \quad (5.7.9)$$

Combining the estimates (5.7.8), (5.7.9) we arrive at

$$I_{\varepsilon,N} \lesssim \int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m+1} f|^2 P_t \nu_N \, \frac{dydt}{t} + \int_{\mathbb{R}^n} H_m^p \, dy + IV_{\varepsilon,N} + V_{\varepsilon,N},$$

where we notice that $I_{\varepsilon,N} < \infty$ since $\nu_N \leq N$ and $\sup_{t>0} \|\Theta_{t,m} f\|_{L^2(\mathbb{R}^n)} < \infty$. We now use Proposition 5.2.37 to get that $|P_t \nu_N(y)| \lesssim \fint_{|x-y|<t} \nu_N \, dx$ and so

$$\int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} |\Theta_{t,m+1} f(y)|^2 P_t \nu_N(y) \, \frac{dydt}{t}$$
$$\lesssim \int_{\mathbb{R}^n} \int_{\varepsilon}^{1/\varepsilon} \fint_{|x-y|<t} |\Theta_{t,m+1} f(y)|^2 \, \frac{dydt}{t} \nu(x) \, dx.$$

Now taking first the limit as $\varepsilon \to 0$ and then as $N \to \infty$, and using (5.7.6), (5.7.7) and the previous equation we see that

$$I \lesssim \int_{\mathbb{R}^n} \mathcal{S}(\Theta_{t,m+1})^2 \, \nu dx + \int_{\mathbb{R}^n} H_m^p \, dx \lesssim \int_{\mathbb{R}^n} H_m^p \, dx.$$

The result now follows from the definition of H_m, more specifically (5.7.4). $\square$

Remark 5.7.10. As far as the hypotheses of the previous result are concerned, we note that we have good control on the quantities involving $\mathbb{V}$, by Theorem 5.7.1. Moreover by [BHL+b, Theorem 6.12] (see also Hypothesis A in Definition 5.4.1), the conditions on the quantities $\|\Theta_{t,m} f\|_{L^2(\mathbb{R}^n)}$ are also satisfied. Therefore, under the same hypotheses as Theorem 5.4.2, we may rewrite the above as

$$\|\mathcal{S}(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{S}(\Theta_{t,m+1} f)\|_{L^p(\mathbb{R}^n)} + \|\tilde{\mathcal{N}}_2(\Theta_{t,m} f)\|_{L^p(\mathbb{R}^n)}.$$

As a Corollary of this result and Lemma 5.6.2 on the boundedness of the non-tangential maximal function we have the following

Theorem 5.7.11. *Suppose $\mathcal{L}$ satisfies the hypotheses of Theorem 5.4.2. Let $p \in (2 - \varepsilon_0, 2 + \varepsilon_0)$, with ε_0 as in Theorem 5.4.2, then for every $f \in C_c^\infty(\mathbb{R}^n)$*

$$\|\mathcal{S}(t\partial_t \nabla \mathcal{S}_t^{\mathcal{L}} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \qquad \|\tilde{\mathcal{N}}_2(\nabla \mathcal{S}_t^{\mathcal{L}} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In addition

$$\|\mathcal{S}(t\nabla \mathcal{D}_t^{\mathcal{L}} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \qquad \|\tilde{\mathcal{N}}_2(\mathcal{D}_t^{\mathcal{L}} u)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We define $\Theta_{t,m}$ to be either $t^m \partial_t^{m-1} \nabla(\mathcal{S}_t^{\mathcal{L}} \nabla)$ or $t^m \partial_t^{m-1} \nabla \mathcal{D}_t^{\mathcal{L}}$. For $p > 2$ we have by Proposition 5.2.4 and Theorem 5.7.1

$$\|\mathcal{S}(\Theta_{t,1} f)\|_{L^p(\mathbb{R}^n)} + \|\mathbb{V}(\Theta_{t,1} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

It remains to show the non-tangential maximal function bound when $p < 2$ and $p > 2$ and the conical square function bound when $p < 2$. We will show both the square function and non-tangential maximal function bounds in the case $p < 2$. (The case of the non-tangential maximal function bounds when $p > 2$ the same.)

We treat the case of the single layer first. By Theorem 5.4.2, together with the traveling down for vertical square functions in Theorem 5.7.1 and Corollary 5.6.7 we see that for some $m_0 \geq 1$,

$$\|\mathcal{S}(\Theta_{t,m_0} f)\|_{L^p(\mathbb{R}^n)} + \|\tilde{\mathcal{N}}_2(\Theta_{t,m_0} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{5.7.12}$$

We shall show that (5.7.12) holds with m_0 replaced by $m_0 - 1$, as long as $m_0 - 1 \geq 1$.

To treat the non-tangential maximal function we appeal to Corollary 5.6.7 and obtain

$$\begin{aligned}\|\tilde{\mathcal{N}}_2(\Theta_{t,m_0-1})\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{S}(\Theta_{t,m_0} f)\|_{L^p(\mathbb{R}^n)} + \sup_{t>0} \|\Theta_{t,m_0-1} f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \\ \lesssim \|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

This gives the desired bound as long as $m_0 - 1 \geq 1$.

By the traveling down procedure for the conical square function (Lemma 5.7.2) we have (recall that the vertical square function is under control for any m by Theorem 5.7.1)

$$\begin{aligned}\|\mathcal{S}(\Theta_{t,m_0-1} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{S}(\Theta_{t,m_0} f)\|_{L^p(\mathbb{R}^n)} + \|\tilde{\mathcal{N}}(\Theta_{t,m_0-1} f)\|_{L^p(\mathbb{R}^n)} \\ + \sup_{t>0} \|\Theta_{t,m_0-1}\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

and this gives the desired square function bound for $m_0 - 1 \geq 1$. Therefore we have shown by induction that

$$\|\mathcal{S}(\Theta_{t,1} f)\|_{L^p(\mathbb{R}^n)}, \|\tilde{\mathcal{N}}_2(\Theta_{t,1})\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

To get the bound for $\tilde{\mathcal{N}}_2(\nabla \mathcal{S}_t^{\mathcal{L}})$ we use Proposition 5.6.1 to get

$$\|\tilde{\mathcal{N}}_2(\nabla \mathcal{S}_t^{\mathcal{L}} f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\tilde{\mathcal{N}}_q(\partial_t \mathcal{S}_t^{\mathcal{L}} f)\|_{L^p(\mathbb{R}^n)} + \sup_{t>0} \|\nabla \mathcal{S}_t^{\mathcal{L}} f\|_{L^p(\mathbb{R}^n)},$$

for any $1 \leq q$. In particular, choosing $q < p$ we can apply directly Lemma 5.6.2 and interpolation to obtain the result.

The double layer is handled in the same way, owing to the appropriate estimates from Theorem 5.4.15, Theorem 5.5.4 and Corollary 5.6.7. □

5.8 Existence

Proposition 5.8.1 (Mapping Properties, Part I)**.** *The operators $\mathcal{S}^{\mathcal{L}} : C_c^\infty(\mathbb{R}^n) \to S_+^2$, $\mathcal{D}^{\mathcal{L},+} : C_c^\infty(\mathbb{R}^n) \to D_+^2$ both have unique continuous extensions to $L^2(\mathbb{R}^n)$; that is,*

$$\mathcal{S}^{\mathcal{L}} : L^2(\mathbb{R}^n) \to S_+^2, \quad \mathcal{D}^{\mathcal{L},+} : L^2(\mathbb{R}^n) \to D_+^2.$$

Moreover, for $f, g \in L^2(\mathbb{R}^n)$ we have that $\mathcal{S}^{\mathcal{L}} g$, $\mathcal{D}^{\mathcal{L},+} f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ are solutions of $\mathcal{L}w = 0$ in $\mathbb{R}^{n+1}_+$, and we have the square function estimates

$$\|\mathcal{S}(t\partial_t \nabla \mathcal{S}_t^{\mathcal{L}} g)\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}^n)}, \qquad \|\mathcal{S}(t\nabla \mathcal{D}_t^{\mathcal{L},+} f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

Similar considerations also hold in the lower half space (in this case we work with $\mathcal{D}^{\mathcal{L},-}$).

The proof is a simple density argument (using the fact that S_+^2 and D_+^2 are Banach spaces), and as such is omitted.

The next result is a statement about Sobolev functions that will allow us to eventually assign boundary values to the extensions of the layer potentials defined above.

Proposition 5.8.2. *Let $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$. The following statements hold.*

(i) If $u \in D_+^2$ then $u_0 := \lim_{t\to 0^+} u(t)$ exists as a weak limit in $L^2(\mathbb{R}^n)$. Moreover u_0 agrees with the trace of u in the sense that for every $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$

$$(u_0, \Phi(\cdot, 0)) = \iint_{\mathbb{R}^{n+1}_+} \Big(u\partial_t \Phi + \partial_t u \Phi\Big),$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^n)$.

(ii) If $u \in S_+^2$ then $U_0 := \lim_{t\to 0^+} u(t)$ exists as a weak limit in $Y^{1,2}(\mathbb{R}^n)$. Moreover, U_0 agrees with the trace of u in the sense described in i).

Proof. To prove i), we start by noticing that since $u \in D_+^2$, there exists a subsequence $t_k \to 0^+$ and a function $u_0 \in L^2(\mathbb{R}^n)$ such that $u_{t_k} \to u_0$ weakly in $L^2(\mathbb{R}^n)$ as $k \to \infty$. Now, again since $u \in D_+^2$, we only need to show that

$$\lim_{t\to 0^+} (u(t), \phi) = (u_0, \phi), \qquad \text{for each } \phi \in C_c^\infty(\mathbb{R}^n).$$

Consider $\Phi(x,t) = \phi(x)\eta(t)$, where $\eta \in C_c^\infty(-2,2)$ is such that $\eta \equiv 1$ in $(-1,1)$, so that $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$. Now, the hypotheses imply that, for fixed $t > 0$, the identity

$$(u(t), \Phi(\cdot, t)) = \iint_{\mathbb{R}^{n+1}_t} \Big(u \partial_t \Phi + \partial_t u \Phi \Big),$$

holds, which, for our particular choice of Φ and $t < 1$, implies that

$$(u(t), \phi) = \iint_{\mathbb{R}^{n+1}_t} \Big(u \partial_t \Phi + \partial_t u \Phi \Big).$$

Hence, an application of the Dominated Convergence Theorem yields the desired result since $u \in L^{2(n+1)/(n-1)}(\mathbb{R}^{n+1}_+)$ and $\nabla u \in L^2(\mathbb{R}^{n+1}_+)$. The second part of the statement in i) now follows by the fact that $\Phi(\cdot, t) \to \Phi(\cdot, 0)$ strongly in $L^2(\mathbb{R}^n)$ for any $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$.

The proof of ii) follows similar ideas. Arguing as before, we can prove that there exists a weak limit $U_0 \in L^{2n/(n-2)}(\mathbb{R}^n)$, and that it agrees with the usual trace in the sense described in i). Similarly, along a subsequence $t_k \to 0$, we have that $\nabla_\| u(t_k) \to v \in L^2(\mathbb{R}^n)$ weakly for some $v \in L^2(\mathbb{R}^n)^n$. If we can show that $v = \nabla_\| U_0$, then the result would follow from the uniqueness of the limit. To this end, we fix $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^n)$ and compute that

$$(v, \phi) = \lim_{k \to \infty} (\nabla_\| u(t_k), \phi) = - \lim_{k \to \infty} (u(t_k), \operatorname{div}_\| \phi) = -(U_0, \operatorname{div}_\| \phi),$$

as desired. $\square$

Combining the two previous propositions, we arrive at the following corollary whose proof is standard and thus omitted.

Corollary 5.8.3. *For every $f, g \in L^2(\mathbb{R}^n)$ we can define the bounded linear operators*

$$\mathcal{S}_0^{\mathcal{L}} : L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n), \quad \mathcal{D}_0^{\mathcal{L},+} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

given by

$$\mathcal{S}_0^{\mathcal{L}} g := \lim_{t \to 0^+} \mathcal{S}_t^{\mathcal{L}} g, \quad \mathcal{D}_0^{\mathcal{L},+} f := \lim_{t \to 0+} \mathcal{D}_t^{\mathcal{L},+} f,$$

where both are weak limits, the first being in $Y^{1,2}(\mathbb{R}^n)$ and the second in $L^2(\mathbb{R}^n)$.

We may remove the condition that $u \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ for solutions satisfying trace decay at infinity.

Proposition 5.8.4. *Suppose that $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+) \cap S^2_+$ satisfies that $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. Then there exists $u_0 \in Y^{1,2}(\mathbb{R}^n)$ such that $\lim_{t\to 0^+} u(t) = u_0$ exists weakly in $Y^{1,2}(\mathbb{R}^n)$. Moreover, since $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1}_+) \cap S^2_+ \subset W^{1,2}(I^+_R)$ for any $R > 0$, the trace $\operatorname{Tr}_0 u$ exists as an element of $L^2_{\text{loc}}(\mathbb{R}^n)$, and $\operatorname{Tr}_0 u = u_0$ as distributions. In particular, the conclusion holds for $u = \mathcal{S}^{\mathcal{L}} g$ for $g \in L^2(\mathbb{R}^n)$ or $u = \mathcal{D}^{\mathcal{L},+} f$ with $f \in Y^{1,2}(\mathbb{R}^n)$ (see Corollary 5.8.9).*

Proof. Since $u \in S^2_+$ there exists a subsequence $t_k \to 0^+$ and $u_0 \in L^{2n/(n-2)}(\mathbb{R}^n)$ such that $\lim_{k\to\infty} u(t_k) = u_0$ weakly in $L^{2n/(n-2)}(\mathbb{R}^n)$. Now, since $u \in Y^{1,2}(\Sigma^b_0)$ for any $b > 0$, we have that for each $\Phi \in C^\infty_c(\mathbb{R}^{n+1})$, there exists $b > 0$ such that

$$(u(t_k), \Phi(t_k)) = -\iint_{\Sigma^b_{t_k}} (uD_{n+1}\Phi + D_{n+1}u\Phi).$$

Fixing $\phi \in C^\infty_c(\mathbb{R}^n)$ and extending it to $\mathbb{R}^{n+1}$ so that $\Phi(\cdot, t) \equiv \phi(\cdot)$ in a neighborhood of $t = 0$, we see that

$$(u_0, \phi) = -\iint_{\Sigma^b_0} (uD_{n+1}\Phi + D_{n+1}u\Phi),$$

which gives the uniqueness of the limit u_0. Therefore, $\lim_{t\to 0} u(t) = u_0$ exists as a weak limit in $L^{2n/(n-2)}(\mathbb{R}^n)$. To see that $u_0 \in Y^{1,2}(\mathbb{R}^n)$, we proceed as follows: Since for any weak limit v in $L^2(\mathbb{R}^n)$ of $\nabla_{\|} u$, we have that for any $\phi \in C^\infty_c(\mathbb{R}^n; \mathbb{C}^n)$, the identity

$$(v, \phi) = \lim_{k\to\infty} (\nabla_{\|} u(t_k), \phi) = -\lim_{k\to\infty} (u(t_k), \operatorname{div}_{\|} \phi) = -(u_0, \operatorname{div}_{\|} \phi),$$

holds, we conclude that $v = \nabla_{\|} u_0$. This shows that the weak limits are unique and thus, since $u \in S^2_+$, the full limit exists; that is, $\lim_{t\to 0} \nabla_{\|} u(t) = \nabla_{\|} u_0$ weakly in $L^2(\mathbb{R}^n)$. Now, we recall that every element $\ell \in Y^{1,2}(\mathbb{R}^n)^*$ can be written in the form

$$\ell(w) = \int_{\mathbb{R}^n} (\psi_0 w + \psi \cdot \nabla_{\|} w), \qquad \text{for all } w \in Y^{1,2}(\mathbb{R}^n),$$

for some $\psi_0 \in L^{2n/(n+2)}(\mathbb{R}^n)$ and $\psi = (\psi_1, \ldots, \psi_n) \in L^2(\mathbb{R}^n)^n$. This gives that $u(t) \to u_0$ weakly in $Y^{1,2}(\mathbb{R}^n)$.

We now turn to the proof of the final statement in the proposition. Since for every

$\Phi(\cdot,t) \in C_c^\infty(\mathbb{R}^{n+1})$ we have that $\Phi(\cdot,t) \to \Phi(\cdot,0)$ strongly in $L^2(\mathbb{R}^n)$, we need only check that $\lim_{t\to 0}(u(t),\Phi(t)) = (u_0,\Phi(0))$. Along these lines it is enough to prove that

$$(u(t),\Phi(t)) = -\iint_{\mathbb{R}^{n+1}_{>t}} (uD_{n+1}\Phi + D_{n+1}u\Phi), \qquad \text{for all } t>0,$$

but this in turn follows from [BHL⁺b, Proposition 2.16], since $u \in W^{1,2}_{\mathrm{loc}}(\Sigma^\infty_{t/2})$. $\square$

Proposition 5.8.5 (Conormal derivative of solutions in slice spaces)**.** *Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+) \cap S^2_+$ satisifies $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. Then there exists g in $L^2(\mathbb{R}^n)$ such that*

$$(g,\phi) = \iint_{\mathbb{R}^{n+1}_+} \Big((A\nabla u + B_1)\overline{\nabla\Phi} + B_2\cdot\nabla u\overline{\Phi}\Big), \qquad \textit{for all } \Phi \in C_c^\infty(\mathbb{R}^{n+1}),$$

where $\phi(\cdot) = \Phi(\cdot,0)$. We write $g = \partial_{\nu^{\mathcal{L}},+}u$. Moreover, $g = \lim_{t\to 0^+} -e_{n+1}\cdot\mathrm{Tr}_t(A\nabla u + B_1u)$, where the limit is taken in the weak sense in the space $L^2(\mathbb{R}^n)$. In particular, this notion of the conormal derivative agrees with our previous definition in $Y^{1,2}(\mathbb{R}^{n+1}_+)$ whenever both exist.

Proof. We follow the proof of [AAA⁺11, Lemma 4.3 (iii)]. We will first show that for any $R>0$, there exists $g_R \in (C_c^\infty(\Delta_R))^*$ such that for any $\Phi \in C_c^\infty(I_R)$,

$$\langle g_R, \Phi(\cdot,0)\rangle = \iint_{\mathbb{R}^{n+1}_+} \Big((A\nabla u + B_1)\cdot\overline{\nabla\Phi} + B_2\cdot\nabla u\overline{\Phi}\Big). \tag{5.8.6}$$

In particular, this allows us to define $g \in (C_c^\infty(\mathbb{R}^n))^*$ such that $g = \lim_{R\uparrow\infty} g_R$ in the sense of distributions and (5.8.6) holds for any $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$ and g in place of g_R. Thus, fix $R>0$, $\phi \in C_c^\infty(\Delta_R)$, and $\Phi \in W_0^{1,2}(I_R)$ any extension of ϕ (that is, $\mathrm{Tr}_0\,\Phi = \phi$). We define the (anti-)linear functional $\Lambda_R : C_c^\infty(\Delta_R) \to \mathbb{C}$ by

$$\Lambda_R(\phi) = \iint_{\mathbb{R}^{n+1}_+} \Big((A\nabla u + B_1u)\cdot\overline{\nabla\Phi} + B_2\cdot\nabla u\overline{\Phi}\Big).$$

To see this is indeed well-defined, that is, it does not depend on the extension Φ, we simply note that for any two extensions Φ_1,Φ_2, we have that $\Phi_1 - \Phi_2 \in W_0^{1,2}(I_R^+)$, and $u \in S^2_+$ solves $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. Now, as in the proof of the Lax-Milgram theorem, we have that $|\Lambda_R(\phi)| \lesssim \|\nabla u\|_{L^2(\Sigma_0^R)}\|\nabla\Phi\|_{L^2(I_R^+)}$. Construct Φ to satisfy that $\Delta\Phi = 0$ in I_R^+, $\Phi = \varphi$ on Δ_R, and $\Phi = 0$ on $\partial I_R \cap \mathbb{R}^{n+1}_+$. In this case, we have that $\|\nabla\Phi\|_{L^2(I_R^+)} \lesssim \|\phi\|_{\dot{H}^{1/2}(\Delta_R)}$ by the usual extension theorem. Combining these last two estimates, we arrive at $|\Lambda_R(\phi)| \lesssim \|\nabla u\|_{L^2(\Sigma_0^R)} \|\phi\|_{\dot{H}^{1/2}(\Delta_R)}$, whence via the Riesz representation theorem there exists $g_R \in (\dot{H}^{1/2}(\Delta_R))^*$ such that $\langle g_R, \phi\rangle = \Lambda_R(\phi)$ for each $\phi \in C_c^\infty(\Delta_R)$. From the definition of Λ_R, we see that the restriction of g_R to $\Delta_{R'}$ equals $g_{R'}$ whenever $R' < R$. In particular, this allows us to define a distribution g such that

$$\langle g, \phi\rangle = \iint_{\mathbb{R}^{n+1}_+} \Big((A\nabla u + B_1)\overline{\nabla\Phi} + B_2 \cdot \nabla u \overline{\Phi} \Big),$$

for all $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$ with $\Phi(\cdot, 0) = \phi$. It remains to show that $g \in L^2(\mathbb{R}^n)$. For this, note that via the previous procedure we can define a conormal at height $t \geq 0$, which we denote as g^t, as the distribution which satisfies

$$\langle g^t, \Phi^t\rangle = \iint_{\mathbb{R}^{n+1}_t} \Big((A\nabla u + B_1)\overline{\nabla\Phi} + B_2 \cdot \nabla u \overline{\Phi} \Big), \tag{5.8.7}$$

for all $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$ where $\Phi^t(\cdot) = \Phi(\cdot, t)$. This formula shows that the conormal $\partial_{\nu,t}^{\mathcal{L},+} u$ in [BHL$^+$b, Definition 4.9] agrees with g^t, as distributions in $\mathbb{R}^n$. In particular, from the proof of [BHL$^+$b, Lemma 4.11 (i)] we see that, for any $t > 0$, $g^t \in L^2(\mathbb{R}^n)$ and $g^t = -e_{n+1} \cdot \mathrm{Tr}_t(A\nabla u + B_1 u)$. Moreover, since $u \in S^2_+$, we have that $\|g^t\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{S^2_+}$. By weak compactness, we can extract a subsequence $t_k \to 0$ and $\tilde{g}$ such that $g^{t_k} \to \tilde{g}$ weakly in $L^2(\mathbb{R}^n)$. From (5.8.7) it is then easy to see that $\tilde{g} = g^0 = g$ as distributions, and the result follows. $\square$

We now take the first step towards proving existence of layer potential solutions, by proving the appropriate so-called jump relations for the Double Layer and the conormal derivative of the Single Layer.

Lemma 5.8.8 (Jump Relations)**.** *There exist bounded linear operators $K, \tilde{K} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that for every $f, g \in L^2(\mathbb{R}^n)$ we have that $\big(\pm \frac{1}{2}I + \tilde{K}\big)g = \partial_\nu^{\mathcal{L},\pm}(\mathcal{S}^{\mathcal{L}} g)$, and $\big(\mp \frac{1}{2}I + K\big)f = \mathcal{D}_0^{\mathcal{L},\pm} f$.*

Proof. First, we note that by [BHL$^+$b, Propositions 4.18 and 4.22], we can define operators $K : \dot{H}^{1/2}(\mathbb{R}^n) \to \dot{H}^{1/2}(\mathbb{R}^n)$, $\tilde{K} : \dot{H}^{-1/2}(\mathbb{R}^n) \to \dot{H}^{-1/2}(\mathbb{R}^n)$, such that the identities in the lemma are satisfied for $f, g \in C_c^\infty(\mathbb{R}^n)$. Moreover, by Propositions 5.8.1 and 5.8.5, we obtain that $K, \tilde{K}$ are $L^2(\mathbb{R}^n)$ bounded (that is, admit a unique linear, continuous extension to $L^2(\mathbb{R}^n)$); the result now follows via a density argument. $\square$

Corollary 5.8.9 (Additional mapping property of $\mathcal{D}$)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Assume further that the operator $(\mathcal{S}_0^{\mathcal{L}})^{-1} : Y^{1,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ exists and is bounded. Then we have*

$$\sup_{t>0} \|\mathcal{D}_t^{\mathcal{L},+} f\|_{Y^{1,2}(\mathbb{R}^n)} \lesssim \|f\|_{Y^{1,2}(\mathbb{R}^n)},$$

with implicit constants depending on dimension, ellipticity of $\mathcal{L}$, and the norm of $(\mathcal{S}_0^{\mathcal{L}})^{-1}$.

Proof. We know by Theorem 5.8.16 that the map

$$\mathcal{S}_0^{\mathcal{L}} : L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n)$$

is bounded and invertible. In particular we have that the set

$$\mathcal{F} := \Big\{ f \in Y^{1,2}(\mathbb{R}^n) : f = \mathcal{S}_0^{\mathcal{L}} \psi,\ \psi \in C_c^\infty(\mathbb{R}^n) \Big\}$$

is dense in $Y^{1,2}(\mathbb{R}^n)$. We note that for $f \in \mathcal{F}$ we have $f \in \dot{H}^{1/2}(\mathbb{R}^n) \cap Y^{1,2}(\mathbb{R}^n)$ by [BHL+b, Proposition 4.7 (iii)]. For such an f and $\psi := (\mathcal{S}_0^{\mathcal{L}})^{-1} f$ we set

$$u(\cdot, \tau) := \mathcal{S}_\tau^{\mathcal{L}} \psi, \quad \tau < 0.$$

Then by [BHL+b, Theorem 4.16 (iv)], applied to u in $\mathbb{R}^{n+1}_-$, we have

$$\mathcal{D}^{\mathcal{L},+}(f) = -\mathcal{S}^{\mathcal{L}}(\partial_{\nu^{\mathcal{L}},-} u), \quad \text{ in } \mathbb{R}^{n+1}_+. \tag{5.8.10}$$

Now recall from the jump relations (see [BHL+b, Proposition 4.22 (ii)]) that

$$\partial_{\nu^{\mathcal{L}},-} u = \Big(-\frac{1}{2} I + \tilde{K} \Big) \psi, \quad \text{ in } \dot{H}^{-1/2}(\mathbb{R}^n),$$

so that, using the definition of u, (5.8.10) becomes

$$\mathcal{D}^{\mathcal{L},+}(f) = -\mathcal{S}^{\mathcal{L}}\Big(\Big(-\frac{1}{2} I + \tilde{K} \Big) \psi \Big) = -\mathcal{S}^{\mathcal{L}}\Big(\Big(-\frac{1}{2} I + \tilde{K} \Big) (\mathcal{S}_0^{\mathcal{L}})^{-1} f \Big). \tag{5.8.11}$$

Finally from Proposition 5.8.1 and Theorem 5.8.16 we know the following maps are bounded

$$\mathcal{S}^{\mathcal{L}} : L^2(\mathbb{R}^n) \to S^2_+, \Big(-\frac{1}{2}I + \tilde{K}\Big) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$
$$(\mathcal{S}_0^{\mathcal{L}})^{-1} : Y^{1,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

which gives the desired bound

$$\|\mathcal{D}^{\mathcal{L},+}(f)\|_{S^2_+} \lesssim \|f\|_{Y^{1,2}(\mathbb{R}^n)}, \quad f \in \mathcal{F}.$$

We conclude the claimed inequality from the density of $\mathcal{F}$ in $Y^{1,2}(\mathbb{R}^n)$. □

Proposition 5.8.12. *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Assume further that $(\mathcal{S}_0^{\mathcal{L}})^{-1} : Y^{1,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ exists and is bounded. Let $f \in Y^{1,2}(\mathbb{R}^n)$, then the operator K from Lemma 5.8.8 extends to a bounded operator from $Y^{1,2}(\mathbb{R}^n)$ to itself. Moreover we retain the jump relation*

$$\Big(-\frac{1}{2}I + K\Big) f = \lim_{t\to 0} \mathcal{D}_t^{\mathcal{L},+}(f),$$

where the limit on the right is a weak limit in $Y^{1,2}(\mathbb{R}^n)$.

Proof. The proof of the first statement follows from Corollary 5.8.9, which together with Proposition 5.8.4 guarantees the existence of a weak limit in $Y^{1,2}(\mathbb{R}^n)$ for $f \in C_c^\infty(\mathbb{R}^n)$, and Lemma 5.8.8 which gives the desired identity.

□

Lemma 5.8.13 (Additional mapping property of $\mathcal{S}$)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis A (see Definition 5.4.1). Assume further that the inverse operators $(\mathcal{S}_0^{\mathcal{L}^*})^{-1}, (\mathcal{S}_0^{\mathcal{L}})^{-1} : L^2(\mathbb{R}^n) \to [Y^{1,2}(\mathbb{R}^n)]^*$ and*

$$\Big(-\frac{1}{2} + K\Big)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

exist and are bounded. Then the operator $\mathcal{S}$ extends as a bounded operator $\mathcal{S} : [Y^{1,2}(\mathbb{R}^n)]^* \to D^2_+$, that is,

$$\sup_{t>0} \|\mathcal{S}_t^{\mathcal{L}} g\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{[Y^{1,2}(\mathbb{R}^n)]^*},$$

with implicit constants depending on dimension, ellipticity of $\mathcal{L}$ and the norm of $(\mathcal{S}_0^{\mathcal{L}})^{-1}$.

Proof. Notice that, by the mapping properties of $-\frac{1}{2}I + K$ (see Corollary 5.8.9), and using the smallness of $\|B_i\|_{L^n(\mathbb{R}^n)}$, we obtain that

$$-\frac{1}{2}I + \tilde{K} : Y^{1,2}(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n)$$

is bounded and invertible. From this and the Green's Formula (see [BHL+b, Theorem 4.16 (iv)]) we have that for any $g \in C_c^\infty(\mathbb{R}^n)$

$$\mathcal{D}^{\mathcal{L},+}(\mathcal{S}_0^{\mathcal{L}} g) = -\mathcal{S}^{\mathcal{L}}\Big(\Big(-\frac{1}{2}I + \tilde{K}\Big)g\Big). \tag{5.8.14}$$

By Proposition 5.8.12 and Corollary 5.8.3 we have that, taking weak limits in $Y^{1,2}(\mathbb{R}^n)$ as $t \to 0$

$$\Big(-\frac{1}{2}I + K\Big)(\mathcal{S}_0^{\mathcal{L}} g) = -\mathcal{S}_0^{\mathcal{L}}\Big(-\frac{1}{2}I + \tilde{K}\Big)g,$$

or equivalently

$$-(\mathcal{S}_0^{\mathcal{L}})^{-1}\Big(-\frac{1}{2}I + K\Big)\mathcal{S}_0^{\mathcal{L}} g = \Big(-\frac{1}{2}I + \tilde{K}\Big)g =: h,$$

which means, using the corresponding mapping properties for $-(1/2)I + K$ and $\mathcal{S}_0^{\mathcal{L}}$ (see Proposition 5.8.12 and the fact that $adj(\mathcal{S}_0^{\mathcal{L}}) = \mathcal{S}_0^{\mathcal{L}^*}$), that we can extend

$$-\frac{1}{2}I + \tilde{K} : Y^{1,2}(\mathbb{R}^n)^* \to Y^{1,2}(\mathbb{R}^n)^*$$

as a bounded and, with smallness of $\|B_i\|_{L^n(\mathbb{R}^n)}$, invertible operator. In particular $\|g\|_{Y^{1,2}(\mathbb{R}^n)^*} \approx \|h\|_{Y^{1,2}(\mathbb{R}^n)^*}$. Using this in (5.8.10) we arrive at the fact that, for $g \in C_c^\infty(\mathbb{R}^n)$ it holds $\mathcal{S}^{\mathcal{L}} h \in D_+^2$ and

$$\|\mathcal{S}_t^{\mathcal{L}} h\|_{D_+^2} \lesssim \|\mathcal{S}_0^{\mathcal{L}} g\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{Y^{1,2}(\mathbb{R}^n)^*} \approx \|h\|_{Y^{1,2}(\mathbb{R}^n)^*}.$$

Since the set

$$\Big\{h = \Big(-\frac{1}{2}I + \tilde{K}\Big)g : g \in C_c^\infty(\mathbb{R}^n)\Big\}$$

is dense in $Y^{1,2}(\mathbb{R}^n)^*$ we conclude the desired property by a density argument. □

Definition 5.8.15 (Hypothesis B)**.** We will say $\mathcal{L}$ satisfies Hypothesis B if the following properties hold.

1. $\mathcal{L}$ satisfies Hypothesis A, along with the hypotheses of Theorem 5.7.1 and Theorem 5.7.11.
2. The following operators are invertible

$$\mathcal{S}_0^{\mathcal{L}^*}, \mathcal{S}_0^{\mathcal{L}} : L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n), \qquad -\frac{1}{2} + K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

3. The following operators are invertible

$$\pm\frac{1}{2} + K : Y^{1,2}(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n),$$

$$\pm\frac{1}{2} + \tilde{K} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

The first condition in Hypothesis B ensures that we have the right square and non-tangential maximal function estimates in $L^p(\mathbb{R}^n)$ for the layer potentials associated to $\mathcal{L}$ and $\mathcal{L}^*$. In particular the first condition implies that the objects in item (2) are well-defined and bounded (not necessarily invertible in general). The objects in item (2), more specifically their inverses, are used in the previous Propositions to define the objects in (3); this is the reason for the statement to be written in this way.

Theorem 5.8.16 (Invertibility of Layer Potentials)**.** *Suppose $\mathcal{L}_0$ satisfies hypothesis B (see Definition 5.8.15), with coefficients A^0, B_i^0 for $i = 1, 2$, and let $\mathcal{L}_1$ be defined by*

$$\mathcal{L}_1 = -\operatorname{div}((A^0 + M)\nabla + (B_1^0 + B_1)) + (B_2^0 + B_2) \cdot \nabla.$$

There exists $\rho > 0$ depending on dimension, ellipticity of $\mathcal{L}_0$, and the norms of the inverse operators in item (2) of Hypothesis B with the property that if

$$\max\{\|M\|_{L^\infty(\mathbb{R}^n)}, \|B_1\|_{L^n(\mathbb{R}^n)}, \|B_2\|_{L^n(\mathbb{R}^n)}\} < \rho,$$

then $\mathcal{L}_1$ satisfies Hypothesis B.

Proof. Set $\|M\|_\infty = 1$ and $\|B_i\|_n = 1$, $i = 1, 2$, and then define the operator

$$\mathcal{L}_z u := -\operatorname{div}((A + zM)\nabla u + (B_1^0 + zB_1)u) + (B_2^0 + zB_2) \cdot \nabla u,$$
$$z \in \mathbb{C},\ u \in Y^{1,2}(\mathbb{R}^{n+1}).$$

We write $\mathcal{L}_z = \mathcal{L}_0 - z\mathcal{M}$. The idea will be to show that K_z, $\tilde{K}_z$ and $\mathcal{S}_0^{\mathcal{L}_z}$ are analytic in z in a neighborhood of the origin. Note that, by Lax-Milgram, $\mathcal{L}_0$ is always invertible, and thus there exists ε_0 such that if $z \in B_{\varepsilon_0} = B(0, \varepsilon_0)$, then $\mathcal{L}_z$ is also invertible, and moreover $\mathcal{L}_z^{-1} = \mathcal{L}_0^{-1} \sum_{k=0}^{\infty} (z\mathcal{M}\mathcal{L}_0^{-1})^k$, the series converging in the operator norm of $\mathcal{B}(Y^{1,2}(\mathbb{R}^{n+1})^*; Y^{1,2}(\mathbb{R}^{n+1}))$. In particular, the map $z \mapsto \mathcal{L}_z^{-1}$ is analytic in B_{ε_0}. Now fix $t \geq 0$. By definition of the single layer, we conclude that $\mathcal{S}_t^{\mathcal{L}_z}$ is also analytic with values in $\mathcal{B}(\dot{H}^{-1/2}(\mathbb{R}^n); \dot{H}^{1/2}(\mathbb{R}^n))$. Since $\nabla_\| : \dot{H}^{1/2}(\mathbb{R}^n) \to \dot{H}^{-1/2}(\mathbb{R}^n)$, we have that $\nabla_\| \mathcal{S}_t^{\mathcal{L}_z}$ is analytic in B_{ε_0} with values in $\mathcal{B}(\dot{H}^{-1/2}(\mathbb{R}^n); \dot{H}^{-1/2}(\mathbb{R}^n))$. Thus, for $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^n)$, we have that the map $z \mapsto (\nabla_\| \mathcal{S}_t^{\mathcal{L}_z} f, g)$ is analytic, and

$$\sup_{z \in B_{\varepsilon_0}} \sup_{t \geq 0} \|\mathcal{S}_t^{\mathcal{L}_z}\|_{L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n)} \lesssim 1.$$

It follows by [Kat95, Theorem 3.12] that the map $z \mapsto \mathcal{S}_t^{\mathcal{L}_z}$ is holomorphic with values in $\mathcal{B}(L^2(\mathbb{R}^n); Y^{1,2}(\mathbb{R}^n))$, for any $t \geq 0$. In particular, we have that $\mathcal{S}_0^{\mathcal{L}_z}$ is analytic. Similarly, $\partial_\nu^{\mathcal{L}_z,+} \mathcal{S}^{\mathcal{L}_z}$ is analytic with values in $\mathcal{B}(\dot{H}^{-1/2}(\mathbb{R}^n); \dot{H}^{-1/2}(\mathbb{R}^n))$, and for $f, g \in C_c^\infty(\mathbb{R}^n)$, we have that

$$\sup_{z \in B_{\varepsilon_0}} \|\partial_\nu^{\mathcal{L}_z,+} \mathcal{S}^{\mathcal{L}_z}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim 1, \tag{5.8.17}$$

and the map $z \mapsto (\partial_{\nu^{\mathcal{L}_z,+}} \mathcal{S}^{\mathcal{L}_z} f, g) = \lim_{t \to 0} (A\nabla \mathcal{S}_t^{\mathcal{L}_z} f + zB_1 \mathcal{S}_t^{\mathcal{L}_z} f, g)$ is analytic. Thus we obtain that $\partial_\nu^{\mathcal{L}_z,+}(\mathcal{S}^{\mathcal{L}_z})$ is analytic with values in $\mathcal{B}(L^2(\mathbb{R}^n))$. By the jump relations in Lemma 5.8.8, $\tilde{K}_z$ is also analytic with values in $\mathcal{B}(L^2(\mathbb{R}^n))$. The analyticity of K_z follows from that of $\tilde{K}_z$ by noting that $\langle \mathcal{D}_0^{\mathcal{L}_z,+} f, g \rangle = \langle f, \partial_\nu^{\mathcal{L}_z^*,+} \mathcal{S}^{\mathcal{L}_z^*} g \rangle - \langle f, g \rangle$ for $f, g \in C_c^\infty(\mathbb{R}^n)$ (see [BHL+b, Proposition 4.18 (ii)]).

We have thus shown that the maps $z \mapsto \mathcal{S}_0^{\mathcal{L}_z}$, $z \mapsto K_z$, $z \mapsto \tilde{K}_z$ are all analytic. Therefore, by the Cauchy integral formula, we obtain that

$$\sup_{z \in B_{\varepsilon_0/2}} \left\| \tfrac{d}{dz} \tilde{K}_z \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim_{\varepsilon_0} \sup_{z \in B_{\varepsilon_0}} \|\tilde{K}_z\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim_{\varepsilon_0} 1,$$

where we used (5.8.17). Consequently, for any $z, w \in B_{\varepsilon_0/2}$, we have that $\|\tilde{K}_z - \tilde{K}_w\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim |z - w|$. This implies that for all z small enough, $\tilde{K}_z$ is invertible. The other boundary operators are treated similarly. $\square$

Theorem 5.8.18 (Existence of Solutions)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Then the boundary value problems* $(\mathrm{D})_2$*,* $(\mathrm{N})_2$*, and* $(\mathrm{R})_2$*, as given in (1.2.13)-(1.2.14), admit a solution.*

Proof. To solve the Dirichlet problem, we fix $f \in L^2(\mathbb{R}^n)$ and set $F = \Big(-\frac{1}{2}I + K\Big)^{-1} f$, which is well-defined by Theorem 5.8.16 as an element of $L^2(\mathbb{R}^n)$. Let $u := \mathcal{D}^{\mathcal{L},+}F$. Then the fact that $u \in D^2_+$ follows from Proposition 5.8.1, the non-tangential maximal function estimate follows from Theorem 5.7.11, while Lemma 5.8.8 gives the weak convergence to f.

To upgrade the convergence of $\mathcal{D}^{\mathcal{L},+}_t f$ to strong convergence in $L^2(\mathbb{R}^n)$, we mimmick the proof of [AAA$^+$11, Lemma 4.23]. First, we note that by Theorem 5.8.16 we have that $\mathcal{A} := \{\mathcal{S}^{\mathcal{L}}_0 \operatorname{div}_\| g : g \in C^\infty_c(\mathbb{R}^n)\}$ is dense in $L^2(\mathbb{R}^n)$. Indeed, since $adj(\mathcal{S}^{\mathcal{L}}_0) = \mathcal{S}^{\mathcal{L}^*}_0$, we have that $\mathcal{S}^{\mathcal{L}}_0 : Y^{1,2}(\mathbb{R}^n)^* \to L^2(\mathbb{R}^n)$ is invertible, and therefore any $h \in L^2(\mathbb{R}^n)$ may be written as $h = \mathcal{S}^{\mathcal{L}}_0 H$ for some $H \in L^2(\mathbb{R}^n)$. Moreover, any $H \in Y^{1,2}(\mathbb{R}^n)^*$ can be written as $H = \operatorname{div}_\| g$ for some $g \in L^2(\mathbb{R}^n)^n$ (as can be seen for instance by embedding $Y^{1,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)^n$ via $u \mapsto \nabla_\| u$ and using the Hahn-Banach and the Riesz Representation Theorems). These observations yield the claim.

Now fix $f = \mathcal{S}^{\mathcal{L}}_0(\operatorname{div}_\| g)$ for some $g \in C^\infty_c(\mathbb{R}^n)$ and define $u = \mathcal{S}^{\mathcal{L}}_s(\operatorname{div}_\| g)$ for $s < 0$. By [BHL$^+$b, Theorem 4.16 (iv)], we have that $\mathcal{D}^{\mathcal{L},+}_t f = -\mathcal{S}^{\mathcal{L}}(\partial^{\mathcal{L},-}_\nu u)$ in $\mathbb{R}^{n+1}_+$. Therefore, for any $0 < t' < t$, we have that

$$\|\mathcal{D}^{\mathcal{L},+}_t f - \mathcal{D}^{\mathcal{L},+}_{t'} f\|_{L^2(\mathbb{R}^n)} = \Big\| \int_{t'}^{t} \partial_\tau \mathcal{S}^{\mathcal{L}}_\tau(\partial^{\mathcal{L},+}_\nu u)\, d\tau \Big\|_{L^2(\mathbb{R}^n)} \lesssim (t-t')\|\partial^{\mathcal{L},+}_\nu u\|_{L^2(\mathbb{R}^n)},$$

where we used the estimates on slices from Theorem 5.5.1. Thus $\{\mathcal{D}^{\mathcal{L},+}_t f\}_t$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ as $t \to 0$.

For the Neumann problem we proceed in a similar way, with $w := \mathcal{S}^{\mathcal{L}}(1/2I + \tilde{K})^{-1}h$, and we appeal to Lemma 5.8.8, Theorem 5.8.16, Proposition 5.8.1, and Theorem 5.7.11.

Finally for the Regularity problem we set $v := \mathcal{S}^{\mathcal{L}}(S^{\mathcal{L}}_0)^{-1}g$, and make use of Corollary 5.8.3, Theorem 5.8.16, and Theorem 5.7.11.

It remains to show the non-tangential convergence statements, to which we now turn.

The convergence for the Regularity Problem goes as follows: By Proposition 5.2.7

we have that the solution v has a non-tangential limit, call it g_0, so we only need to show $g = g_0$. We know that $v(\cdot,t)$ converges weakly to g in $L^{2^*}(\mathbb{R}^n)$ as $t \to 0^+$. Define

$$v'(\cdot,t) = \fint_{t/2}^{3t/2} v(\cdot,s)\,ds$$

then $w'(\cdot,t)$ converges weakly to g in $L^{2^*}(\mathbb{R}^n)$ as $t \to 0^+$. Indeed, for fixed $\varphi \in L^{2_*}$ we have

$$\Big| \int_{\mathbb{R}^n} v(x,t)\varphi(x) - \int_{\mathbb{R}^n} g(x)\varphi(x) \Big| \le \epsilon_\varphi(t),$$

where $\epsilon_\varphi(t) \downarrow 0$ as $t \to 0^+$. From this one may establish $v'(\cdot,t)$ converges weakly to g in $L^{2^*}(\mathbb{R}^n)$. Moreover, if for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define

$$(A_t f)(x) := \fint_{|y-x|_\infty < t} f(y)\,dy$$

then $\tilde{v}(x,t) = (A_t v')(x)$ and it follows that $\tilde{v}(x,t)$ converges weakly to g in $L^{2^*}(\mathbb{R}^n)$ as $t \to 0^+$. Indeed, for if $\varphi \in L^{2_*}$ then

$$\int_{\mathbb{R}^n} (A_t v')(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} v'(x,t)(A_t\varphi)(x)\,dx$$

and since $v'(\cdot,t)$ converges weakly to g in $L^{2^*}(\mathbb{R}^n)$ as $t \to 0^+$ and $(A_t\varphi)(x)$ converges strongly in L^{2_*} as $t \to 0^+$ we have that

$$\int_{\mathbb{R}^n} v'(x,t)(A_t\varphi)(x)\,dx \to \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx, \quad \text{as } t \to 0^+.$$

It follows that $g_0(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$.

For the Dirichlet problem, we use compatible well-posedness (see below in the proof) to get that for smooth initial data $f \in C_c^\infty(\mathbb{R}^n)$ the solutions to the Dirichlet and Regularity problems obtained via layer potentials agree. In particular, if $u_f = \mathcal{D}^{\mathcal{L},+}(-1/2+K)^{-1}f$, then u_f has a non-tangential limit. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and we have the maximal function estimate

$$\|\tilde{\mathcal{N}}_2(u_f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

the existence of a limit for general $f \in L^2(\mathbb{R}^n)$ follows a standard argument.

Now we turn to the compatible well-posedness statement: If $f \in C_c^\infty(\mathbb{R}^n)$ and we set

$$u_f := \mathcal{D}^{\mathcal{L}}\Big(-\frac{1}{2}+K\Big)^{-1} f, \qquad v_f := \mathcal{S}^{\mathcal{L}}(\mathcal{S}_0^{\mathcal{L}})^{-1} f,$$

the layer potential solutions of the Dirichlet and Regularity problems with data f respectively, we claim then $u_f = v_f$ and both agree with the solution furnished via Lax-Milgram with Dirichlet data f.

We first prove that u_f agrees with the Lax-Milgram solution. For this, by the mapping properties of the double layer (see [BHL+b, Definition 4.6]) it is enough to show that

$$Tf := \Big(-\frac{1}{2}+K\Big)^{-1} f \in H_0^{1/2}(\mathbb{R}^n).$$

We know (see Theorem 5.8.16 and Proposition 5.8.12) that T maps $L^2(\mathbb{R}^n)$ and $Y^{1,2}(\mathbb{R}^n)$ to itself, so in particular $Tf \in W^{1,2}(\mathbb{R}^n) \subset H_0^{1/2}$.

For v_f we proceed similarly, noting that $(\mathcal{S}_0^{\mathcal{L}})^{-1}$ maps $Y^{1,2}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n) \to [Y^{1,2}(\mathbb{R}^n)]^*$ (see Theorem 5.8.16 and Lemma 5.8.13). It's thus enough, by the mapping properties of the single layer (see [BHL+b, Proposition 4.2]), to prove that

$$[Y^{1,2}(\mathbb{R}^n)]^* \cap L^2(\mathbb{R}^n) \subset H^{-1/2}(\mathbb{R}^n). \tag{5.8.19}$$

This follows from the fact that elements of the first space are of the form $G \in L^2(\mathbb{R}^n)$ such that $G = \operatorname{div} H$ for some $H \in L^2(\mathbb{R}^n;\mathbb{C}^n)$, while the second space contains all elements of the form $(-\Delta)^{1/2}F$ with $F \in H_0^{1/2}(\mathbb{R}^n)$. Fix G, H as above. By the Riesz representation theorem in $Y^{1,2}(\mathbb{R}^n)$ there exists a weak solution $F_1 \in Y^{1,2}(\mathbb{R}^n)$ of the problem $G = \operatorname{div} H = -\Delta F_1$; set $F := (-\Delta)^{1/2}F_1$, so that it's enough to prove $F \in H^{1/2}(\mathbb{R}^n)$. First, clearly $F \in L^2(\mathbb{R}^n)$ by Plancherel's Theorem, since $\nabla F_1 \in L^2(\mathbb{R}^n)$; moreover, since $(-\Delta)^{1/2}F = G \in L^2(\mathbb{R}^n)$, we have that $F \in W^{1,2}(\mathbb{R}^n)$, and by interpolation, $F \in H^{1/2}(\mathbb{R}^n)$ as desired. $\square$

5.9 Uniqueness

Lemma 5.9.1. *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Assume u is a good $\mathcal{D}$ solution. Then for every $\tau > 0$, $\partial_{\nu_\tau} u \in [Y^{1,2}(\mathbb{R}^n)]^*$, with the bound*

$$\sup_{\tau>0} \|\partial_{\nu_\tau} u\|_{[Y^{1,2}(\mathbb{R}^n)]^*} \leq C \sup_{t>0} \|u\|_{L^2(\mathbb{R}^n)}.$$

Proof. By symmetry of hypotheses and Lemma 5.8.13 and Theorem 5.8.16 the operator $S_0^{\mathcal{L}^*} : L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n)$ is bounded and invertible. Then the collection of functions $\mathcal{F} := \{\varphi \in Y^{1,2}(\mathbb{R}^n) : \varphi = S_0^{\mathcal{L}^*} f, f \in C_c^\infty\}$ is dense in $Y^{1,2}(\mathbb{R}^n)$. Notice that $v_\varphi = \mathcal{S}^{\mathcal{L}^*}([S_0^{\mathcal{L}^*}]^{-1}\varphi) = S^{\mathcal{L}^*} f \in Y^{1,2}(\mathbb{R}^{n+1})$ since $f \in C_c^\infty(\mathbb{R}^n) \subset H^{-1/2}(\mathbb{R}^n)$. Also, $\mathrm{Tr}_0 u_\tau \in H_0^{1/2}(\mathbb{R}^n)$ since $u_\tau \in Y^{1,2}(\mathbb{R}^{n+1}_+)$, where, as above $u_\tau(\cdot,\cdot) := u(\cdot, \cdot + \tau)$. Then by definition of $\partial_{\nu^*} v_\varphi \in H^{-1/2}(\mathbb{R}^n)$ (see [BHL+b, Definition 4.9]) with

$$\begin{aligned}(\mathrm{Tr}_0 u_\tau, \partial_{\nu^*} v_\varphi) &= \overline{(\partial_{\nu^*} v_\varphi, \mathrm{Tr}_0 u_\tau)} \\ &= \overline{B_{\mathcal{L}^*}[v_\varphi, u_\tau]} = B_{\mathcal{L}}[u_\tau, v_\varphi].\end{aligned}$$

Now $B_{\mathcal{L}}[u_\tau, v_\varphi] = (\partial_{\nu_\tau} u, \varphi)$, since v_φ solves the regularity problem with data φ by Theorem 5.8.18. In particular φ is the weak limit of $v_\varphi(\cdot, t)$ in $Y^{1,2}(\mathbb{R}^n)$ as $t \to 0$.

Having established

$$(\partial_{\nu_\tau} u, \varphi) = (\mathrm{Tr}_0 u_\tau, \partial_{\nu^*} v_\varphi)$$

for φ in $\mathcal{F}$ we see that $\partial_{\nu_\tau} u \in [Y^{1,2}(\mathbb{R}^n)]^*$ by the fact that the map

$$F_\varphi := \partial_{\nu^*} v_\varphi = \partial_{\nu^*} \mathcal{S}^{\mathcal{L}^*}([S_0^{\mathcal{L}^*}]^{-1}\varphi)$$

maps $Y^{1,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, by Proposition 5.8.5, the mapping property mentioned at the start of the proof for $\mathcal{S}_0^{\mathcal{L}^*}$, and the density of $\mathcal{F}$ in $Y^{1,2}(\mathbb{R}^n)$. $\square$

Finally we state a technical lemma that will allow us to prove a representation formula for solutions to the regularity and Neumann problems

Proposition 5.9.2. *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+) \cap S^2_+$ be a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$. Then for every $\tau_0 > 0$ and every $t > 0$ we have*

$$\partial_\tau|_{\tau=\tau_0} \mathcal{D}_t^{\mathcal{L},+}(\mathrm{Tr}_0\, u_\tau) = \mathcal{D}_t^{\mathcal{L},+}(\mathrm{Tr}_0(D_{n+1} u_{\tau_0})),$$

and

$$\partial_\tau|_{\tau=\tau_0}\mathcal{S}_t^{\mathcal{L}}(\partial_{\nu^{\mathcal{L}},+}u_\tau) = \mathcal{S}_t^{\mathcal{L}}(\partial_{\nu^{\mathcal{L}},+}(D_{n+1}u_{\tau_0})).$$

Proof. We work with the double layer first. For this we consider, for $t > 0$ fixed, the following functions:

$$f(\tau) := \operatorname{Tr}_0 u_\tau = \operatorname{Tr}_\tau u, \quad H(\tau) := \mathcal{D}_t^{\mathcal{L},+}(f(\tau)).$$

We note that, by hypothesis and Corollary 5.8.9, we have

$$f \in C((0,\infty); Y^{1,2}(\mathbb{R}^n)), \quad H \in C((0,\infty); Y^{1,2}(\mathbb{R}^n)).$$

The idea is now to use [BHL$^+$b, Theorem 2.14] to get the desired differentiability of f. For this purpose define

$$\varphi(\tau) := \|\operatorname{Tr}_\tau(D_{n+1}u)\|_{Y^{1,2}(\mathbb{R}^n)} = \|\nabla_\| \operatorname{Tr}_\tau(D_{n+1}u)\|_{L^2(\mathbb{R}^n)} \in L^2_{\mathrm{loc}}((0,\infty);\mathbb{R}). \tag{5.9.3}$$

First we note that by [BHL$^+$b, Lemma 2.3] we have that $f : (0,\infty) \to Y^{1,2}(\mathbb{R}^n)$ and $\varphi : (0,\infty) \to \mathbb{R}$ are continuous functions. By the Lebesgue Differentiation Theorem it is then enough to show

$$\left(\fint_{-\varepsilon}^{\varepsilon} \|f(\tau_2+s) - f(\tau_1+s)\|^2_{Y^{1,2}(\mathbb{R}^n)}\,ds\right)^{1/2} \leq \int_{\tau_1}^{\tau_2}\left(\fint_{-\varepsilon}^{\varepsilon}\varphi^2(s+\tau)\,ds\right)^{1/2} d\tau, \tag{5.9.4}$$

for all ε small enough (depending on τ_1 and τ_2). For this purpose we compute, calling I the left hand side of (5.9.4),

$$\begin{aligned}
I = &\left(\fint_{-\varepsilon}^{\varepsilon}\int_{\mathbb{R}^n} |\nabla_\| \operatorname{Tr}_{\tau_2+s} u(x) - \nabla_\| \operatorname{Tr}_{\tau_1+s} u(x)|^2\,dxds\right)^{1/2} \\
&= \left(\fint_{-\varepsilon}^{\varepsilon}\int_{\mathbb{R}^n} |\nabla_\| u_{\tau_2}(x,s) - \nabla_\| u_{\tau_1}(x,s)|^2\,dxds\right)^{1/2} \\
&= \left(\fint_{-\varepsilon}^{\varepsilon}\int_{\mathbb{R}^n} \left|\int_{\tau_1}^{\tau_2}\nabla_\|\partial_\tau u(x,\tau+s)\,d\tau\right|^2 dxds\right)^{1/2} \\
\leq &\int_{\tau_1}^{\tau_2}\left(\fint_{-\varepsilon}^{\varepsilon}\int_{\mathbb{R}^n}|\nabla_\|\partial_\tau u(x,\tau+s)|^2\,dxds\right)^{1/2} d\tau = \int_{\tau_1}^{\tau_2}\left(\fint_{-\varepsilon}^{\varepsilon}\varphi^2(s+\tau)\,ds\right)^{1/2} d\tau,
\end{aligned}$$

where we used the Fundamental Theorem of Calculus in the third line and Minkowski's inequality in the fourth.

As mentioned above this shows that $f \in W^{1,2}_{\mathrm{loc}}((0,\infty); Y^{1,2}(\mathbb{R}^n))$. Now we will show that

$$f'(\tau) = \mathrm{Tr}_\tau(D_{n+1}u), \qquad \text{for each } \tau > 0, \tag{5.9.5}$$

and moreover the difference quotients converge weakly

$$\Delta^h f(\tau) \to f'(\tau), \qquad \text{for every } \tau > 0.$$

For this fix $\psi \in C_c^\infty(0,\infty)$, $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^n)$ and let $\ell := -\operatorname{div}_\parallel \phi \in Y^{1,2}(\mathbb{R}^n)^*$. Using that the function $\tau \mapsto f(\tau)\psi'(\tau) \in Y^{1,2}(\mathbb{R}^n)$ is continuous (see again [BHL+b, Lemma 2.3]) and compactly supported on $(0,\infty)$ and properties of the Bochner integral (see for instance [CH98, Proposition 1.4.22]) we obtain

$$\begin{aligned}
&\Big\langle \int_0^\infty f(\tau)\psi'(\tau)\,d\tau, \ell\Big\rangle \\
&\quad = \int_0^\infty \langle f(\tau)\psi'(\tau), \ell\rangle\,d\tau = \int_0^\infty \int_{\mathbb{R}^n} \nabla_\parallel \mathrm{Tr}_\tau\, u(x)\psi'(\tau)\phi(x)\,dxd\tau \\
&= \int_0^\infty \int_{\mathbb{R}^n} \nabla_\parallel u(x,\tau)\psi'(\tau)\phi(x)\,dxd\tau = -\int_0^\infty \int_{\mathbb{R}^n} \nabla_\parallel D_{n+1}u(x,\tau)\psi(\tau)\phi(x)\,dxd\tau \\
&\quad = -\int_0^\infty \int_{\mathbb{R}^n} \nabla_\parallel \mathrm{Tr}_\tau(D_{n+1}u)(x)\psi(\tau)\phi(x)\,dxd\tau \\
&\quad = -\int_0^\infty \langle \mathrm{Tr}_\tau(D_{n+1}u)\psi(\tau), \ell\rangle\,d\tau = \Big\langle -\int_0^\infty \mathrm{Tr}_\tau(D_{n+1}u)\psi(\tau)\,d\tau, \ell\Big\rangle,
\end{aligned}$$

where we used integration by parts in the fourth line. Now we conclude, since the collection $\{\operatorname{div}_\parallel \phi : \phi \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^n)\}$ is dense in $Y^{1,2}(\mathbb{R}^n)^*$, that indeed (5.9.5) holds.

The convergence of the difference quotients is a consequence of the fact that $f' \in C((0,\infty); Y^{1,2}(\mathbb{R}^n))$ and the Fundamental Theorem of Calculus. In fact we get strong convergence in $Y^{1,2}(\mathbb{R}^n)$ as $h \to 0$ of $\Delta^h f(\tau)$ for every $\tau > 0$.

With this we can conclude the argument for the Double Layer: Define

$$H(\tau) := \mathcal{D}_t^{\mathcal{L},+} \mathrm{Tr}_0\, u_\tau = \mathcal{D}_t^{\mathcal{L},+} \mathrm{Tr}_\tau\, u.$$

We claim that $H \in C^1((0,\infty); Y^{1,2}(\mathbb{R}^n))$ and

$$H'(\tau) = \mathcal{D}_t^{\mathcal{L},+}(\mathrm{Tr}_0(D_{n+1}u_\tau)) = \mathcal{D}_t^{\mathcal{L},+}(\mathrm{Tr}_\tau(D_{n+1}u)).$$

Notice first that $H \in C((0,\infty); Y^{1,2}(\mathbb{R}^n))$ by the mapping properties of the Double Layer (see Corollary 5.8.9) and the fact that $H(\tau) = \mathcal{D}_t^{\mathcal{L},+}(f(\tau))$ (recall that $t > 0$ is fixed throughout). Morever, using these two facts again we see

$$\|H(\tau_1) - H(\tau_2)\|_{Y^{1,2}(\mathbb{R}^n)} \lesssim \|f(\tau_1) - f(\tau_2)\|_{Y^{1,2}(\mathbb{R}^n)} \leq \Big| \int_{\tau_1}^{\tau_2} \varphi(s)\, ds \Big|,$$

where φ is defined in (5.9.3). This shows that $H \in W^{1,2}_{\mathrm{loc}}((0,\infty); Y^{1,2}(\mathbb{R}^n))$. Moreover we have

$$\Delta^h H(\tau) = \mathcal{D}_t^{\mathcal{L},+}(\Delta^h f(\tau)),$$

so that, by the linearity and continuity of $\mathcal{D}_t^{\mathcal{L},+}$ in $Y^{1,2}(\mathbb{R}^n)$ and the weak convergence of $\Delta^h f(\tau)$ in $Y^{1,2}(\mathbb{R}^n)$, we obtain for every $\tau > 0$

$$\Delta^h H(\tau) \to \mathcal{D}^{\mathcal{L},+}(f'(\tau)) = \mathcal{D}_t^{\mathcal{L},+}(\mathrm{Tr}_0(D_{n+1}u_\tau))$$

weakly in $Y^{1,2}(\mathbb{R}^n)$ as $h \to 0$

The proof for the Single Layer follows the same lines. Define, for $t > 0$ fixed and $\tau > 0$,

$$g(\tau) := \partial_{\nu^{\mathcal{L},+}} u_\tau = \partial_{\nu_\tau^{\mathcal{L},+}} u,$$

where the second equality follows from [BHL+b, Lemma 4.11 (i)]. As before we first claim that $g \in C^1((0,\infty); L^2(\mathbb{R}^n))$ and we have

$$g'(\tau) = \partial_{\nu^{\mathcal{L},+}}(D_{n+1}u_\tau) = \partial_{\nu_\tau^{\mathcal{L},+}}(D_{n+1}u).$$

For this purpose we use the L^2 characterization of the conormal derivative (see again [BHL+b, Lemma 4.11]) so that

$$\begin{aligned} g(\tau) &= N \cdot \mathrm{Tr}_0(A\nabla u_\tau + B_1 u_\tau) = N \cdot \mathrm{Tr}_\tau(A\nabla u + B_1 u) \\ &= N \cdot \mathrm{Tr}_\tau(\tilde{A}\nabla_{\|} u + B_1 u) + N \cdot \mathrm{Tr}_\tau(\vec{a} D_{n+1} u) =: g_1(\tau) + g_2(\tau), \end{aligned}$$

where $\tilde{A} := (a_{ij})_{1\le i\le n+1, 1\le j\le n}$ and $\vec{a} := (a_{i,n+1})_{1\le i\le n+1}$. We note that by the Hölder's and Sobolev's inequalities

$$\|g_1(\tau_2) - g_1(\tau_1)\|_{L^2(\mathbb{R}^n)} \lesssim \|f(\tau_2) - f(\tau_1)\|_{Y^{1,2}(\mathbb{R}^n)} \le \Big| \int_{\tau_1}^{\tau_2} \varphi(s)\, ds \Big|,$$

where f, φ are as in the proof for the Double Layer. Therefore it is enough to control g_2, and for this we can proceed exactly in the same way as we did for f: For fixed τ_2, τ_1 and $\varepsilon > 0$ small

$$\fint_{-\varepsilon}^{\varepsilon} \|g(\tau_2 + s) - g(\tau_1 + s)\|^2_{L^2(\mathbb{R}^n)}\, ds = \fint_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}^n} |D_{n+1}(u_{\tau_2}(x,s) - u_{\tau_1}(x,s))|^2\, dxds$$
$$= \fint_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}^n} \Big| \int_{\tau_1}^{\tau_2} D^2_{n+1} u(s+\tau)\, d\tau \Big|^2 dxds \lesssim \fint_{-\varepsilon}^{\varepsilon} \int_{\tau_1}^{\tau_2} \| \operatorname{Tr}_{\tau+s}(D^2_{n+1}u)\|_{L^2(\mathbb{R}^n)}\, d\tau ds,$$

and $\tilde{\varphi}(\tau) := \| \operatorname{Tr}_\tau(D^2_{n+1}u)\|_{L^2(\mathbb{R}^n)} \in L^2_{\text{loc}}(0,\infty)$. Therefore by [BHL+b, Theorem 2.14] we get that $g \in W^{1,2}_{\text{loc}}((0,\infty); L^2(\mathbb{R}^n))$ and the difference quotients converge a.e. to g'. To verify the formula for g' we compute, for $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in C_c^\infty(0,\infty)$,

$$\Big\langle \int_0^\infty g(\tau)\psi'(\tau)\, d\tau, \phi \Big\rangle_{L^2(\mathbb{R}^n)}$$
$$= \int_0^\infty \int_{\mathbb{R}^n} N \cdot (A\nabla u(x,\tau) + B_1 u(x,\tau))\psi'(\tau)\phi(x)\, dxd\tau$$
$$= -\int_0^\infty \int_{\mathbb{R}^n} N \cdot (A\nabla D_{n+1} u(x,\tau) + B_1 D_{n+1} u(x,\tau))\psi(\tau)\phi(x)\, dxd\tau$$
$$= \Big\langle \int_0^\infty \partial_{\nu_\tau^{\mathcal{L},+}}(D_{n+1}u)\psi(\tau), \phi \Big\rangle_{L^2(\mathbb{R}^n)}.$$

This gives the desired representation for $g'(\tau)$. Moreover, using this representation we see that $g' \in C((0,\infty); L^2(\mathbb{R}^n))$ and so the difference quotients satisfy $\Delta^h g(\tau) \to g'(\tau)$ weakly for every $\tau > 0$. The result now follows from the mapping properties of the single layer (see Proposition 5.8.1). □

5.9.1 Neumann and Regularity Problems

We begin with a lemma that gives a representation of good $\mathcal{N}/\mathcal{R}$ solutions above a positive height.

Lemma 5.9.6. *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Let u be a good $\mathcal{N}/\mathcal{R}$ solution and $u_\tau(\cdot,\cdot) = u(\cdot, \cdot+\tau)$, as above. Then*

$$u_\tau = -\mathcal{D}(Tr_0 u_\tau) + \mathcal{S}(\partial_\nu u_\tau), \tag{5.9.7}$$

where $Tr_0 u_\tau \in Y^{1,2}(\mathbb{R}^n)$, $\partial_\nu u_\tau \in L^2(\mathbb{R}^n)$, and $\mathcal{D}$ and $\mathcal{S}$ are viewed (as their natural extensions) from these spaces mapping into S^2_+.

Proof. We have $\mathrm{Tr}_0 u_\tau(\cdot) = u(\cdot,\tau) \in Y^{1,2}(\mathbb{R}^n)$ (by the fact that $u \in S^2_+$) and $\partial_\nu u_\tau \in L^2(\mathbb{R}^n)$ (by Proposition 5.8.5) . For the latter we may consider $u_{\tau/2} \in Y^{1,2}(\mathbb{R}^{n+1}_+)$, a solution in $\mathbb{R}^{n+1}_+$, since the operator $\mathcal{L}$ is t-independent.

By Proposition 5.9.2 together with the Green's formula for $\partial_t u_\tau \in Y^{1,2}(\mathbb{R}^{n+1})$ (see [BHL+b, Theorem 4.16 (ii)]) we know that for $\tau_0 > 0$

$$\partial_\tau u_\tau|_{\tau=\tau_0}(x,t) = -\partial_\tau[\mathcal{D}(\mathrm{Tr}_0 u_\tau) + \mathcal{S}(\partial_\nu u_\tau)]\big|_{\tau=\tau_0}(x,t),$$

as functions in S^2_+. We may now integrate in τ_0 to obtain

$$u_\tau = -\mathcal{D}(\mathrm{Tr}_0 u_\tau) + \mathcal{S}(\partial_\nu u_\tau),$$

where we must use the decay at infinity hypothesis in the definition of S^2_+. □

Now we push the representation above down to the boundary.

Lemma 5.9.8. *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Suppose that u is a good $\mathcal{N}/\mathcal{R}$ solution, then*

$$u = -\mathcal{D}f + \mathcal{S}g, \tag{5.9.9}$$

where $f \in Y^{1,2}(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are as in Propositions 5.8.4 and 5.8.5 respectively.

Proof. By Propositions 5.8.4 and 5.8.5, $u_\tau(\cdot,0) \to f \in Y^{1,2}(\mathbb{R}^n)$ and $\partial_\nu u_\tau \to g \in L^2(\mathbb{R}^n)$ weakly in $Y^{1,2}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ respectively as $\tau \to 0^+$. Set $u_\tau(\cdot,0) = f_\tau$ and $\partial_\nu u_\tau = g_\tau$, then rephrasing the above, we have $\vec{h}_\tau = (f_\tau, g_\tau)$ converges to $(f,g) =: \vec{h}$ weakly in $Y^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Let $\tau_k \downarrow 0$ then by Mazur's lemma there exists a sequence

$\{\tilde{h}_l\}_{l=1}^{\infty} \subset Y^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that $\tilde{h}_l \to \vec{h}$ *strongly* in $Y^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with

$$\tilde{h}_l = \sum_{k=l}^{N(l)} \lambda_{k,l} \vec{h}_{\tau_k},$$

where $l \leq N(l) < \infty$, $\lambda_{k,l} \in [0,1]$ and $\sum_{k=l}^{N(l)} \lambda_{k,l} = 1$. Set

$$\tilde{u} := \mathcal{D}f + \mathcal{S}g.$$

To prove the lemma it is enough to show for $t > 0$, $\tilde{u}(\cdot, t) = u(\cdot, t)$ as functions in $Y^{1,2}(\mathbb{R}^n)$.

We have from Lemma 5.9.6 that

$$u_\tau = -\mathcal{D}(f_\tau) + S(g_\tau).$$

Set

$$u_l := \sum_{k=l}^{N(l)} \lambda_{k,l} u_{\tau_k}.$$

We show $u_l(\cdot, t)$ converges strongly to both $u(\cdot, t)$ and $\tilde{u}(\cdot, t)$ in $Y^{1,2}(\mathbb{R}^n)$. From the bounded mappings $\mathcal{D} : Y^{1,2}(\mathbb{R}^n) \to S_+^2$ and $\mathcal{S} : L^2(\mathbb{R}^n) \to S_+^2$ we have

$$\|\nabla[\tilde{u}(\cdot, t) - u_l(\cdot, t)]\|_{L^2(\mathbb{R}^n)} \leq \|\vec{h} - \tilde{h}_l\|_{Y^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \to 0 \text{ as } l \to \infty,$$

where we used the strong convergence of $\tilde{h}_l$ to $\vec{h}$. To show $u_l(\cdot, t)$ converges strongly to $u(\cdot, t)$ in $Y^{1,2}(\mathbb{R}^n)$ we write for $l \geq 0$,

$$\begin{aligned}
\|\nabla u_l(\cdot, t) - \nabla u(\cdot, t)\|_{L^2(\mathbb{R}^n)} &= \|\sum_{k=l}^{N(l)} \lambda_{k,l} \nabla[u_{\tau_k} - u](\cdot, t)\|_{L^2(\mathbb{R}^n)} \\
&\leq \sum_{k=l}^{N(l)} \lambda_{k,l} \|\nabla[u_{\tau_k} - u](\cdot, t)\|_{L^2(\mathbb{R}^n)} \\
&\leq \sup_{k \geq l} \|\nabla[u(\cdot, t + \tau_k) - u(\cdot, t)]\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where we used $\sum_{k=l}^{N(l)} \lambda_{k,l} = 1$ and $u(\cdot, \cdot + \tau) = u_\tau(\cdot, \cdot) = \mathcal{D}(f_\tau) + S(g_\tau)$. We can then use the continuity of $\nabla u(\cdot,t)$ in $L^2(\mathbb{R}^n)$ (see [BHL$^+$b, Lemma 2.3]) along with $\tau_k \downarrow 0$ to obtain $\|\nabla u_l(\cdot,t) - \nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0$ as l tends to infinity. $\square$

Theorem 5.9.10 (Uniqueness of the Regularity Problem Among Good $\mathcal{N}/\mathcal{R}$ solutions)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Suppose u is a good $\mathcal{N}/\mathcal{R}$ solution, with $u(\cdot,0)=0$, interpreted in the sense of Proposition 5.8.4 (i.e. $\lim_{t\to 0} u(t) = 0$ weakly in $Y^{1,2}(\mathbb{R}^n)$). Then $u \equiv 0$ in $\mathbb{R}^{n+1}_+$*

Proof. By Lemma 5.9.8, we have $u = -\mathcal{D}f + \mathcal{S}^{\mathcal{L}}g$, where f and g are as in Lemma 5.9.8. It follows that $u = \mathcal{S}^{\mathcal{L}}g$, since $f=0$ (see the proof of Lemma 5.9.8). Moreover, by taking traces (in the sense of Proposition 5.8.4) in $Y^{1,2}(\mathbb{R}^n)$ we obtain $0 = \mathcal{S}_0^{\mathcal{L}}g$, for $g \in L^2(\mathbb{R}^n)$. It follows from the invertibility of $\mathcal{S}_0^{\mathcal{L}} : L^2(\mathbb{R}^n) \to Y^{1,2}(\mathbb{R}^n)$ that $g=0$. This gives $u \equiv 0$. $\square$

Theorem 5.9.11 (Uniqueness of the Neumann problem among good $\mathcal{N}/\mathcal{R}$ solutions)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Suppose u is a good $\mathcal{N}/\mathcal{R}$ solution, with $\partial_\nu u = 0$, interpreted in the sense of Proposition 5.8.5. Then $u \equiv 0$ in $\mathbb{R}^{n+1}_+$.*

Proof. By Lemma 5.9.8, we have $u = -\mathcal{D}f + \mathcal{S}g$, where f and g are as in Lemma 5.9.8. It follows that $u = -\mathcal{D}f$, since $g = 0$ (see the proof of Lemma 5.9.8), where $f \in Y^{1,2}(\mathbb{R}^n)$. From (5.8.11), we have after taking conormal derivatives, in the sense of Proposition 5.8.5[12], and using the jump relations for the conormal of the single layer potential

$$0 = \partial_{\nu^{\mathcal{L}},+}u = -\partial_{\nu^{\mathcal{L}},+}\mathcal{D}f = -(-\tfrac{1}{2}I + \widetilde{K})(\tfrac{1}{2}I + \widetilde{K})S_0^{-1}f \quad \text{in } L^2(\mathbb{R}^n).$$

The invertibility of $\pm\frac{1}{2}I + \widetilde{K} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $S_0^{-1} : Y^{1,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ yields that $f=0$ and hence $u \equiv 0$. $\square$

[12] We note that, having obtained the mapping property $\mathcal{D} \to S_+^2$, the equality of (5.8.11) holds on every t-slice in the space $Y^{1,2}(\mathbb{R}^n)$ therefore the weak $L^2(\mathbb{R}^n)$ limits, in t, of the co-normal derivatives ∂_{ν_t} are the same.

5.9.2 The Dirichlet Problem

Lemma 5.9.12. *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Let u be a good $\mathcal{D}$ solution. For $\tau > 0$, set $(f_\tau, g_\tau) := (Tr_0 u_\tau, \partial_\nu u_\tau) = (Tr_0 u_\tau, \partial_{\nu_\tau} u) \in L^2(\mathbb{R}^n) \times [Y^{1,2}(\mathbb{R}^n)]^*$, where we use Lemma 5.9.1 to identify $\partial_{\nu_\tau} u$ as an element of $[Y^{1,2}(\mathbb{R}^n)]^*$. Then*

$$u = -\mathcal{D}f + \mathcal{S}g,$$

where the pair $(f, g) \in L^2(\mathbb{R}^n) \times [Y^{1,2}(\mathbb{R}^n)]^*$ is **any** *convergent weak limit of (f_{τ_k}, g_{τ_k}), $\tau_k \downarrow 0$ in the space $L^2(\mathbb{R}^n) \times [Y^{1,2}(\mathbb{R}^n)]^*$.*

Remark 5.9.13. We note that the existence of at least one such limiting pair (f, g) is guaranteed by the fact that f_τ, g_τ are uniformly bounded in $L^2(\mathbb{R}^n)$ and $Y^{1,2}(\mathbb{R}^n)^*$ respectively (the first by the hypothesis $u \in D^2_+$ and the second by Lemma 5.9.1) together with the fact that both of these spaces are reflexive.

Unlike the case of good $\mathcal{N}/\mathcal{R}$ solutions, here we make no assertion about the uniqueness of such a limiting pair.

Proof. The proof is quite similar to Lemma 5.9.8, but we provide the details here. We have that $u_\tau \in Y^{1,2}(\mathbb{R}^{n+1}_+)$ with $\mathcal{L}u_\tau = 0$ we have

$$u_\tau = -\mathcal{D}(\mathrm{Tr}_0 u_\tau) + S(\partial_\nu u_\tau)$$

for all $\tau > 0$. Let $\vec{h}_{\tau_k} := (f_{\tau_k}, g_{\tau_k}) \rightharpoonup (f, g) =: \vec{h} \in L^2(\mathbb{R}^n) \times [Y^{1,2}(\mathbb{R}^n)]^*$ be as in the statement of the lemma. Using Mazur's lemma there exists a sequence $\{\tilde{h}_l\}_{l=1}^\infty \subset L^2(\mathbb{R}^n) \times [Y^{1,2}(\mathbb{R}^n)]^*$ such that $\tilde{h}_l \to \vec{h}$ *strongly* in $L^2(\mathbb{R}^n) \times [Y^{1,2}(\mathbb{R}^n)]^*$ with

$$\tilde{h}_l = \sum_{k=l}^{N(l)} \lambda_{k,l} \vec{h}_{\tau_k},$$

where $l \leq N(l) < \infty$, $\lambda_{k,l} \in [0, 1]$ and $\sum_{k=l}^{N(l)} \lambda_{k,l} = 1$. Set

$$\tilde{u} := \mathcal{D}f + \mathcal{S}g$$

and

$$u_l := \sum_{k=l}^{N(l)} \lambda_{k,l} u_{\tau_k}.$$

We show $u_l(\cdot, t)$ converges strongly to both $u(\cdot, t)$ and $\tilde{u}(\cdot, t)$ in $L^2(\mathbb{R}^n)$. From the bounded mappings $\mathcal{D} : L^2(\mathbb{R}^n) \to D^2_+$ and $\mathcal{S} : [Y^{1,2}(\mathbb{R}^n)]^* \to D^2_+$ we have

$$\|\tilde{u}(\cdot, t) - u_l(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \|\vec{h} - \tilde{h}_l\|_{L^2(\mathbb{R}^n)\times[Y^{1,2}(\mathbb{R}^n)]^*} \to 0 \text{ as } l \to \infty.$$

To show $u_l(\cdot, t)$ converges strongly to $u(\cdot, t)$ in $L^2(\mathbb{R}^n)$ we write for $l \geq 0$,

$$\begin{aligned}
\|u_l(\cdot, t) - u(\cdot, t)\|_{L^2(\mathbb{R}^n)} &= \| \sum_{k=l}^{N(l)} \lambda_{k,l}[u_{\tau_k} - u](\cdot, t)\|_{L^2(\mathbb{R}^n)} \\
&\leq \sum_{k=l}^{N(l)} \lambda_{k,l} \|[u_{\tau_k} - u](\cdot, t)\|_{L^2(\mathbb{R}^n)} \\
&\leq \sup_{k \geq l} \|u(\cdot, t + \tau_k) - u(\cdot, t)\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where we used $\sum_{k=l}^{N(l)} \lambda_{k,l} = 1$ and $u(\cdot, \cdot + \tau) = u_\tau(\cdot, \cdot) = \mathcal{D}(f_\tau) + S(g_\tau)$. We can then use the continuity of $u(\cdot, t)$ in $L^2(\mathbb{R}^n)$ (see [BHL$^+$b, Lemma 2.3])[13] along with $\tau_k \downarrow 0$ to obtain $\|u_n(\cdot, t) - u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$ as n tends to infinity. Therefore $u = \tilde{u}$ in D^2_+ and the lemma is shown. □

Theorem 5.9.14 (Uniqueness of the Dirichlet problem among good $\mathcal{D}$ solutions)**.** *Suppose $\mathcal{L}$ satisfies Hypothesis B (see Definition 5.8.15). Suppose u is a good $\mathcal{D}$ solution, with $u(\cdot, t) \to 0$ weakly in $L^2(\mathbb{R}^n)$. Then $u \equiv 0$.*

Proof. By Lemma 5.9.12, we have that $u = \mathcal{S}g$ for some $g \in [Y^{1,2}(\mathbb{R}^n)]^*$, where $g \in [Y^{1,2}(\mathbb{R}^n)]^*$ is any weak limit of $g_{\tau_k} = \partial_{\nu_{\mathcal{L},+}} u_{\tau_k}$, $\tau_k \downarrow 0$ as in Lemma 5.9.12. We also have (see (5.8.14))

$$\mathcal{S}_t g = \mathcal{D}_t(S_0[-\tfrac{1}{2}I + \tilde{K}]^{-1} g),$$

where we used $[-\frac{1}{2}I + \tilde{K}]^{-1} : [Y^{1,2}(\mathbb{R}^n)]^* \to [Y^{1,2}(\mathbb{R}^n)]^*$ and $S_0 : [Y^{1,2}(\mathbb{R}^n)]^* \to$

[13] We may modify this Lemma, using now the function space $W^{1,2}(\Sigma_a^b)$ instead of $Y^{1,2}(\Sigma_a^b)$ to obtain the desired continuity in $L^2(\mathbb{R}^n)$ instead of $L^{2^*}(\mathbb{R}^n)$.

$L^2(\mathbb{R}^n)$. Taking weak limits in $L^2(\mathbb{R}^n)$ we obtain

$$0 = [-\tfrac{1}{2}I + K]S_0[-\tfrac{1}{2}I + \tilde{K}]^{-1}g$$

The invertibility of the mappings $-\frac{1}{2}I + K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $S_0 : [Y^{1,2}(\mathbb{R}^n)]^* \to L^2(\mathbb{R}^n)$ and $[-\frac{1}{2}I + \tilde{K}]^{-1} : [Y^{1,2}(\mathbb{R}^n)]^* \to [Y^{1,2}(\mathbb{R}^n)]^*$ give that $g = 0$ in $[Y^{1,2}(\mathbb{R}^n)]^*$ and hence $u \equiv 0$. $\square$

Chapter 6

Exponential decay estimates for fundamental solutions of Schrödinger-type operators

The research in this chapter was done in collaboration with S. Mayboroda [MP19].

6.1 Introduction

In this section, we continue the introduction to this chapter, already started in Section 1.2.4. Relevant literature review may be found in Section 1.3.6.

Let $A = (A_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with complex bounded measurable coefficients satisfying the uniform ellipticity condition (1.1.4). Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a vector of real-valued $L^2_{\text{loc}}(\mathbb{R}^n)$ functions and assume that $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is scalar, real valued, and positive almost everywhere on $\mathbb{R}^n$. Following the tradition and physical significance, we will refer to $\mathbf{a}$ as the magnetic potential and to V as the electric potential. We consider the generalized Schrödinger operator, formally given by (1.1.8). Let us further denote

$$D_{\mathbf{a}} = \nabla - i\mathbf{a},$$

and let $\mathbf{B}$ be the magnetic field, as given in (1.2.22). Due to the gauge invariance property, one expects $\mathbf{B}$ rather than $\mathbf{a}$ to be the primary relevant parameter. The most important particular cases that will be highlighted throughout the chapter are the magnetic Schrödinger operator $-(\nabla - i\mathbf{a})^2 + V$ and the generalized electric Schrödinger operator $-\mathrm{div} A\nabla + V$. We remark that our operators are not necessarily self-adjoint, in particular, the matrix A is not required to be symmetric (or even real-valued in the first part of the chapter).

Our goal is to treat as general a situation as possible taking no regularity assumptions on A or on V. The Fefferman-Phong Uncertainty Principle requires a mild control on oscillations of V and $\mathbf{B}$, manifested, for instance, in terms of a membership to suitable weight spaces. We say that $w \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, with $w > 0$ a.e., belongs to the Reverse Hölder class $RH_p = RH_p(\mathbb{R}^n)$ if there exists a constant C so that for any ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B w^p \right)^{1/p} \leq C \fint_B w, \tag{6.1.1}$$

and the reader will witness below that we typically assume that $V + |\mathbf{B}| \in RH_{n/2}$. Obviously, such potentials are not necessarily smooth and not necessarily bounded. Let us briefly mention here that a lot of attention has been devoted to certain polynomial potentials - in particular, because they serve as a toy model in related problems in semiclassical analysis. Any non-negative polynomial belongs to $RH_{n/2}$ class, with the constant depending on the degree and the dimension. In fact, for any polynomial P and $\alpha > 0$ we have $|P|^\alpha \in RH_p$ for any $p > 1$, with the constant depending only on α, n, and the degree of P, and $|X|^\alpha \in RH_{n/2}$ for any $\alpha > -2$.

For a function $w \in RH_p, p \geq \frac{n}{2}$, define the maximal function $m(X, w)$ by

$$\frac{1}{m(X,w)} := \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(X,r)} w \leq 1 \right\}, \tag{6.1.2}$$

and the distance function

$$d(X,Y,w) = \inf_\gamma \int_0^1 m(\gamma(t), w) |\gamma'(t)| \, dt, \tag{6.1.3}$$

where $\gamma : [0,1] \to \mathbb{R}^n$ is absolutely continuous and $\gamma(0) = X, \gamma(1) = Y$. Finally, for any $U \subset \mathbb{R}^n$, we let $d(X, U, w) := \inf_{Y \in U} d(X,Y,w)$. The function m comes from the Uncertainty Principle, which is generalized in the present chapter to serve the operators (1.1.8).

Its explicit formula (6.1.2) was introduced in [She95]. Since it is one of the main points underpinning many of our results, let us say a few more words. The function m measures the sum of the contributions of the kinetic energy $\Re e AD_{\mathbf{a}} f \overline{D_{\mathbf{a}} f}$ and potential energy $V|f|^2$, reaching optimum when f is a bump. In this vein, we find the definition (6.1.2) more telling than any of the particular representations, but let us mention nonetheless that in the aforementioned case $V = |P|^\alpha$, $\alpha > 0$, we have $m(\cdot, V) \approx \sum_{|\beta| \leq k} |\partial^\beta P|^{\frac{\alpha}{\alpha|\beta|+2}}$, where k is the degree of P [She96a]. In the context of polynomial-like potentials, by methods crucially relying on smoothness, the Uncertainty Principle has been proved in [Smi98, HM88, HN85, MN91]. However, the real breakthrough in this direction came when Fefferman and Phong treated the non-smooth potentials [Fef83]. This approach has been further formalized in [She95] to address $V \in RH_{n/2}$ and in [She96a], [BA10], to treat magnetic potentials.

With this notation in mind, we list our main results.

For any operator L given by (1.1.8) and for any $f \in L^2(\mathbb{R}^n)$ with compact support, there exist constants $\tilde{d}, \varepsilon, C > 0$ such that

$$\int_{\left\{X \in \mathbb{R}^n | d(X, \operatorname{supp} f, V+|\mathbf{B}|) \geq \tilde{d}\right\}} m\left(\cdot, V + |\mathbf{B}|\right)^2 |L^{-1} f|^2 e^{2\varepsilon d(\cdot, \operatorname{supp} f, V+|\mathbf{B}|)} \leq C \int_{\mathbb{R}^n} |f|^2 \frac{1}{m(X, V+|\mathbf{B}|)^2}, \quad (6.1.4)$$

provided that A is an elliptic matrix with complex bounded measurable coefficients, and either $\mathbf{a} = 0$ and $V \in RH_{n/2}$, or, more generally, $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, $V > 0$ a.e. on $\mathbb{R}^n$, and

$$\begin{cases} V + |\mathbf{B}| \in RH_{n/2}, \\ 0 \leq V \leq c\, m(\cdot, V + |\mathbf{B}|)^2, \\ |\nabla \mathbf{B}| \leq c'\, m(\cdot, V + |\mathbf{B}|)^3. \end{cases} \quad (6.1.5)$$

An analogous estimate holds for the resolvent operator $(I + t^2 L)^{-1}$, $t > 0$:

$$\int_{\{X \in \mathbb{R}^n \,:\, d(X, \operatorname{supp} f, V + |\mathbf{B}| + \frac{1}{t^2}) \geq \tilde{d}\}} m(\cdot, V + |\mathbf{B}| + \tfrac{1}{t^2})^2 \left|(I + t^2 L)^{-1} f\right|^2 e^{2\varepsilon d(\cdot, \operatorname{supp} f, V+|\mathbf{B}|+\frac{1}{t^2})} \leq C \int_{\mathbb{R}^n} |f|^2 m\Big(\cdot, V + |\mathbf{B}| + \frac{1}{t^2}\Big)^2. \quad (6.1.6)$$

In other words, $L^{-1}f$ decays as $e^{-\varepsilon d(\cdot, \operatorname{supp} f, V+|\mathbf{B}|)}$ away from the support of f and the resolvent decays as $e^{-\varepsilon d(\cdot, \operatorname{supp} f, V+|\mathbf{B}|+\frac{1}{t^2})}$. The strongest previously known result for the resolvent (almost) in this generality is due to Germinet and Klein [GK03]. Their work is restricted to self-adjoint operators, but otherwise, modulo some technical differences, they treat considerably more general elliptic systems than we do, including the Maxwell equation, and they go much farther towards the Combes-Thomas estimates. However, the exponential decay that they postulate is a much weaker estimate with $\frac{1}{t^2}$ in place of our $V + |\mathbf{B}| + \frac{1}{t^2}$. They do not treat the operator L^{-1}. Actually, an estimate with $\frac{1}{t^2}$ in place of our $V + |\mathbf{B}| + \frac{1}{t^2}$ has also appeared in many sources before but under stronger assumptions on the operator, and we do not attempt to review the corresponding literature.

Due to a possible lack of local boundedness of solutions to (1.1.8), the L^2 estimates in (6.1.4) are of the nature of the best possible. However, for operators whose solutions satisfy Moser and/or Harnack inequality stronger pointwise bounds can be obtained. For instance, if $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, and assumptions (6.1.5) are satisfied, then

$$|\Gamma_M(X,Y)| \le \frac{Ce^{-\varepsilon d(X,Y,V+|\mathbf{B}|)}}{|X-Y|^{n-2}} \quad \text{for a.e. } X,Y \in \mathbb{R}^n, \tag{6.1.7}$$

where Γ_M is an integral kernel of the magnetic Schrödinger operator (1.1.7), that is, the solution to $L_M\Gamma(X,Y) = \delta_y(X)$, $X,Y \in \mathbb{R}^n$, interpreted in a suitable weak sense. More generally, this bound is valid for any operator (1.1.8) with an elliptic matrix A of complex bounded measurable coefficients and (6.1.5), assuming, in addition, local boundedness of solutions and a classical estimate on the fundamental solution by $|X-Y|^{2-n}$ – see Theorem 6.6.7.

Finally, if fundamental solutions are bounded from above and below by a multiple of $|X-Y|^{2-n}$, e.g., if A is a real, bounded, elliptic matrix, $V \in RH_{\frac{n}{2}}$, and $\mathbf{a} = 0$, then we establish both upper and lower estimates (1.2.18) – see Corollaries 6.6.16 and 6.7.35. This covers the case of the generalized electric Schrödinger operator.

The outline of this chapter is as follows. In Section 6.2, we present a theory of a generalized magnetic Schrödinger operator L from (1.1.8), we define the resolvent, the heat semigroup, the inverse of L, and other notions. In Section 6.3, we provide auxiliary estimates on the maximal function m and distance d; most of the material in this section is well-known. In Section 6.4, we establish exponential decay in L^2 for the resolvent of the operator L and for L^{-1}, including (6.1.4) and (6.1.6). In Section 6.5, we provide a construction of the fundamental solution to the magnetic Schrödinger operator with $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $V > 0$ a.e., together with the basic bound by $C|X - Y|^{2-n}$. In Section 6.6, we establish exponential upper pointwise bounds on the fundamental solution, including, in particular, (6.1.7) and the upper bound in (1.2.18). In Section 6.7, we give exponential lower bounds for the fundamental solution, including, in particular, the lower bound in (1.2.18).

6.2 The theory of the generalized magnetic Schrödinger operator

6.2.1 Preliminaries

We always assume $n \geq 3$. Let L be the operator formally given by (1.1.8), where $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is a real-valued vector function, A is an elliptic matrix with complex, bounded measurable coefficients, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is scalar, complex. We will write $\Re e\, V = V^+ - V^-$ where $V^{\pm} \geq 0$. The negative part, V^-, must satisfy

$$\int_{\mathbb{R}^n} V^-|u|^2 \leq c_1 \int_{\mathbb{R}^n} \Re e\, AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}u} \,+\, c_2\|u\|^2_{L^2(\mathbb{R}^n)}, \qquad \text{for each } u \in C_c^\infty(\mathbb{R}^n), \tag{6.2.1}$$

where $c_1 \in [0, 1)$ and $c_2 \in [0, \infty)$.

Let $H := L^2(\mathbb{R}^n)$. Corresponding to the operator L given in (1.1.8), we consider for $u, v \in C_c^\infty(\mathbb{R}^n)$ the form

$$\mathfrak{l}(u, v) := \int_{\mathbb{R}^n} AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}v} + Vu\bar{v}. \tag{6.2.2}$$

and then define the domain of $\mathfrak{l}$ as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathfrak{l}} := \sqrt{\Re e\, \mathfrak{l}(u, u) + (1 + c_2)\|u\|^2_H}, \tag{6.2.3}$$

which will henceforth be known as $D(\mathfrak{l})$. This can be done because by adding $0 \leq c_1 \int_{\mathbb{R}^n}[\Re e\, V + V^-]|u|^2$ to (6.2.1), we see that

$$\int_{\mathbb{R}^n} V^-|u|^2 \leq \frac{c_1}{1 - c_1}\Re e\, \mathfrak{l}(u, u) \,+\, \frac{c_2}{1 - c_1}\|u\|^2_H, \qquad \text{for each } u \in D(\mathfrak{l}), \tag{6.2.4}$$

and so $\Re e\, \mathfrak{l}(u,u) + c_2 \|u\|_H^2 \geq 0$. It immediately follows that

$$\|u\|_{\mathfrak{l}} = 0 \implies u = 0 \text{ a.e. on } \mathbb{R}^n,$$

and $(D(\mathfrak{l}), \|\cdot\|_{\mathfrak{l}})$ is a normed space. From (6.2.4) we can see that

$$\int_{\mathbb{R}^n} V^-|u|^2 \leq \frac{1}{1-c_1}\|u\|_{\mathfrak{l}}^2, \qquad \text{for each } u \in D(\mathfrak{l}). \tag{6.2.5}$$

Moreover, using (6.2.1) and the fact that $\Re e\, V = V^+ - V^-$, it is easy to conclude that

$$\int_{\mathbb{R}^n} V^+|u|^2 \leq \|u\|_{\mathfrak{l}}^2, \qquad \text{for each } u \in D(\mathfrak{l}). \tag{6.2.6}$$

We will also need to consider the following condition on the imaginary part of V:

$$\int_{\mathbb{R}^n} |\mathrm{Im}\, V|\, |u||v| \leq c_3\sqrt{\Re e\, \mathfrak{l}(u,u) + c_4\|u\|_H^2}\,\sqrt{\Re e\, \mathfrak{l}(v,v) + c_4\|v\|_H^2}, \tag{6.2.7}$$

for each $u, v \in D(\mathfrak{l})$, where $c_3 > 0$ and c_4 is either 0 or 1. Of course, the condition (6.2.7) with $c_4 = 0$ implies the one with $c_4 = 1$, but we'll see that for the non-homogeneous setting, the case $c_4 = 1$ is enough for us. As a start, we recall the *diamagnetic inequality*, which can be formulated as

Lemma 6.2.8. *Suppose that* $\mathbf{a}$ *is a measurable function on* $\mathbb{R}^n$ *and let* u *be a measurable function on* $\mathbb{R}^n$ *such that* $D_{\mathbf{a}}u$ *is a measurable function on* $\mathbb{R}^n$*. Then* $\nabla|u|$ *is a measurable function on* $\mathbb{R}^n$*, and*

$$\Big|\nabla|u|(X)\Big| \leq \Big|D_{\mathbf{a}}u(X)\Big|, \tag{6.2.9}$$

for almost every $X \in \mathbb{R}^n$.

Let $2^* := \frac{2n}{n-2}$. By the Sobolev Embedding, we have that

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^n)}.$$

Observe that the map $\|\nabla \cdot\|_{L^2(\mathbb{R}^n)}$ is a norm on $C_c^\infty(\mathbb{R}^n)$. Define $Y^{1,2}$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under this norm. The diamagnetic inequality implies

Corollary 6.2.10. *Suppose that* $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $u \in C_c^\infty(\mathbb{R}^n)$*. Then*

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C \|D_{\mathbf{a}} u\|_{L^2(\mathbb{R}^n)}.$$

In particular, $D(\mathfrak{l}) \hookrightarrow L^{2^*}(\mathbb{R}^n)$.

The diamagnetic inequality is especially useful for the aforementioned embedding. At the moment, we turn to the theory of the form $\mathfrak{l}$: the proof of the following proposition is standard:

Proposition 6.2.11. *Let* $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$*, let* A *be an elliptic matrix with complex, bounded measurable coefficients, and let* $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *satisfying (6.2.1) and (6.2.7) with* c_4 *either* 0 *or* 1*. Then the form* $\mathfrak{l}$ *is densely defined, bounded below, continuous, and closed. If* $c_2 \equiv 0$ *in (6.2.1), then* $\mathfrak{l}$ *is accretive.*

Unless stated otherwise, take all the assumptions of the previous proposition, with $c_2 \equiv 0$ in (6.2.1). Using Definition 1.21 of [Ouh05] (with $A \mapsto L$, $\mathfrak{a} \mapsto \mathfrak{l}$ in the notation of [Ouh05]), we can define an unbounded operator $L : D(L) \to H$ where $D(L)$ is given as

$$D(L) = \Big\{ u \in D(\mathfrak{l}) \text{ s.t. } \exists v \in H : \mathfrak{l}(u, \phi) = (v, \phi)_H \ \forall \phi \in D(\mathfrak{l}) \Big\}, \qquad Lu := v.$$

The operator L is called the *operator associated with the form* $\mathfrak{l}$. Then Proposition 1.22 in [Ouh05] applies and we conclude that L is densely defined, for every $\varepsilon > 0$ the operator $L + \varepsilon$ is invertible from $D(L)$ into H, and its inverse $(L + \varepsilon)^{-1}$ is a bounded operator on H. In addition,

$$\|\varepsilon (L + \varepsilon)^{-1} f\|_H \leq \|f\|_H, \qquad \text{for each } \varepsilon > 0, f \in H.$$

We will denote by $\mathfrak{l}^*$ the adjoint form of $\mathfrak{l}$, and by L^* the operator associated to $\mathfrak{l}^*$ (see Proposition 1.24 in [Ouh05]). We also note (Lemma 1.25 in [Ouh05]) that $D(L)$ is a core of $\mathfrak{l}$; that is, $D(L)$ is dense in $D(\mathfrak{l})$ under the norm $\|\cdot\|_{\mathfrak{l}}$. Moreover, since by the aforementioned results, the resolvent set $\rho(-L)$ is not empty, then by Proposition 1.35 in [Ouh05], we see that $-L$ is a closed operator.

We now see that L is an accretive operator (see Definition 1.46 in [Ouh05]) since $\mathfrak{l}$ is an accretive form. Since $(L + \varepsilon)$ is invertible, then in particular it has dense range. So, by

Theorem 1.49 in [Ouh05], it follows that L is m-accretive, and that $-L$ is the generator of a strongly continuous contraction semigroup on H.

6.2.2 The homogeneous operator

So far we have seen that the expression $(L+\varepsilon)^{-1}f$ makes sense for $\varepsilon > 0$, but our previous construction cannot work for the homogeneous case: the operator L as defined above is not necessarily invertible as a map from $D(L)$ to $H = L^2(\mathbb{R}^n)$. It is imperative therefore to construct a homogeneous theory. The following argument is inspired by that of Section 3 of [ABA07].

Let us use the notation $\mathcal{V}_{\mathbf{a},V} := D(\mathfrak{l})$, where we will omit the subscript if the magnetic and electric potentials are clear from context. Observe that by the diamagnetic inequality we have that $(\Re e \int AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}u})^{\frac{1}{2}}$ is a norm on $C_c^\infty(\mathbb{R}^n)$. If $c_2 \equiv 0$, we define the space $\dot{\mathcal{V}}$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{\dot{\mathcal{V}}} := \sqrt{\Re e\Big(\int_{\mathbb{R}^n} AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}u} + V|u|^2\Big)}.$$

Indeed, if $u \in \dot{\mathcal{V}}$ with $\|u\|_{\dot{\mathcal{V}}} = 0$, then by (6.2.1), the diamagnetic inequality and the Sobolev embedding we obtain

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \lesssim \|D_{\mathbf{a}}u\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{\dot{\mathcal{V}}} = 0,$$

whence we must have $u = 0$ a.e. on $\mathbb{R}^n$. Thus we have $\dot{\mathcal{V}} \hookrightarrow L^{2^*}(\mathbb{R}^n)$. For instance, if $\mathbf{a} \equiv 0, V \equiv 0$, then $\dot{\mathcal{V}} = Y^{1,2}$. The form $\dot{\mathfrak{l}}$ is given by the same formula as $\mathfrak{l}$ in (6.2.2) for $u, v \in \dot{\mathcal{V}}$, and $\dot{\mathfrak{l}}$ is a coercive, bounded form on $\dot{\mathcal{V}}$. Now also suppose that $c_4 \equiv 0$. The estimates

$$\|D_{\mathbf{a}}u\|^2_{L^2(\mathbb{R}^n)} \leq C(C_A, c_1)\|u\|^2_{\dot{\mathcal{V}}},$$

$$\int_{\mathbb{R}^n} |V||u|^2 \leq C(c_1, c_3)\|u\|^2_{\dot{\mathcal{V}}},$$

$$\|u\|_{\dot{\mathcal{V}}} \leq C_A\|D_{\mathbf{a}}u\|_{L^2(\mathbb{R}^n)} + \||V|^{\frac{1}{2}}u\|_{L^2(\mathbb{R}^n)}$$

hold for $u \in \dot{\mathcal{V}}$. Actually, if we further assume that V is real-valued, then the map

$$\langle u, v\rangle_{\dot{\mathfrak{l}}_I} := \int_{\mathbb{R}^n} D_{\mathbf{a}}u\overline{D_{\mathbf{a}}v} + Vu\bar{v}, \qquad \text{for each } u, v \in \dot{\mathcal{V}},$$

is an inner product on $\dot{\mathcal{V}}$, and the induced norm is equivalent to $\|\cdot\|_{\mathfrak{l}}$. Hence $\dot{\mathcal{V}}$ can be seen as a Hilbert space when V is real-valued.

Define the operator $\dot{L} : \dot{\mathcal{V}} \to \dot{\mathcal{V}}'$ in the following way: for $u \in \dot{\mathcal{V}}$, $\dot{L}u$ is the functional $f \in \dot{\mathcal{V}}'$ given by

$$f(v) = \dot{\mathfrak{l}}(u, v), \qquad \text{for each } v \in \dot{\mathcal{V}}.$$

Clearly, $\dot{L}$ is a linear operator, which is bounded on $\dot{\mathcal{V}}$ (this is proven similarly to the continuity of $\mathfrak{l}$). By the Lax-Milgram Theorem, it is also invertible, so that $\dot{L}^{-1} : \dot{\mathcal{V}}' \to \dot{\mathcal{V}}$ exists and is unique. This means that for all $f \in \dot{\mathcal{V}}'$, there exists a unique $u \in \dot{\mathcal{V}}$ such that

$$\int_{\mathbb{R}^n} AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}v} + Vu\bar{v} = (f, v), \qquad \text{for each } v \in \dot{\mathcal{V}},$$

where (f, v) is the duality pairing of $\dot{\mathcal{V}}'$ with $\dot{\mathcal{V}}$.

The following proposition says that the space of compactly supported $L^2(\mathbb{R}^n)$ functions can be seen as a subspace of $\dot{\mathcal{V}}'$. The proof is omitted.

Proposition 6.2.12. *Assume that $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, A is an elliptic matrix with complex, bounded, measurable coefficients, $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies (6.2.7) and (6.2.1) with $c_2 \equiv c_4 \equiv 0$. Suppose $f \in L^2(\mathbb{R}^n)$ is compactly supported. Then $f \in \dot{\mathcal{V}}'$, and*

$$\|f\|_{\dot{\mathcal{V}}'} \le C|\operatorname{supp} f|^{\frac{1}{n}}\|f\|_{L^2(\mathbb{R}^n)}, \tag{6.2.13}$$

$$\|\dot{L}^{-1}f\|_{\dot{\mathcal{V}}} \le C|\operatorname{supp} f|^{\frac{1}{n}}\|f\|_{L^2(\mathbb{R}^n)}, \tag{6.2.14}$$

where C is a constant depending only on C_A, c_1, and n.

We now prove

Lemma 6.2.15. *Assume that $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, A is an elliptic matrix with complex, bounded, measurable coefficients, $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies (6.2.7) and (6.2.1) with $c_2 \equiv c_4 \equiv 0$. Suppose $f \in \dot{\mathcal{V}}' \cap L^2(\mathbb{R}^n)$. For $\varepsilon > 0$, let $u_\varepsilon = (L+\varepsilon)^{-1}f \in D(\mathfrak{l})$. Then $\{u_\varepsilon\}$ is* *a bounded sequence in $\dot{\mathcal{V}}$ which converges strongly in $\dot{\mathcal{V}}$ to $\dot{L}^{-1}f$. In particular, $\{u_\varepsilon\}$ converges to $\dot{L}^{-1}f$ strongly in the topology of $L^2_{\text{loc}}(\mathbb{R}^n)$, and a subsequence converges pointwise a.e. on $\mathbb{R}^n$.*

Proof. By definition of the sequence $\{u_\varepsilon\}$, we have

$$\int_{\mathbb{R}^n} AD_{\mathbf{a}}u_\varepsilon\overline{D_{\mathbf{a}}v} + (V+\varepsilon)u_\varepsilon\overline{v} = \int_{\mathbb{R}^n} f\,\overline{v}, \qquad \text{for each } v \in \mathcal{V} = D(\mathfrak{l}), \tag{6.2.16}$$

and in particular, since $u_\varepsilon \in \mathcal{V}$, we can write

$$\int_{\mathbb{R}^n} AD_{\mathbf{a}}u_\varepsilon\overline{D_{\mathbf{a}}u_\varepsilon} + (V+\varepsilon)|u_\varepsilon|^2 = \int_{\mathbb{R}^n} f\,\overline{u}_\varepsilon,$$

and from this we obtain (since the right-hand side can be re-written as the duality pairing $(f, \overline{u_\varepsilon})$)

$$\|u_\varepsilon\|^2_{\dot{\mathcal{V}}} = \Re e \int_{\mathbb{R}^n} AD_{\mathbf{a}}u_\varepsilon\overline{D_{\mathbf{a}}u_\varepsilon} + V|u_\varepsilon|^2 \leq \|f\|_{\dot{\mathcal{V}}'}\|u_\varepsilon\|_{\dot{\mathcal{V}}},$$

yielding the boundedness of the sequence $\{u_\varepsilon\}$ in $\dot{\mathcal{V}}$, with

$$\|u_\varepsilon\|_{\dot{\mathcal{V}}} \leq \|f\|_{\dot{\mathcal{V}}'}. \tag{6.2.17}$$

Hence, the sequence has a weak limit, say, $u \in \dot{\mathcal{V}}$. Since by the diamagnetic inequality (6.2.9) we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \varepsilon u_\varepsilon\overline{\phi}\Big| &\leq C_\phi\varepsilon\|u_\varepsilon\|_{L^{2^*}(\mathbb{R}^n)} \leq C_\phi\varepsilon\|\nabla|u_\varepsilon|\|_{L^2(\mathbb{R}^n)} \\ &\leq C_\phi\varepsilon\|D_{\mathbf{a}}u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C_\phi\varepsilon\|u_\varepsilon\|_{\dot{\mathcal{V}}} \leq C_{\phi,f}\varepsilon, \qquad \text{for each } \phi \in C_c^\infty(\mathbb{R}^n),\end{aligned}$$

then by taking limit as $\varepsilon \to 0$ on (6.2.16), we get that

$$\int_{\mathbb{R}^n} AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}\phi} + Vu\overline{\phi} = \int_{\mathbb{R}^n} f\,\overline{\phi}, \qquad \text{for each } \phi \in C_c^\infty(\mathbb{R}^n),$$

and hence for all $\phi \in \dot{\mathcal{V}}$, as $u \in \dot{\mathcal{V}}$. In other words, $u = \dot{L}^{-1}f \in \dot{\mathcal{V}}$. By the uniqueness of $\dot{L}^{-1}f$, it follows the whole sequence $\{u_\varepsilon\}$ converges weakly to $\dot{L}^{-1}f$. Now,

$$\Re e\,(f,u) = \|u\|^2_{\dot{\mathcal{V}}} \leq \liminf_{\varepsilon\to 0}\|u_\varepsilon\|^2_{\dot{\mathcal{V}}} \leq \limsup_{\varepsilon\to 0}\|u_\varepsilon\|^2_{\dot{\mathcal{V}}}$$

$$\leq \limsup_{\varepsilon\to 0}\left[\|u_\varepsilon\|^2_{\dot{\mathcal{V}}}+\int_{\mathbb{R}^n}\varepsilon|u_\varepsilon|^2\right]=\limsup_{\varepsilon\to 0}\Re e\,(f,u_\varepsilon)=\Re e\,(f,u), \tag{6.2.18}$$

which implies $\|u_\varepsilon\|_{\dot{\mathcal{V}}}\to\|u\|_{\dot{\mathcal{V}}}$. This, together with the weak convergence, gives the strong convergence in $\dot{\mathcal{V}}$. □

6.2.3 Local solutions to the magnetic Schrödinger operator and their properties

It will also be of interest to define local solutions for the operator L given in (1.1.8); under certain conditions, such local solutions will enjoy a Caccioppoli-type estimate and a Moser estimate (but we don't prove the latter result until Section 6.5). First, if $\Omega\subset\mathbb{R}^n$, we define

$$\mathcal{V}_{\mathbf{a},V}(\Omega)=\left\{u \text{ measurable s.t. } D_{\mathbf{a}}u\in L^2(\Omega) \text{ and } |V|^{\frac{1}{2}}u\in L^2(\Omega)\right\},$$

and also

$$\mathcal{V}_{\mathbf{a},V,loc}(\Omega)=\left\{u \text{ measurable s.t. } D_{\mathbf{a}}u\in L^2_{\text{loc}}(\Omega) \text{ and } |V|^{\frac{1}{2}}u\in L^2_{\text{loc}}(\Omega)\right\},$$

with $\mathcal{V}_{\mathbf{a},V,0}(\Omega)$, the completion of $C_c^\infty(\Omega)$ under the topology associated to $\mathcal{V}_{\mathbf{a},V}(\Omega)$. We omit the subscripts when possibility of confusion is slim. We remark that, in particular, the space of functions with compact support which lie in $\mathcal{V}(\Omega)$ is a subset of $\mathcal{V}_0(\Omega)$, and if (6.2.1), (6.2.7) are satisfied by V with $c_2\equiv c_4\equiv 0$, then the elements of $\dot{\mathcal{V}}$ lie in $\mathcal{V}(\Omega)$ for any $\Omega\subset\mathbb{R}^n$. Furthermore, elements of $\mathcal{V}_{\text{loc}}(\Omega)$ are locally square integrable. Indeed, since $D_{\mathbf{a}}u\in L^2_{\text{loc}}(\Omega)$, then by the diamagnetic inequality (6.2.9), $\nabla|u|\in L^2_{\text{loc}}(\Omega)$, and from this, we conclude $u\in L^2_{\text{loc}}(\Omega)$. Now, if $f\in(\mathcal{V}_0(\Omega))'$, we say that a measurable function u solves $Lu=f$ on Ω *in the weak sense* if $u\in\mathcal{V}_{\text{loc}}(\Omega)$ and

$$\int_\Omega AD_{\mathbf{a}}u\overline{D_{\mathbf{a}}\phi}+Vu\overline{\phi}=(f,\phi) \tag{6.2.19}$$

for every $\phi\in C_c^\infty(\Omega)$. By a standard limiting argument it is clear that if u solves $Lu=f$ on Ω in the weak sense, then (6.2.19) is satisfied for $\phi\in\mathcal{V}_0(\Omega)$. It's also clear that

$u = \dot{L}^{-1}f$ also solves $Lu = f$ on Ω in the weak sense. We present a generalized Caccioppoli inequality, whose proof is standard:

Theorem 6.2.20. *Assume that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$*,* A *is an elliptic matrix with complex, bounded, measurable coefficients,* $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ *satisfies (6.2.1) and (6.2.7) (with* $c_2 \in [0, \infty)$ *and* c_4 *either* 0 *or* 1*). Suppose* $f \in \Big(\mathcal{V}(B_R)\Big)' \cap L^2_{\text{loc}}(B_R)$*, and that* $Lu = f$ *on* $B(X_0, R) \subset \mathbb{R}^n$ *in the weak sense. Then*

$$\int_{B(X_0,r)} |D_{\mathbf{a}} u|^2 \le C\Big\{\Big[\frac{1}{(R-r)^2} + c_2\Big] \int_{B(X_0,R)} |u|^2 + \int_{B(X_0,R)} |f||u|\Big\}. \tag{6.2.21}$$

for every r*,* $0 < r < R$*, where* C *is a constant depending only on* c_1 *and* C_A*.*

6.3 The Fefferman-Phong-Shen maximal function and related properties

Let $\mathscr{B}$ be the collection of all balls in $\mathbb{R}^n$, and define

$$\|w\|_{RH_p} := \sup_{B \in \mathscr{B}} \frac{\left(\fint_B w^p\right)^{1/p}}{\fint_B w}.$$

The following results are well-known in the theory of the Reverse-Hölder classes (see, for instance, [Ste93a]):

Proposition 6.3.1. *If* $w \in RH_p$ *for some* $p \ge 1$*, then* $w \in RH_r$ *for each* $r \in [1, p]$*.*

Proposition 6.3.2. [Geh73] *If* $w \in RH_p$*, for some* $p \ge 1$*, then there exists* $\varepsilon > 0$ *depending only on* $\|w\|_{RH_p}, p$*, and* n*, such that* $w \in RH_r$ *for every* $r \in [p, p+\varepsilon)$*. Moreover,* $\|w\|_{RH_r}$ *depends only on* $\|w\|_{RH_p}, p$*, and* n*.*

Proposition 6.3.3. *If* $w \in RH_p$ *for some* $p > 1$*, then there exists a constant* C_0 *depending only on* $\|w\|_{RH_p}, p$*, and* n *so that for any ball* $B \subset \mathbb{R}^n$*,*

$$\int_{2B} w \le C_0 \int_B w, \tag{6.3.4}$$

where $2B$ *denotes the ball with same center as* B *and twice the radius of* B*.*

Proof. Fix a point $X \in \mathbb{R}^n$ and $r > 0$. Then, for any $R > r$, we observe that

$$\begin{aligned}
\int_{B(X,R)\backslash B(X,r)} w &\leq \Big(\int_{B(X,R)\backslash B(X,r)} w^p \Big)^{1/p} |B(X,R)\backslash B(X,r)|^{1-\frac{1}{p}} \\
&\leq \Big(\frac{R^n}{|B(0,1)|R^n} \int_{B(X,R)} w^p \Big)^{1/p} |B(0,1)|(R^n - r^n)^{1-\frac{1}{p}} \\
&\leq \Big(\frac{\|w\|_{RH_p}}{|B(0,1)|R^n} \int_{B(X,R)} w^p \Big) |B(0,1)| R^{n/p}(R^n - r^n)^{1-\frac{1}{p}} \\
&= \|w\|_{RH_p} \left[1 - \frac{r^n}{R^n} \right]^{1-\frac{1}{p}} \int_{B(X,R)} w^p, \qquad (6.3.5)
\end{aligned}$$

where in the first inequality we used Hölder's Inequality, and in the third one we used the fact that $w \in RH_p$. Let $R_w > 0$ satisfy

$$\|w\|_{RH_p} \left[1 - \frac{r^n}{R_w^n} \right]^{1-\frac{1}{p}} = \frac{1}{2},$$

that is,

$$\frac{R_w}{r} = \left[1 - \left(\frac{1}{2\|w\|_{RH_p}} \right)^{\frac{p}{p-1}} \right]^{-1/n} =: \alpha, \qquad (6.3.6)$$

and we note that α does not depend on X or r, and $\alpha > 1$. Then, for each $R \in (r, \alpha r]$, due to (6.3.5) we observe that

$$\begin{aligned}
\int_{B(X,R)} w &= \int_{B(X,r)} w + \int_{B(X,R)\backslash B(X,r)} w \\
&\leq \int_{B(X,r)} w + \frac{1}{2} \int_{B(X,R)} w \\
\implies \int_{B(X,R)} w &\leq 2 \int_{B(X,r)} w. \qquad (6.3.7)
\end{aligned}$$

Estimate (6.3.7) holds independently of X and r, provided $R \in (r, \alpha r]$. Since $\alpha > 1$, there exists $M \in \mathbb{N}$ such that $2 \leq \alpha^M$. For example, we can take

$$M := \left\lfloor \frac{\ln 2}{\ln \alpha} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ is the floor function. Thus, upon applying (6.3.7) M times, we have

$$\int_{B(X,2r)} w \leq \int_{B(X,\alpha^M r)} w \leq 2^M \int_{B(X,r)} w \leq 2^{1+\frac{\ln 2}{\ln \alpha}} \int_{B(X,r)} w,$$

which gives (6.3.4) with

$$C_0 := 2^{1+\frac{\ln 2}{\ln \alpha}}.$$

$\square$

For a function $w \in RH_p, p \geq \frac{n}{2}$, recall we define the maximal function $m(X, w)$ by (6.1.2), and the distance function $d(X, Y, w)$ by (6.1.3). For each $r > 0$, let

$$\Phi(r) := \frac{1}{r^{n-2}} \int_{B(X,r)} w.$$

Whenever w is understood from context, we will simply write $d(X, Y) = d(X, Y, w)$. It is straightforward to prove that d satisfies the triangle inequality.

The following results were proven in [She95]; here we expose the results while keeping a careful account of the constants.

Proposition 6.3.8. [She95] *Suppose $w \in L^1_{\text{loc}}(\mathbb{R}^n)$. Fix $X \in \mathbb{R}^n$, and let $\hat{r} = \frac{1}{m(X,w)}$. Suppose that $\hat{r} \in (0, +\infty)$. Then*

$$\Phi(\hat{r}) = \frac{1}{\hat{r}^{n-2}} \int_{B(X,\hat{r})} w = 1.$$

Proof. Suppose otherwise. We note that $\Phi(r)$ is continuous on $(0, +\infty)$ (because it is the product of two continuous functions). So if $\Phi(\hat{r}) < 1$, then there exists $\delta > 0$ such that $\Phi(r) < 1$ for all $r \in (\hat{r} - \delta, \hat{r} + \delta)$. If $\hat{r} < +\infty$, then this contradicts the definition of $\hat{r}$ as the supremum of $\mathscr{R} := \{r > 0 \mid \Phi(r) \leq 1\}$. So $\Phi(\hat{r}) \geq 1$. But if $\Phi(\hat{r}) > 1$, then again there exists $\delta > 0$ such that $\Phi(r) > 1$ for all $r \in (\hat{r} - \delta, \hat{r} + \delta)$, so there can be no sequence $\{r_n\} \subset \mathscr{R}$ which converges to $\hat{r}$. It follows that $\hat{r}$ cannot be a supremum of $\mathscr{R}$. Thus $\Phi(\hat{r}) = 1$. $\square$

Lemma 6.3.9. [She95] *Let $w \in RH_p,\ p \geq \frac{n}{2}$, and $0 < r < R$. Then*

$$\Phi(r) \leq \|w\|_{RH_p} \Big(\frac{R}{r}\Big)^{\frac{n}{p}-2} \Phi(R).$$

That is,

$$\frac{1}{r^{n-2}}\int_{B(X,r)} w \le \|w\|_{RH_p}\left(\frac{R}{r}\right)^{\frac{n}{p}-2}\frac{1}{R^{n-2}}\int_{B(X,R)} w. \tag{6.3.10}$$

Proof. We observe that

$$\begin{aligned}\frac{1}{|B(X,r)|}\int_{B(X,r)} w &\le \Big(\frac{1}{|B(X,r)|}\int_{B(X,r)} w^p\Big)^{\frac{1}{p}}\\ &\le \Big(\frac{R}{r}\Big)^{\frac{n}{p}}\Big(\frac{1}{|B(X,R)|}\int_{B(X,R)} w^p\Big)^{\frac{1}{p}} \le \|w\|_{RH_p}\Big(\frac{R}{r}\Big)^{\frac{n}{p}}\Big(\frac{1}{|B(X,R)|}\int_{B(X,R)} w\Big),\end{aligned}$$

where the first inequality follows from Hölder's Inequality, and the last one is due to $w \in RH_p$. Multiplying both sides by $R^2 r^2$ now gives (6.3.10). ☐

Remark 6.3.11. From Lemma 6.3.9, it is easy to conclude that, if $w \in RH_p$, $p \ge \frac{n}{2}$ and $w > 0$ a.e. on $\mathbb{R}^n$, then $m(\cdot, w)$ only takes values in $(0, +\infty)$.

Observe that by Proposition 6.3.2, if $w \in RH_{\frac{n}{2}}$, we may actually assume that $w \in RH_p$ for some $p > \frac{n}{2}$.

Proposition 6.3.12. [She95] *Let $w \in RH_p, p > \frac{n}{2}$. Fix $X \in \mathbb{R}^n$ and let $\hat{r} = \frac{1}{m(X,w)}$. Suppose that $r > 0$ satisfies $\Phi(r) \sim 1$. That is, there exists a constant $\hat{C} \ge 1$ such that $\frac{1}{\hat{C}} \le \Phi(r) \le \hat{C}$. Then $r \sim \hat{r}$, with constant $\tilde{C} := \left(\|w\|_{RH_p}\hat{C}\right)^{\frac{1}{2-\frac{n}{p}}}$. Furthermore, the converse is true: if $r \sim \hat{r}$, then $\Phi(r) \sim 1$.*

Proof. First suppose that $r < \hat{r}$. Then we can apply Lemma 6.3.9 to see that

$$\frac{1}{\hat{C}} \le \Phi(r) \le \|w\|_{RH_p}\left(\frac{\hat{r}}{r}\right)^{\frac{n}{p}-2}\Phi(\hat{r}) = \|w\|_{RH_p}\left(\frac{\hat{r}}{r}\right)^{\frac{n}{p}-2}$$

since $\Phi(\hat{r}) = 1$. Now, $p > \frac{n}{2}$ implies $\frac{n}{p} - 2 < 0$. As such, we have

$$\hat{r} \le \left(\|w\|_{RH_p}\hat{C}\right)^{\frac{1}{2-\frac{n}{p}}} r.$$

So in this case we are able to write

$$\frac{1}{\left(\|w\|_{RH_p}\hat{C}\right)^{\frac{1}{2-\frac{n}{p}}}} \leq \frac{\hat{r}}{r} \leq \left(\|w\|_{RH_p}\hat{C}\right)^{\frac{1}{2-\frac{n}{p}}}. \tag{6.3.13}$$

If instead we have $r > \hat{r}$, similarly we get

$$\frac{1}{\left(\|w\|_{RH_p}\hat{C}\right)^{\frac{1}{2-\frac{n}{p}}}} < \frac{r}{\hat{r}} \leq \left(\|w\|_{RH_p}\hat{C}\right)^{\frac{1}{2-\frac{n}{p}}}$$

which gives (6.3.13) in this case too. It follows that (6.3.13) is true regardless of whether $r < \hat{r}$. This proves the result. $\square$

Lemma 6.3.14. *Let $w \in RH_p$, $p > \frac{n}{2}$. Then, for any constant $C > 0$ and any $X, Y \in \mathbb{R}^n$ we have*

$$\left(\|w\|_{RH_p} C_0^{1+\log_2(C+1)}\right)^{-\frac{1}{2-\frac{n}{p}}} \leq \frac{m(X,w)}{m(Y,w)} \leq \left(\|w\|_{RH_p} C_0^{1+\log_2(C+1)}\right)^{\frac{1}{2-\frac{n}{p}}}$$
$$\text{if} \quad |X-Y| \leq \frac{C}{m(X,w)},$$

where C_0 is the constant from (6.3.4), which depends on $\|w\|_{RH_p}$, p, and n only. Therefore,

$$m(X,w) \sim m(Y,w) \quad \text{if} \quad |X-Y| \leq \frac{C}{m(X,w)}. \tag{6.3.15}$$

Proof. Let $r := \frac{1}{m(X,w)}$. Suppose that $|X-Y| \leq Cr$. Then $B(Y,r) \subset B(X,(C+1)r)$. Therefore,

$$\frac{1}{r^{n-2}}\int_{B(Y,r)} w \leq \frac{1}{r^{n-2}}\int_{B(X,(C+1)r)} w$$
$$\leq C_0^{1+\log_2(C+1)} \frac{1}{r^{n-2}}\int_{B(X,r)} w = C_0^{1+\log_2(C+1)}.$$

Similarly, we have

$$1 = \frac{1}{r^{n-2}}\int_{B(X,r)} w \leq C_0^{1+\log_2(C+1)} \frac{1}{r^{n-2}}\int_{B(Y,r)} w,$$

so that

$$\frac{1}{C_0^{1+\log_2(C+1)}} \leq \frac{1}{r^{n-2}} \int_{B(Y,r)} w \leq C_0^{1+\log_2(C+1)},$$

which by Proposition 6.3.12 implies that

$$\frac{1}{\left(\|w\|_{RH_p} C_0^{1+\log_2(C+1)}\right)^{\frac{1}{2-\frac{n}{p}}}} \leq rm(Y,w) \leq \left(\|w\|_{RH_p} C_0^{1+\log_2(C+1)}\right)^{\frac{1}{2-\frac{n}{p}}}.$$

This means $r \sim \frac{1}{m(Y,w)}$, or equivalently, $m(X,w) \sim m(Y,w)$. □

Remark 6.3.16. From Lemma 6.3.14 it follows that if $\Omega \subset \mathbb{R}^n$ is a bounded set, then there exists a constant C, depending on Ω, such that

$$\frac{1}{C} \leq m(X,w) \leq C. \tag{6.3.17}$$

The following results are proved in [She95].

Lemma 6.3.18. [She95] *Let $w \in RH_p, p \geq \frac{n}{2}$. Then there exist constants $C, c, k_0 > 0$, depending only on p, $\|w\|_p, n$, such that for any $X, Y \in \mathbb{R}^n$,*

$$m(X,w) \leq Cm(Y,w)[1 + |X-Y|m(Y,w)]^{k_0}, \tag{6.3.19}$$

and

$$m(X,w) \geq \frac{cm(Y,w)}{\left[1 + |X-Y|m(Y,w)\right]^{\frac{k_0}{k_0+1}}}. \tag{6.3.20}$$

Proposition 6.3.21. [She95] *Let $n \geq 3$, $w \in RH_p$, $p > \frac{n}{2}$. Then there is a constant C, which is a numerical multiple of C_0 from (6.3.4), such that for every $X \in \mathbb{R}^n$ and every $R > 0$,*

$$\int_{B(X,R)} \frac{w(Z)\,dz}{|Z-X|^{n-2}} \leq C \frac{1}{R^{n-2}} \int_{B(X,R)} w(Z)\,dz. \tag{6.3.22}$$

Proof. If $p = \frac{n}{2}$, then by Proposition 6.3.2, $w \in RH_{p+\varepsilon}$ for some $\varepsilon > 0$. Therefore, we can assume $p > \frac{n}{2}$ without loss of generality. Let q be the Hölder conjugate of p. Since $p > \frac{n}{2}$, then $q < \frac{n}{n-2}$. By Hölder's Inequality we observe

$$\int_{B(X,R)} \frac{w(Z)\,dz}{|Z-X|^{n-2}} \leq \left(\int_{B(X,R)} w^p\right)^{\frac{1}{p}} \left(\int_{B(X,R)} \frac{dz}{|Z-X|^{q(n-2)}}\right)^{\frac{1}{q}}$$

$$\le \left(\int_{B(X,R)} \frac{dz}{|Z-X|^{q(n-2)}} \right)^{\frac{1}{q}} C_0 |B(X,R)|^{\frac{1}{p}-1} \int_{B(X,R)} w.$$

But the integral on the right-hand side can be estimated in spherical coordinates:

$$\int_{B(X,R)} \frac{dz}{|Z-X|^{q(n-2)}} \le C(n) \int_0^R \frac{1}{r^{q(n-2)}}\, r^{n-1}\, dr$$
$$= C(n) \frac{1}{n-q(n-2)}\, r^{n-q(n-2)} \Big|_0^R \le C(n) \frac{1}{n-q(n-2)}\, R^{n-q(n-2)}.$$

It follows that

$$\int_{B(X,R)} \frac{w(Z)\, dz}{|Z-X|^{n-2}}$$
$$\le \left(C(n) \frac{1}{n-q(n-2)} \right)^{\frac{1}{q}} R^{\frac{n}{q}} R^{-(n-2)} C_0 (C(n))^{-\frac{1}{q}} R^{-\frac{n}{q}} \int_{B(X,R)} w,$$

which gives (6.3.22), with constant $C = \Big(n-q(n-2)\Big)^{-\frac{1}{q}} C_0$. $\square$

We also note the following useful observation:

Proposition 6.3.23. *Let $w \in RH_p$, $p > \frac{n}{2}$. Let $X, Y \in \mathbb{R}^n$. Then for any constant $C > 0$ there exists a constant $\mathcal{C}$ depending on C, $\|w\|_{RH_p}$, p, and n only such that*

$$d(X,Y,w) \le \mathcal{C} \qquad \text{if} \qquad |X-Y| \le \frac{C}{m(X,w)}. \tag{6.3.24}$$

Proof. Let γ be the straight line that connects X to Y. Then

$$|X-\gamma(t)| \le |X-Y| \le \frac{C}{m(X,w)}, \qquad \text{for each } t \in [0,1],$$

per the hypothesis. By (6.3.15), we see that $m(\gamma(t),w) \sim m(X,w)$ for each $t \in [0,1]$, so that there exists a constant $C_1 > 0$ satisfying $m(\gamma(t),w) \le C_1 m(X,w)$ for all $t \in [0,1]$. By definition of d, we have

$$d(X,Y,w) \le \int_0^1 m(\gamma(t),w) |\gamma'(t)|\, dt$$

$$\leq C_1 \int_0^1 m(X,w)|\gamma'(t)|\, dt = C_1 m(X,w)|X-Y| \leq C_1 C$$

Finally, take $\mathcal{C} := C_1 C$. $\square$

The following proposition will be useful in proving the lower bound exponential decay estimate of Section 6.7.

Proposition 6.3.25. [She99] *Let $w \in RH_p,\ p > \frac{n}{2}$. Then there exists a constant $\mathcal{C} > 0$, depending on $\|w\|_{RH_p}$ and n only, which satisfies*

$$X \notin B\Big(Y, \frac{2}{m(Y,w)}\Big) \qquad \textit{whenever } \ |X-Y| \geq \frac{\mathcal{C}}{m(X,w)}. \tag{6.3.26}$$

Proof. Suppose otherwise, so that for each $j \in \mathbb{N}$, there are points $X_j, Y_j \in \mathbb{R}^n$ which satisfy

$$|X_j - Y_j| \geq \frac{j}{m(X_j,w)} \qquad \text{but } \ X_j \in B\Big(Y_j, \frac{2}{m(Y_j,w)}\Big)$$

whence we have

$$\frac{j}{m(X_j,w)} \leq |X_j - Y_j| \leq \frac{2}{m(Y_j,w)}, \qquad \text{for each } j \in \mathbb{N}. \tag{6.3.27}$$

Now, the fact that $X_j \in B\Big(Y_j, \frac{2}{m(Y_j,w)}\Big)$ for each $j \in \mathbb{N}$ implies by Lemma 6.3.14 that there is a constant C, independent of j, such that

$$m(X_j,w) \leq C m(Y_j,w), \qquad \text{for each } j \in \mathbb{N}.$$

Using this result on (6.3.27) gives

$$m(Y_j,w)j \leq 2m(X_j,w) \leq 2Cm(Y_j,w), \quad \text{for each } j \in \mathbb{N}$$

$$\implies j \leq 2C, \qquad \text{for each } j \in \mathbb{N}$$

which is a contradiction. This establishes the result. $\square$

Next, we present an estimate often known as a *Fefferman-Phong inequality* for the magnetic Schrödinger operator. We cite the statement from [BA10]; the proof of the statement is an easy generalization of the proof first written in [She96a]:

Theorem 6.3.28. *Suppose that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)^n$*, and moreover assume (6.1.5). Then, for all* $u \in C^1_c(\mathbb{R}^n)$*,*

$$\int_{\mathbb{R}^n} m^2(X, V + |\mathbf{B}|)|u|^2 \, dx \leq C \int_{\mathbb{R}^n} (|D_{\mathbf{a}} u|^2 + V|u|^2) \, dx, \tag{6.3.29}$$

where C *depends on the constants* c, c' *from (6.1.5) and on* $\|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$.

At this juncture we observe that the Fefferman Phong Inequality of Theorem 6.3.28 is preserved in the class of functions $\dot{\mathcal{V}}_{\mathbf{a},V}$:

Corollary 6.3.30. *Suppose that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$*,* A *is an elliptic matrix with complex, bounded, measurable coefficients, and assume (6.1.5). Then, for all* $u \in \dot{\mathcal{V}}_{\mathbf{a},V}$*,*

$$\int_{\mathbb{R}^n} m^2(X, V + |\mathbf{B}|)|u|^2 \, dx \leq C \int_{\mathbb{R}^n} (\Re e \, A D_{\mathbf{a}} u \overline{D_{\mathbf{a}} u} + V|u|^2) \, dx, \tag{6.3.31}$$

where C *depends on the constants* c, c' *from (6.1.5) and on* $C_A, \|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$*. Note that* V *is a real (rather than complex) valued function, which is, in addition, non-negative.*

6.4 L^2 Exponential decay

For notational simplicity, for $t > 0$ we will write

$$L_t := L + \frac{1}{t^2}, \qquad V_t := V + \frac{1}{t^2}, \qquad R_t := (1 + t^2 L)^{-1} = \frac{1}{t^2} L_t^{-1},$$

and identify the case $t = \infty$ with $L_\infty := L, V_\infty = V$. Per the discussion in Section 6.2, the family of operators $\{R_t\}$ is uniformly bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. We assume that V is real-valued, $V \geq 0$, and emphasize that for U an open bounded set in $\mathbb{R}^n$, in the definition of the spaces $\mathcal{V}_{\mathbf{a},V,loc}(U)$ we always take the weight to be V, even when studying the operators L_t. This can be done, because if $\left(V + \frac{1}{t^2}\right)^{\frac{1}{2}} u \in L^2_{\text{loc}}(U)$, then it necessarily follows that $V^{\frac{1}{2}} u \in L^2_{\text{loc}}(U)$. On the other hand, if $V^{\frac{1}{2}} u \in L^2_{\text{loc}}(U)$ and $D_{\mathbf{a}} u \in L^2_{\text{loc}}(U)$, then by the diamagnetic inequality it follows that $u \in L^2_{\text{loc}}(U)$, which implies $\frac{1}{t^2} u \in L^2_{\text{loc}}(U)$, and so $\left(V + \frac{1}{t^2}\right)^{\frac{1}{2}} u \in L^2_{\text{loc}}(U)$. For a bounded set $U \subset \mathbb{R}^n$ and $c > 1$, we define

$$cU := \left\{ X \in \mathbb{R}^n \mid \exists Y \in U \text{ s.t. } |X - Y| < \frac{c-1}{2} \operatorname{diam} U \right\},$$

that is, cU is a $\frac{c-1}{2}\mathrm{diam}\,U-$ neighborhood of U. Since for each $t>0$, $V_t+|\mathbf{B}|\geq V+|\mathbf{B}|$ pointwise a.e. on $\mathbb{R}^n$, then by the definition of the Fefferman-Phong-Shen maximal function, it is easy to see that

$$m(X, V+|\mathbf{B}|) \leq m(X, V_t+|\mathbf{B}|), \qquad \text{for each } X \in \mathbb{R}^n, \tag{6.4.1}$$

and hence also

$$d(X, Y, V+|\mathbf{B}|) \leq d(X, Y, V_t+|\mathbf{B}|), \qquad \text{for each } X, Y \in \mathbb{R}^n. \tag{6.4.2}$$

Furthermore, if for some $p \in [1,\infty)$ we have $V+|\mathbf{B}| \in RH_p$ and $V+|\mathbf{B}| \neq 0$ a.e. on $\mathbb{R}^n$, then for any $t>0$, $V_t+|\mathbf{B}| \in RH_p$ with its RH_p-norm controlled by that of the former function.

Proposition 6.4.3. *Assume that $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, A is an elliptic matrix with complex, bounded, measurable coefficients, and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is real-valued with $V \geq 0$ a.e. on $\mathbb{R}^n$. If $\mathbf{a} \equiv 0$, assume $V \in RH_{\frac{n}{2}}$, otherwise take assumptions (6.1.5). Suppose $U \subset \mathbb{R}^n$ is a bounded open set. Let $t \in (0,\infty]$ and let $u \in \mathcal{V}_{\mathrm{loc}}(\mathbb{R}^n\backslash U)$ be a solution to $L_t u = 0$ in the weak sense on $\mathbb{R}^n\backslash U$. Suppose $\phi \in C_c^\infty(\mathbb{R}^n)$ is such that $\phi \equiv 0$ on $2U$. Let $g = g_t \in C^\infty(\mathbb{R}^n)$ be a non-negative function satisfying $|\nabla g(X)| \leq C_2 m(X, V_t+|\mathbf{B}|)$ for every $X \in \mathbb{R}^n$ (C_2 independent of t). Then*

$$\int_{\mathbb{R}^n} m(X, V_t+|\mathbf{B}|)^2 |u\phi|^2 e^{2\varepsilon g}\, dx \leq C \int_{\mathbb{R}^n} |u|^2 |\nabla\phi|^2 e^{2\varepsilon g}\, dx, \tag{6.4.4}$$

for any $\varepsilon \in (0,\varepsilon_0)$, where ε_0 and C depend only on $C_A, C_2, n, \|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, and the constants from (6.1.5), but not on t.

Proof. Let $f = \phi e^{\varepsilon g}$ and $\psi = \psi_\varepsilon = uf$. Let us prove that $\psi \in \mathcal{V}_0(\mathbb{R}^n\backslash U) \cap \dot{\mathcal{V}}$. Since $u \in \mathcal{V}_{\mathrm{loc}}(\mathbb{R}^n\backslash U)$, then $D_{\mathbf{a}} u \in L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash U)$ and $V^{\frac{1}{2}} u \in L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash U)$. Since $f \in C_c^\infty(\mathbb{R}^n)$, we have that $V^{\frac{1}{2}} uf \in L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash U)$. Owing to Corollary 6.2.10, we see that $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash U)$. Now, for any $\eta \in C_c^\infty(\mathbb{R}^n\backslash U)$,

$$\int_{\mathbb{R}^n} (f u_{X_i} + u f_{X_i})\eta\, dx = -\int_{\mathbb{R}^n} u (f\eta)_{X_i}\, dx + \int_{\mathbb{R}^n} u f_{X_i} \eta\, dx = -\int_{\mathbb{R}^n} u f \eta_{X_i}\, dx,$$

which proves that uf is weakly differentiable with gradient $f\nabla u + u\nabla f$. Since

$$D_{\mathbf{a}}\psi = D_{\mathbf{a}}(uf) = fD_{\mathbf{a}}u + u\nabla f,$$

it follows that $D_{\mathbf{a}}\psi \in L^2_{\text{loc}}(\mathbb{R}^n \backslash U)$. Hence $\psi \in \mathcal{V}_{\text{loc}}(\mathbb{R}^n \backslash U)$. The fact that ψ has compact support within $\mathbb{R}^n \backslash U$ now implies that $\psi \in \mathcal{V}_0(\mathbb{R}^n \backslash U)$. Since f is compactly supported in $\mathbb{R}^n \backslash U$, we have that $V|\psi|^2 = V|u|^2 f^2 \in L^1(\mathbb{R}^n)$. Similarly, $D_{\mathbf{a}}\psi \in L^2(\mathbb{R}^n)$. Hence, we actually have $\psi \in \dot{\mathcal{V}}$.

By a virtually identical argument, it is easy to see that $uf^2 \in \mathcal{V}_0(\mathbb{R}^n \backslash U)$. We note that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |D_{\mathbf{a}}\psi|^2 + V_t|\psi|^2 \leq C_A \int_{\mathbb{R}^n} \Re e\, A D_{\mathbf{a}}\psi \overline{D_{\mathbf{a}}\psi} + V_t|\psi|^2 \\
&= C_A \int_{\mathbb{R}^n} \Re e\, A\Big[f^2 D_{\mathbf{a}}u\overline{D_{\mathbf{a}}u} + f\overline{u}D_{\mathbf{a}}u\nabla f + fu\nabla f\overline{D_{\mathbf{a}}u} + |u|^2\nabla f\nabla f\Big] + V_t u\overline{uf^2} \\
&= C_A \int_{\mathbb{R}^n} \Re e\, A\Big[D_{\mathbf{a}}u\overline{D_{\mathbf{a}}(uf^2)} + fu\nabla f\overline{D_{\mathbf{a}}u} - f\overline{u}D_{\mathbf{a}}u\nabla f + |u|^2\nabla f\nabla f\Big] + V_t u\overline{uf^2}.
\end{aligned} \tag{6.4.5}$$

Since u solves $L_t u = 0$ on $\mathbb{R}^n \backslash U$ in the weak sense and $uf^2 \in \mathcal{V}_0(\mathbb{R}^n \backslash U)$, (6.4.5) reduces to

$$\begin{aligned}
&\int_{\mathbb{R}^n} |D_{\mathbf{a}}\psi|^2 + V_t|\psi|^2 \\
&\quad \leq C\int_{\mathbb{R}^n} |u|^2|\nabla f|^2 + C\int_{\mathbb{R}^n} \Re e\, A\Big[fu\nabla f\overline{D_{\mathbf{a}}u} - f\overline{u}D_{\mathbf{a}}u\nabla f\Big] = I + II.
\end{aligned} \tag{6.4.6}$$

By the boundedness of A and the Cauchy inequality with $\delta > 0$, we observe that

$$|II| \leq \int_{\mathbb{R}^n} \delta f^2|D_{\mathbf{a}}u|^2 + \int_{\mathbb{R}^n} \frac{1}{\delta}|u|^2|\nabla f|^2. \tag{6.4.7}$$

Moreover, a straightforward computation yields

$$|D_{\mathbf{a}}(uf)|^2 = f^2|D_{\mathbf{a}}u|^2 + 2f\nabla f \Re e\,(\bar{u}D_{\mathbf{a}}u) + |u|^2|\nabla f|^2,$$

so that applying the Cauchy inequality we see that

$$\frac{1}{2}f^2|D_{\mathbf{a}}u|^2 \le |D_{\mathbf{a}}(uf)|^2 + |u|^2|\nabla f|^2. \tag{6.4.8}$$

Putting together (6.4.8) with (6.4.7), (6.4.6), and the fact that $\psi = uf$, we achieve

$$\int_{\mathbb{R}^n} |D_{\mathbf{a}}\psi|^2 + V_t|\psi|^2 \le C\int_{\mathbb{R}^n} |u|^2|\nabla f|^2, \tag{6.4.9}$$

where C is a constant which depends only on C_A, n, and not on t. In passing, we note that by the diamagnetic inequality, (6.4.9) yields

$$\int_{\mathbb{R}^n} |\nabla|\psi||^2 + V_t|\psi|^2 \le C\int_{\mathbb{R}^n} |u|^2|\nabla f|^2.$$

Using the Fefferman-Phong inequality (6.3.31) and (6.4.9), it follows that we can write

$$\begin{aligned}\int_{\mathbb{R}^n} m(\cdot, V_t + |\mathbf{B}|)^2|\psi|^2 &\le C\int_{\mathbb{R}^n} |u|^2|\nabla f|^2 \\ &\le C\int_{\mathbb{R}^n} |u|^2|\nabla\phi|^2 e^{2\varepsilon g} + C\varepsilon^2\int_{\mathbb{R}^n} |u\phi|^2 e^{2\varepsilon g}|\nabla g|^2 \\ &\le C\int_{\mathbb{R}^n} |u|^2|\nabla\phi|^2 e^{2\varepsilon g} + CC_2^2\varepsilon^2\int_{\mathbb{R}^n} m(\cdot, V_t + |\mathbf{B}|)^2|u\phi|^2 e^{2\varepsilon g},\end{aligned}$$

with C independent of t. This implies that for ε small enough, we can absorb the right-most term into the left-hand side:

$$\int m(\cdot, V_t + |\mathbf{B}|)^2|u\phi|^2 e^{2\varepsilon g} \le C\int |u|^2|\nabla\phi|^2 e^{2\varepsilon g},$$

and this proves the proposition. □

Heuristically, the idea to prove (6.1.6) is that for fixed $Y \in \mathbb{R}^n$, $g = d(\cdot, Y, V_t + |\mathbf{B}|)$. However, the Agmon distance d is not necessarily a smooth function. The next proposition shows that we can procure a continuous function which is close to d in a uniform way, and which can be approximated by a sequence of bounded continuous functions. Both results are given in [She96a].

Proposition 6.4.10. *Let $w \in RH_p$, $p \ge \frac{n}{2}$. Then for each $Y \in \mathbb{R}^n$, there exists a non-negative function $\varphi(\cdot, Y) = \varphi(\cdot, Y, w) \in C^\infty(\mathbb{R}^n)$ such that for every $X \in \mathbb{R}^n$ and*

$Y \in \mathbb{R}^n$,

$$|\varphi(X,Y) - d(X,Y,w)| \le C, \tag{6.4.11}$$

and

$$|\nabla_x \varphi(X,Y)| \le C m(X,w), \tag{6.4.12}$$

where the constants in (6.4.11) and (6.4.12) depend only on $\|w\|_{RH_p}$, p, and n.

Proposition 6.4.13. *Let $w \in RH_p$, $p \ge \frac{n}{2}$. For each $Y \in \mathbb{R}^n$, there exists a sequence of non-negative bounded C^∞ functions $\{\varphi_j(\cdot,Y)\} = \{\varphi_j(\cdot,Y,w)\}$ such that, for every $X \in \mathbb{R}^n$,*

$$\varphi_j(X,Y) \le \varphi(X,Y) \quad \textit{and} \quad \varphi_j(X,Y) \to \varphi(X,Y) \quad \textit{as } j \to \infty,$$

and

$$|\nabla_x \varphi_j(X,Y)| \le C m(X,w), \tag{6.4.14}$$

where C depends on $\|w\|_{RH_p}$, p, and n only.

For an open bounded set $U \subset \mathbb{R}^n$, let

$$\mathscr{B}_{U,t} := \left\{ B\Big(X, \frac{1}{m(X, V_t + |\mathbf{B}|)}\Big) \;\Big|\; X \in U \right\}.$$

By the Besicovitch Covering Theorem (see [DiB16] for a proof), there exists a countable subcollection $\mathscr{B}'_{U,t}$ of $\mathscr{B}_{U,t}$ which covers U, and for which there is uniformly finite overlap of the balls,

$$\sum_{B \in \mathscr{B}'_{U,t}} \chi_B \le c_n,$$

with c_n depending only on the dimension n and not on U nor t. Since $\overline{U}$ is compact, there exists a finite subcollection $\mathscr{B}''_{U,t} = \{B_k\}_{k=1}^K$ of $\mathscr{B}'_{U,t}$ which covers U. Let $\mathscr{F}_{U,t}$ be the family of finite subcollections of $\mathscr{B}_{U,t}$ which cover U with finite overlap at most c_n. Clearly, $\mathscr{B}''_{U,t} \in \mathscr{F}_{U,t}$, so this family is not empty. Let us first show

Proposition 6.4.15. *Assume that $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $V \ge 0$ a.e. on $\mathbb{R}^n$. If $\mathbf{a} \equiv 0$, assume $V \in RH_{\frac{n}{2}}$, otherwise take assumptions (6.1.5). Then there exists*

$\tilde{d}$ depending on $\|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$ and n only, such that

$$\left\{X\in\mathbb{R}^n \;\middle|\; d(X,U,V_t+|\mathbf{B}|)\geq\tilde{d}\right\}\subseteq\mathbb{R}^n\setminus\bigcup_{B\in\mathscr{B}''_{U,t}}4B,\qquad \textit{for any } \mathscr{B}''_{U,t}\in\mathscr{F}_{U,t}.$$

Theorem 6.4.16. *Assume that $\mathbf{a}\in L^2_{\text{loc}}(\mathbb{R}^n)$, A is an elliptic matrix with complex, bounded, measurable coefficients, and $V\in L^1_{\text{loc}}(\mathbb{R}^n)$ with $V\geq 0$ a.e. on $\mathbb{R}^n$. If $\mathbf{a}\equiv 0$, assume $V\in RH_{\frac{n}{2}}$, otherwise take assumptions (6.1.5). Suppose $f\in L^2(\mathbb{R}^n)$ is compactly supported, and let $t\in(0,\infty)$. Then there exists $\tilde{d}>0$, depending on $\|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$ and n only, and there exists $\varepsilon>0$ such that (6.1.6) holds, where ε and C depend on $C_A, n, \|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, and the constants from (6.1.5), but they are independent of $\operatorname{supp} f, t$. Moreover, there exists $\varepsilon>0$ such that*

$$\int_{\left\{X\in\mathbb{R}^n|d(X,\operatorname{supp} f,V_t+|\mathbf{B}|)\geq\tilde{d}\right\}} m\left(\cdot,V_t+|\mathbf{B}|\right)^2\left|L_t^{-1}f\right|^2 e^{2\varepsilon d(\cdot,\operatorname{supp} f,V_t+|\mathbf{B}|)}$$
$$\leq C\int_{\mathbb{R}^n}|f|^2\frac{1}{m(\cdot,V_t+|\mathbf{B}|)^2},\quad (6.4.17)$$

where ε and C depend on $C_A, n, \|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, and the constants from (6.1.5), but they are independent of $\operatorname{supp} f, t$.

Proof. If $f=0$ on $\mathbb{R}^n$, then there is nothing to show, so suppose that $|\operatorname{supp} f|>0$. Fix $t>0$. Let U be any open ball such that $\operatorname{supp} f\subset U$, and write $u:=R_t f$. By construction, u is a weak solution to $L_t u=0$ on $\mathbb{R}^n\setminus U$ in the weak sense, since

$$u=R_t f=\frac{1}{t^2}L_t^{-1}f\implies L_t u=\frac{f}{t^2}.$$

Let $M>0$ such that $4U\subset B_M=B(0,M)$. Take $\phi\in C_c^\infty(\mathbb{R}^n)$ with $\phi\equiv 0$ on $2U$. Furthermore, suppose $\phi\equiv 1$ on $B_M\setminus 4U$, $\phi\equiv 0$ on $\mathbb{R}^n\setminus 2B_M$, and

$$|\nabla\phi|\leq\frac{2}{\operatorname{diam} U}\quad\text{on } 4U\setminus 2U,$$
$$|\nabla\phi|\leq\frac{2}{M}\quad\text{on } 2B_M\setminus B_M.$$

Fix $Y\in\operatorname{supp} f$, $j\in\mathbb{N}$, and let $g=\varphi_j(X,Y)=\varphi_j(X,Y;V_t+|\mathbf{B}|)$ as in Proposition

6.4.13. For each $j \in \mathbb{N}$, $g = \varphi_j(X, Y)$ is an admissible function in Proposition 6.4.3. Then by 6.4.4, we have

$$\begin{aligned}
&\int_{B_M \setminus 4U} m(\cdot, V_t + |\mathbf{B}|)^2 |u|^2 e^{2\varepsilon g} \\
&\qquad \le \int_{\mathbb{R}^n} m(\cdot, V_t + |\mathbf{B}|)^2 |u\phi|^2 e^{2\varepsilon g} \le C \int_{\mathbb{R}^n} |u|^2 |\nabla \phi|^2 e^{2\varepsilon g} \\
&\qquad \le C \Big\{ \int_{4U \setminus 2U} |u|^2 \frac{1}{|\operatorname{diam} U|^2} e^{2\varepsilon g} + \int_{2B_M \setminus B_M} |u|^2 \frac{1}{M^2} e^{2\varepsilon g} \Big\}, \quad (6.4.18)
\end{aligned}$$

where C is independent of j, M, Y, U, t. Since $g = \varphi_j(X, Y)$ is a bounded function on $\mathbb{R}^n$ and by definition $u \in L^2(\mathbb{R}^n)$ because the resolvent maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, it follows that

$$\Big| \int_{2B_M \setminus B_M} |u|^2 \frac{1}{M^2} e^{2\varepsilon g} \Big| \le C_j \frac{1}{M^2} \int_{\mathbb{R}^n} |u|^2 \longrightarrow 0 \quad \text{as} \quad M \to \infty.$$

Consequenty, by Fatou's lemma,

$$\int_{\mathbb{R}^n \setminus 4U} m(\cdot, V_t + |\mathbf{B}|)^2 |u|^2 e^{2\varepsilon g} \le C \int_{4U \setminus 2U} |u|^2 \frac{1}{|\operatorname{diam} U|^2} e^{2\varepsilon g}, \qquad (6.4.19)$$

with C independent of j, Y, U, t.

Now let $\mathscr{B}'' \in \mathscr{F}_{\operatorname{supp} f, t}$, and we can write $\mathscr{B}'' = \{B_k\}_{k=1}^K$. Let $\Omega = \bigcup_{k=1}^K 4B_k$, $f_k = f\chi_{B_k}$, and $h = \max\Big\{1, \sum_{k=1}^K \chi_{B_k}\Big\}$. Then it is clear that $4 \operatorname{supp} f \subset \Omega$, $\operatorname{supp} f_k \subset \overline{B_k}$, and

$$1 \le h(X) \le c_n, \qquad \text{for each } X \in \mathbb{R}^n,$$

$$\frac{1}{h} \sum_{k=1}^K f_k = \frac{1}{\max\Big\{1, \sum_{k=1}^K \chi_{B_k}\Big\}} \sum_{k=1}^K f\chi_{B_k} = f.$$

Let $u_k := R_t\left(\frac{1}{h} f_k\right)$. Since

$$R_t^{-1}\Big(\sum_{k=1}^K u_k\Big) = \sum_{k=1}^K R_t^{-1} u_k = \sum_{k=1}^K \frac{1}{h} f_k = f,$$

it follows that $u = R_t f = \sum_{k=1}^{K} u_k$ by the uniqueness of such a solution in $\mathcal{V}$. Let Y_k be the center of B_k, so that $B_k = B\Big(Y_k, \frac{1}{m(Y_k, V_t + |\mathbf{B}|)}\Big)$. By (6.4.19), for each $j \in \mathbb{N}$ and $k = 1, \dots, K$ we have

$$\int_{\mathbb{R}^n \setminus 4B_k} m(\cdot, V_t + |\mathbf{B}|)^2 |u_k|^2 e^{2\varepsilon\varphi_j(\cdot, Y_k)} \leq C \int_{4B_k \setminus 2B_k} |u_k|^2 \frac{1}{|\operatorname{diam} B_k|^2} e^{2\varepsilon\varphi_j(\cdot, Y_k)}. \tag{6.4.20}$$

We note that for $X \in 4B_k \setminus 2B_k$, by definition we have $|X - Y_k| \leq \frac{4}{m(Y_k, V_t + |\mathbf{B}|)}$, which implies that $d(X, Y_k) \leq C$, and so by (6.4.11) and Proposition 6.4.13, it follows that $\varphi_j(\cdot, Y_k) \leq C$ on $4B_k \setminus 2B_k$, with C independent of j, Y_k. Hence we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus 4B_k} m(\cdot, V_t + |\mathbf{B}|)^2 |u_k|^2 e^{2\varepsilon\varphi_j(\cdot, Y_k)} &\leq C m\big(Y_k, V_t + |\mathbf{B}|\big)^2 \int_{\mathbb{R}^n} |u_k|^2 \\ &\leq C m\big(Y_k, V_t + |\mathbf{B}|\big)^2 \int_{\mathbb{R}^n} \frac{1}{h^2} |f_k|^2 \leq C \int_{B_k} |f_k|^2 m\big(\cdot, V_t + |\mathbf{B}|\big)^2, \end{aligned} \tag{6.4.21}$$

where in the second inequality we used the uniform boundedness of the resolvents R_t from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and in the third inequality we used the fact that f_k is supported on B_k and Lemma 6.3.14. Here C is independent of j, Y_k and t. Letting $j \to \infty$ on (6.4.21), using Fatou's Lemma and (6.4.11), we conclude that for each $k = 1, \dots, K$,

$$\int_{\mathbb{R}^n \setminus 4B_k} m(\cdot, V_t + |\mathbf{B}|)^2 |u_k|^2 e^{2\varepsilon d(\cdot, Y_k, V_t + |\mathbf{B}|)} \leq C \int_{B_k} |f_k|^2 m\big(\cdot, V_t + |\mathbf{B}|\big)^2. \tag{6.4.22}$$

We remark that, by using L_t^{-1} instead of R_t whenever it has appeared up to this point, we can prove all the results so far up to (6.4.20). In this case, note that

$$\begin{aligned} &\Big(\int_{4B_k \setminus 2B_k} \big|L_t^{-1} \frac{1}{h} f_k\big|^2 \frac{1}{|\operatorname{diam} B_k|^2}\Big)^{\frac{1}{2}} \\ &\qquad\qquad \leq C \big\|L_t^{-1} \frac{1}{h} f_k\big\|_{L^{2^*}(\mathbb{R}^n)} \leq C \big\|D_{\mathbf{a}} L_t^{-1} \frac{1}{h} f_k\big\|_{L^2(\mathbb{R}^n)} \\ &\leq C \big\|L_t^{-1} \frac{1}{h} f_k\big\|_{\dot{\mathcal{V}}} \leq C |\operatorname{supp} f_k|^{\frac{1}{n}} \|f_k\|_{L^2(\mathbb{R}^n)} \leq C \Big(\int_{B_k} |f_k|^2 \frac{1}{m(\cdot, V_t + |\mathbf{B}|)^2}\Big)^{\frac{1}{2}}, \end{aligned} \tag{6.4.23}$$

where we first used Hölder's inequality, then the Sobolev inequality and the diamagnetic inequality, then the definition of the norm $\|\cdot\|_{\dot{\mathcal{V}}}$, then (6.2.17) and (6.2.13), and finally Lemma 6.3.14 and the fact that $\operatorname{supp} f_k \subset B_k$. Using (6.4.23) and the analogous (6.4.20), we can thus prove

$$\int_{\mathbb{R}^n\setminus 4B_k} m(\cdot, V_t+|\mathbf{B}|)^2 \Big|L_t^{-1}\Big(\frac{1}{h}f_k\Big)\Big|^2 e^{2\varepsilon\varphi_j(\cdot,Y_k)} \leq C\int_{B_k}|f_k|^2\frac{1}{m(\cdot,V_t+|\mathbf{B}|)^2}, \tag{6.4.24}$$

where C here is independent of t, k and f_k.

Let us prove that for each $X \in \mathbb{R}^n\setminus\Omega$,

$$\sum_{k=1}^{K} e^{-2\varepsilon d(X,Y_k)} \leq C_\varepsilon, \tag{6.4.25}$$

where C_ε is independent of X, t, K, and the support of f. Fix $X \in \mathbb{R}^n\setminus\Omega$ and let

$$A_\ell = \Big\{Y \in \mathbb{R}^n \,:\, |X-Y| \in \Big[\frac{\ell}{m(X,V_t+|\mathbf{B}|)}, \frac{\ell+1}{m(X,V_t+|\mathbf{B}|)}\Big)\Big\}, \qquad \ell\in\mathbb{Z}, \ell\geq 0.$$

Now, since $X \notin 4B_k$ for any k, note that

$$\sum_{k=1}^{K} e^{-2\varepsilon d(X,Y_k)} = \sum_{\ell=4}^{\infty}\sum_{Y_k\in A_\ell} e^{-2\varepsilon d(X,Y_k)} \leq \sum_{\ell=4}^{\infty}\Big(\max_{Y_k\in A_\ell} e^{-2\varepsilon d(X,Y_k)}\Big)\sum_{Y_k\in A_\ell} 1. \tag{6.4.26}$$

Since for all $Y_k \in A_\ell$ we have that $|X-Y_k| \approx \frac{\ell}{m(X,V_t+|\mathbf{B}|)}$, it follows by using Lemma 6.3.18 that $d(X,Y_k) \gtrsim \ell^\delta$ for all $Y_k \in A_\ell$ and for some $\delta > 0$. Hence

$$\max_{Y_k\in A_\ell} e^{-2\varepsilon d(X,Y_k)} \leq e^{-c_\varepsilon \ell^\delta}, \qquad \text{for each } \ell \geq 4.$$

For each $\ell \geq 4$ and $Y \in A_\ell$, write $B_y = B(Y, \frac{1}{m(Y,V_t+|\mathbf{B}|)})$. Then

$$\begin{aligned}\sum_{Y_k\in A_\ell} 1 = \sum_{Y_k\in A_\ell}\frac{|B_k|}{|B_k|} &\leq \frac{1}{\min_{Y\in A_\ell}|B_y|}\sum_{Y_k\in A_\ell}|B_k| \\ &\lesssim \frac{1}{\min_{Y\in A_\ell}|B_y|}\Big|B\Big(X, \frac{\ell}{m(X,V_t+|\mathbf{B}|)}\Big)\Big| \\ &\lesssim \max_{Y\in A_\ell}[m(Y,V_t+|\mathbf{B}|)^n]\frac{\ell^n}{m(X,V_t+|\mathbf{B}|)^n}\end{aligned}$$

$$\lesssim (m(X, V_t + |\mathbf{B}|)^n \ell^{k_0 n}) \frac{\ell^n}{m(X, V_t + |\mathbf{B}|)^n} \lesssim \ell^{(k_0+1)n}.$$

Therefore, from (6.4.26) we conclude that

$$\sum_{k=1}^{K} e^{-2\varepsilon d(X,Y_k)} \lesssim \sum_{\ell=4}^{\infty} \ell^{(k_0+1)n} e^{-c_\varepsilon \ell^\delta} \lesssim C_\varepsilon,$$

as claimed.

Next, since $4B_k \subset \Omega$, and $d(\cdot, \operatorname{supp} f, V_t + |\mathbf{B}|) \leq d(\cdot, Y_k, V_t + |\mathbf{B}|)$, we note that for $\varepsilon' = \frac{1}{2}\varepsilon$ and ε small enough,

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \Omega} m(\cdot, V_t + |\mathbf{B}|)^2 |u|^2 e^{2\varepsilon' d(\cdot, \operatorname{supp} f)} \\
&\quad = \int_{\mathbb{R}^n \setminus \Omega} \Big| \sum_{k=1}^{K} m(\cdot, V_t + |\mathbf{B}|) u_k e^{\varepsilon' [d(\cdot, Y_k) + d(\cdot, \operatorname{supp} f)]} e^{-\varepsilon' d(\cdot, Y_k))} \Big|^2 \\
&\quad \leq \int_{\mathbb{R}^n \setminus \Omega} \Big[\sum_{k=1}^{K} m(\cdot, V_t + |\mathbf{B}|)^2 |u_k|^2 e^{2\varepsilon d(\cdot, Y_k)} \Big] \sum_{j=1}^{K} e^{-2\varepsilon' d(\cdot, Y_j)} \\
&\quad \leq C \sum_{k=1}^{K} \int_{\mathbb{R}^n \setminus \Omega} m(\cdot, V_t + |\mathbf{B}|)^2 |u_k|^2 e^{2\varepsilon d(\cdot, Y_k)} \\
&\quad \leq C \sum_{k=1}^{K} \int_{\mathbb{R}^n \setminus 4B_k} m(\cdot, V_t + |\mathbf{B}|)^2 |u_k|^2 e^{2\varepsilon d(\cdot, Y_k)} \\
&\quad \leq C \sum_{k=1}^{K} \int_{B_k} |f_k|^2 m(\cdot, V_t + |\mathbf{B}|)^2 \\
&\quad \leq C \int_{\mathbb{R}^n} |f|^2 m(\cdot, V_t + |\mathbf{B}|)^2, \qquad (6.4.27)
\end{aligned}$$

where first we used the Cauchy-Schwartz inequality and (6.4.25), then (6.4.22). We note that C does not depend on $\operatorname{supp} f$, nor on the subcollection $\mathscr{B}'' \in \mathscr{F}$ used. Upon using Proposition 6.4.15, (6.1.6) follows immediately. Similarly, by using (6.4.24) in place of (6.4.22) in the argument leading up to (6.4.27), it is clear that we achieve (6.4.17). $\square$

We remark that (6.1.6) implies a Gaffney-type estimate:

$$\left\| (I+t^2L)^{-1}f \right\|_{L^2(E)}$$
$$\leq Ct^2 \exp\Big(-\varepsilon \frac{\operatorname{dist}(E,F)}{t}\Big) \Big\| m\Big(\cdot, V+|\mathbf{B}|+\frac{1}{t^2}\Big) f\Big\|_{L^2(F)} \quad (6.4.28)$$

where $f \in L^2(\mathbb{R}^n)$ is compactly supported with $\operatorname{supp} f \subset F$, and $E \subset \mathbb{R}^n$ satisfies that $d(E,F,V+|\mathbf{B}|)$ is large enough (here and elsewhere, $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean distance). We prove (6.4.28) in the following corollary, which also includes a proof of (6.1.4).

Corollary 6.4.29. *Assume that $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, A is an elliptic matrix with complex, bounded, measurable coefficients, and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $V \geq 0$ a.e. on $\mathbb{R}^n$. If $\mathbf{a} \equiv 0$, assume $V \in RH_{\frac{n}{2}}$, otherwise take assumptions (6.1.5). Suppose $f \in L^2(\mathbb{R}^n)$ is compactly supported. Then there exists $\tilde{d} > 0$, depending on $\|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$ and n only, and there exists $\varepsilon > 0$ such that (6.1.4) holds, where ε, C depend on $C_A, n, \varepsilon, \|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, and the constants from (6.1.5), and are independent of $\operatorname{supp} f$. Moreover, suppose E, F are open sets in $\mathbb{R}^n$ where $\operatorname{supp} f \subset F$ and $d(E,F,V+|\mathbf{B}|) > \tilde{d}$. Then for each $t > 0$, (6.4.28) holds with constants ε, C depending only on $C_A, n, \varepsilon, \|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, and the constants from (6.1.5).*

Proof. Using (6.4.1) and (6.4.2) on (6.4.17), we observe that

$$\int_{\left\{X\in\mathbb{R}^n \mid d(X,\operatorname{supp} f, V+|\mathbf{B}|)\geq \tilde{d}\right\}} m(\cdot, V+|\mathbf{B}|)^2 \left|L_t^{-1} f\right|^2 e^{2\varepsilon d(\cdot,\operatorname{supp} f, V+|\mathbf{B}|)}$$
$$\leq C \int_{\mathbb{R}^n} |f|^2 \frac{1}{m(X,V_t+|\mathbf{B}|)^2}. \quad (6.4.30)$$

The right-hand side of (6.4.30) converges to the right-hand side of (6.1.4) as $t \to \infty$ by the Lebesgue Monotone Convergence Theorem, since it can be proven that for each $X \in \mathbb{R}^n$,

$$m(X, V_t+|\mathbf{B}|) \searrow m(X, V+|\mathbf{B}|), \qquad \text{as } t \to \infty.$$

Therefore, we can now achieve (6.1.4) by using Lemma 6.2.15 and Fatou's Lemma on the left-hand side of (6.4.30).

To see that (6.4.28) is true, first note that for all $X \in \mathbb{R}^n$, if we let $r_x = \frac{1}{m(X,V_t+|\mathbf{B}|)}$, then

$$1 = \frac{1}{r_X^{n-2}} \int_{B(X,r_x)} V + \frac{1}{t^2} \geq \frac{r_X^2}{t^2} |B(0,1)|,$$

and so

$$\frac{1}{m(X, V_t + |\mathbf{B}|)} \leq \frac{1}{|B(0,1)|^{\frac{1}{2}}}\, t.$$

Using the above fact, (6.1.6), (6.4.2), and the hypothesis on the sets E, F, we can write

$$\int_E |(I + t^2 L)^{-1} f|^2 e^{2\varepsilon \frac{\operatorname{dist}(X,F)}{t}}\, dx \leq C t^2 \int_F |f|^2 m(X, V_t + |\mathbf{B}|)^2\, dx,$$

and from the above inequality, it is clear that (6.4.28) follows. $\square$

6.5 The fundamental solution of the magnetic Schrödinger operator and its properties

In this section we aim to pass to the pointwise estimates.

6.5.1 The semigroup theory, Kato-Simon inequality, and the heat kernel

We use all the notation and definitions from Section 6.2. Let $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, let A be an elliptic matrix with complex, bounded measurable coefficients, and let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy (6.2.1) and (6.2.7) with $c_2 \geq 0$ and c_4 either 0 or 1. First, recall the Resolvent formula (see the remarks following Theorem 1.43 in [Ouh05]):

$$(L + \varepsilon)^{-1} f = \int_0^\infty e^{-\varepsilon t} e^{-tL} f\, dt, \qquad \text{for each } \varepsilon > 0,\ f \in H = L^2(\mathbb{R}^n) \tag{6.5.1}$$

and the following identity

$$e^{-tL} f = \lim_{m \to \infty} \Big(1 + \frac{t}{m} L\Big)^{-m} f, \qquad \text{for each } f \in H \tag{6.5.2}$$

where the limits are in the topology of H (that is, in the $L^2(\mathbb{R}^n)-$sense). Define the form

$$\mathfrak{b}(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \overline{\nabla v}, \qquad \text{for each } u, v \in D(\mathfrak{b}) = W^{1,2}(\mathbb{R}^n).$$

In a very similar fashion as in Section 6.2, it is easy to see that $\mathfrak{b}$ is densely defined, continuous, closed, and accretive. Accordingly, we can define the unbounded operator

$-\Delta : D(-\Delta) \to H$, where $D(-\Delta)$ is given as

$$D(-\Delta) = \Big\{u \in D(\mathfrak{b}) \text{ s.t. } \exists v \in H : \mathfrak{b}(u, \phi) = (v, \phi)_H \ \forall \phi \in D(\mathfrak{b})\Big\}, \qquad -\Delta u := v.$$

Hence $D(-\Delta)$ is dense in $W^{1,2}(\mathbb{R}^n)$, $(-\Delta+\varepsilon)^{-1}$ is invertible on H for every $\varepsilon > 0$, and there is a strongly continuous contraction semigroup $e^{t\Delta}$ associated to $-\Delta$. Accordingly, identities analogous to (6.5.1) and (6.5.2) hold. Furthermore, the operator $-\Delta$ is known to enjoy several more properties: its heat semigroup is given by integration against a non-negative heat kernel $p_{-\Delta}$; that is, $p_{-\Delta}(\cdot, \cdot\,; t)$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$e^{t\Delta} f(X) = \int_{\mathbb{R}^n} p_{-\Delta}(X, Y; t) f(Y)\, dy,$$

for a.e. $X \in \mathbb{R}^n$ and for every $t > 0$, $f \in L^2(\mathbb{R}^n)$, and

$$|p_{-\Delta}(X, Y; t)| \le Ct^{-n/2} \qquad \text{for all } t > 0.$$

The operator $-\Delta$ also has a homogeneous realization, which by a slight abuse of notation we denote as $-\Delta$. The *fundamental solution* of $-\Delta$, $\Gamma_{-\Delta}$, is well-known to exist and to satisfy

$$\frac{c_n}{|X-Y|^{n-2}} = \Gamma_{-\Delta}(X, Y) = \int_0^\infty p_{-\Delta}(X, Y; t)\, dt,$$

where c_n is a dimensional constant. For each compactly supported $f \in L^\infty(\mathbb{R}^n)$, we can write

$$(-\Delta)^{-1} f(X) = \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X, Y) f(Y)\, dy, \qquad \text{a.e. } X \in \mathbb{R}^n.$$

From the non-negativity of the heat kernel $p_{-\Delta}$ and the resolvent formula, we deduce that if $0 \le f \le g$ with $f, g \in L^2(\mathbb{R}^n)$, then for all $\varepsilon > 0$ one has

$$(-\Delta+\varepsilon)^{-1} f \le (-\Delta+\varepsilon)^{-1} g \tag{6.5.3}$$

in the a.e. sense on $\mathbb{R}^n$.

In the case that $A \equiv I$, $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$ is real-valued, and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $V \ge 0$ a.e. on $\mathbb{R}^n$, we have the Kato-Simon inequality, the following formulation of which can be found in [LS81], Lemma 6:

$$|(L+\varepsilon)^{-1}f| \le (-\Delta+\varepsilon)^{-1}|f|, \qquad \text{for each } f \in H = L^2(\mathbb{R}^n). \tag{6.5.4}$$

A function $p_L(X,Y;t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a *heat kernel* of the semigroup e^{-tL} if for each $t>0$, $p_L(\cdot,\cdot\,;t)$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$, and

$$e^{-tL}f(X) = \int_{\mathbb{R}^n} p_L(X,Y;t) f(Y)\,dy,$$

for a.e. $X \in \mathbb{R}^n$ and for every $t>0$, $f \in L^2(\mathbb{R}^n)$. It is clear that if a heat kernel to the semigroup e^{-tL} exists, then it must be unique. Let us prove

Proposition 6.5.5. *Suppose that* $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$*,* $A \equiv I$*, and* $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* V *real-valued and* $V \ge 0$ *a.e. on* $\mathbb{R}^n$*. Then*

$$|e^{-tL}f| \le e^{t\Delta}|f|, \qquad \text{for each } t>0, f \in H. \tag{6.5.6}$$

Furthermore, e^{-tL} *can be seen as a bounded map from* $L^1(\mathbb{R}^n)$ *into* $L^\infty(\mathbb{R}^n)$*. The heat kernel* p_L *of the semigroup* e^{-tL} *exists, and moreover it satisfies*

$$|p_L(X,Y;t)| \le p_{-\Delta}(X,Y;t) \tag{6.5.7}$$

for all $t>0$*, a.e.* $X \in \mathbb{R}^n$*, a.e.* $Y \in \mathbb{R}^n$*.*

Proof. It is immediate that (6.5.4) can be rewritten as

$$\Big|\Big(\frac{1}{\varepsilon}L+1\Big)^{-1}f\Big| \le \Big(\frac{1}{\varepsilon}(-\Delta)+1\Big)^{-1}|f|,$$

for $\varepsilon>0$ and $f \in H$. Using the observation (6.5.3) and the Kato-Simon inequality, we note that

$$\Big|\Big(\frac{1}{\varepsilon}L+1\Big)^{-m}f\Big| \le \Big(\frac{1}{\varepsilon}(-\Delta)+1\Big)^{-1}\Big|\Big(\frac{1}{\varepsilon}L+1\Big)^{-m+1}f\Big| \le \cdots \le \Big(\frac{1}{\varepsilon}(-\Delta)+1\Big)^{-m}|f|$$

for any $m \in \mathbb{N}$, $\varepsilon>0$ and $f \in H$. It thus follows that

$$\Big|\Big(\frac{t}{m}L+1\Big)^{-m}f\Big| \le \Big(\frac{t}{m}(-\Delta)+1\Big)^{-m}|f|$$

for all $t > 0$, all $m \in \mathbb{N}$ and all $f \in H$. We take limit as $m \to \infty$ using (6.5.2), and (6.5.6) follows.

Now, we'll show e^{-tL} can be seen as a bounded operator mapping $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. To see this, note that for $f \in C_c^\infty(\mathbb{R}^n)$, we have by (6.5.6),

$$|e^{-tL} f(X)| \leq e^{t\Delta}|f|(X) \leq \|e^{t\Delta}|f|\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}}\|f\|_{L^1(\mathbb{R}^n)}, \qquad \text{a.e. } X \in \mathbb{R}^n$$

whence

$$\|e^{-tL} f\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}}\|f\|_{L^1(\mathbb{R}^n)}, \qquad \text{for each } f \in C_c^\infty(\mathbb{R}^n),$$

and therefore, using the density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$, it follows that we can see e^{-tL} as a map of $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, and a similar argument gives that $e^{-tL} : L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is also a bounded map, with

$$\|e^{-tL}\|_{L^2 \to L^\infty} \leq Ct^{-\frac{n}{4}}, \qquad \text{for each } t > 0.$$

By Dunford's Theorem (see [DP40]), this implies the existence of a measurable function $p_L(\cdot, \cdot, t)$ on $\mathbb{R}^n \times \mathbb{R}^n$, which satisfies

$$e^{-tL} f(X) = \int_{\mathbb{R}^n} p_L(X, Y; t) f(Y)\, dy, \qquad \text{for each } t > 0, \text{a.e. } X \in \mathbb{R}^n, f \in L^2(\mathbb{R}^n). \tag{6.5.8}$$

At this point we show the domination of p_L by $p_{-\Delta}$. Fix $(X, t) \in \mathbb{R}^n \times (0, \infty)$ such that $p_L(X, \cdot\,; t)$ is measurable in $\mathbb{R}^n$. Suppose there exists $B \subset \mathbb{R}^n$ such that

$$|p_L(X, Y; t)| > p_{-\Delta}(X, Y; t), \qquad Y \in B.$$

Using (6.5.6) and (6.5.8) with $f = \chi_B \frac{\overline{p_L}}{|p_L|}$, it is readily observed that for a.e. $X \in \mathbb{R}^n$, all $t > 0$,

$$\int_B |p_L(X, Y; t)|\, dy \leq \int_B p_{-\Delta}(X, Y; t)\, dy,$$

and so it follows that $|B| = 0$. More precisely, for all $t > 0$, for almost every $X \in \mathbb{R}^n$, for almost every $Y \in \mathbb{R}^n$, the inequality (6.5.7) holds. $\square$

6.5.2 The fundamental solution of the magnetic Schrödinger operator, part I

In the following definition, $\mathscr{L}$ is either $L+\varepsilon$ for $\varepsilon>0$ or $\dot{L}$, with $\mathscr{H},\mathscr{R}$ the domain and range respectively of these operators. Specifically, when $\mathscr{L}=L+\varepsilon$, we write $\mathscr{H}=D(L)$, $\mathscr{R}=L^2(\mathbb{R}^n)$. When $\mathscr{L}=\dot{L}$, we write $\mathscr{H}=\dot{\mathcal{V}}$, $\mathscr{R}=\dot{\mathcal{V}}'$.

Definition 6.5.9. We say that a measurable function $\Gamma(X,Y)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ is the fundamental solution to the invertible operator $\mathscr{L}:\mathscr{H}\to\mathscr{R}$ if the following conditions are satisfied:

1. For each $f\in C_c^\infty(\mathbb{R}^n)$, the function

$$u_f=\int_{\mathbb{R}^n}\Gamma(\cdot,Y)f(Y)\,dy$$

   lies in $\mathscr{H}$ and satisfies $u_f=\mathscr{L}^{-1}f$.
2. For a.e. $Y\in\mathbb{R}^n$, $\Gamma(\cdot,Y)$ solves $\mathscr{L}\Gamma=0$ in the weak sense locally on $\mathbb{R}^n\backslash\{Y\}$.

We carefully note that we avoid for now a stronger traditional statement that $\mathscr{L}\Gamma=\delta$ in the sense of distributions. We will discuss this further below.

At this point, for $\varepsilon\geq 0$ note due to (6.5.7) that

$$\int_0^\infty e^{-\varepsilon t}|p_L(X,Y;t)|\,dt \\ \leq\int_0^\infty|p_{-\Delta}(X,Y;t)|\,dt=\int_0^\infty p_{-\Delta}(X,Y;t)\,dt=\Gamma_{-\Delta}(X,Y)<\infty \quad (6.5.10)$$

for a.e. $X,Y\in\mathbb{R}^n, X\neq Y$. For each $\varepsilon\geq 0$, we define the measurable function $\Gamma_{L+\varepsilon}(X,Y)$ given by

$$\Gamma_{L+\varepsilon}(X,Y):=\int_0^\infty e^{-\varepsilon t}p_L(X,Y;t)\,dt, \quad (6.5.11)$$

and we will eventually see that this function is the fundamental solution to the operator $L+\varepsilon$ (we use the notation Γ_L for $\varepsilon=0$; Γ_L will be seen to be the fundamental solution to the operator $\dot{L}$). Due to (6.5.10), a function given by (6.5.11) is well-defined and finite a.e.. The following lemma captures the expected convergence result:

Lemma 6.5.12. *Suppose that* $\mathbf{a} \in L^2_{\rm loc}(\mathbb{R}^n)$*,* $A \equiv I$*, and* $V \in L^1_{\rm loc}(\mathbb{R}^n)$ *with* V *real-valued and* $V \geq 0$ *a.e. on* $\mathbb{R}^n$*. Then for every compactly supported* $f \in L^\infty(\mathbb{R}^n)$*,*

$$(L+\varepsilon)^{-1} f(X) \longrightarrow \int_{\mathbb{R}^n} \Gamma_L(X,Y) f(Y)\, dy, \quad \textit{for a.e. } X \in \mathbb{R}^n, \tag{6.5.13}$$

and

$$\Gamma_{L+\varepsilon}(X,Y) \longrightarrow \Gamma_L(X,Y) \quad \textit{for a.e. } X, Y \in \mathbb{R}^n, X \neq Y, \tag{6.5.14}$$

as $\varepsilon \searrow 0$*.*

Proof. Fix $f \in L^\infty(\mathbb{R}^n)$ with compact support. For each $\varepsilon \geq 0$, we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} e^{-\varepsilon t} |p_L||f|\, dy\, dt \leq \int_0^\infty \int_{\mathbb{R}^n} e^{-\varepsilon t} p_{-\Delta} |f|\, dy\, dt \\ \leq \int_{\mathbb{R}^n} \Gamma_{-\Delta} |f|\, dy = (-\Delta)^{-1} |f|,\end{aligned} \tag{6.5.15}$$

where in the second inequality we used Tonelli's Theorem. Since $(-\Delta)^{-1}|f|$ is a measurable finite a.e. function, it follows that for almost every $X \in \mathbb{R}^n$ and every $\varepsilon > 0$, Fubini's Theorem can be applied to (6.5.1) when (6.5.8) is used. Hence, using (6.5.11), we arrive at the identity

$$(L+\varepsilon)^{-1} f(X) = \int_{\mathbb{R}^n} \Gamma_{L+\varepsilon}(X,Y) f(Y)\, dy, \tag{6.5.16}$$

valid a.e. in $\mathbb{R}^n$, for $\varepsilon > 0$ and $f \in L^\infty(\mathbb{R}^n)$ with compact support. It is clear that

$$e^{-\varepsilon t} p_L(X,Y;t) f(Y) \longrightarrow p_L(X,Y;t) f(Y)$$

pointwise in (Y,t) for almost every $X \in \mathbb{R}^n$, as $\varepsilon \searrow 0$. Moreover, note that for a.e. $Y \in \mathbb{R}^n$,

$$|e^{-\varepsilon t} p_L(X,Y;t) f(Y)| \leq |p_L(X,Y;t) f(Y)| \leq p_{-\Delta}(X,Y;t) |f(Y)|$$

and, since for almost every $X \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \int_0^\infty p_{-\Delta}(X,Y;t) |f(Y)|\, dt\, dy < +\infty,$$

then for almost every $X \in \mathbb{R}^n$, we can apply the Lebesgue Dominated Convergence Theorem to get

$$\int_{\mathbb{R}^n} \int_0^\infty e^{-\varepsilon t} p_{-H}(X,Y;t) f(Y)\, dt\, dy \longrightarrow \int_{\mathbb{R}^n} \int_0^\infty p_{-H}(X,Y;t) f(Y)\, dt\, dy,$$

which is (6.5.13). A very similar convergence argument delivers (6.5.14). $\square$

Combining the results of this section with Lemma 6.2.15, we have

Theorem 6.5.17. *Suppose* $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, $A \equiv I$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $V \geq 0$ *a.e. on* $\mathbb{R}^n$. *Then the function* $\Gamma_L(X,Y)$ *given in (6.5.11) satisfies the following properties:*

a) For any $f \in L^\infty(\mathbb{R}^n)$ *with compact support, the function defined by*

$$u_f = \int_{\mathbb{R}^n} \Gamma_L(\cdot, Y) f(Y)\, dy$$

lies in $\dot{\mathcal{V}}$ *and is the unique element in* $\dot{\mathcal{V}}$ *satisfying* $Lu_f = f$ *in the sense of distributions on* $\mathbb{R}^n$. *Moreover the sequence* $\{(L+\varepsilon)^{-1} f\}$ *converges pointwise almost everywhere on* $\mathbb{R}^n$, *and strongly in* $\dot{\mathcal{V}}$, *to* u_f *as* $\varepsilon \searrow 0$.

b) There exists a constant C *depending only on* n, C_A, *such that for a.e.* $X, Y \in \mathbb{R}^n$,

$$|\Gamma_L(X,Y)| \leq \frac{C}{|X-Y|^{n-2}}. \tag{6.5.18}$$

c) For a.e. $Y \in \mathbb{R}^n$, $\Gamma_L(\cdot, Y) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *and* $\Gamma_L(\cdot, Y) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \backslash \{Y\})$.

Proof. Statements b) and c) hold by the definition of Γ_L and (6.5.7). Statement a) holds by (6.5.13) and Lemma 6.2.15. $\square$

Theorem 6.5.17 doesn't yet give the existence of a fundamental solution, but it does give the existence of an integral kernel of the operator $\dot{L}^{-1}$. Missing from the theorem is another important aspect of the fundamental solution, which is that $L\Gamma_L(\cdot, Y) = \delta_y$ in the sense of distributions on $\mathbb{R}^n$. Though this latter fact may not be accessible to us in the full generality, we will need some variation of $L\Gamma_L(\cdot, Y) = 0$ in the weak sense on $\mathbb{R}^n \backslash \{Y\}$ to satisfy the conditions of Definition 6.5.9. For this purpose, it is necessary to prove $D_{\mathbf{a}} \Gamma_L(\cdot, Y) \in L^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \{Y\})$.

At this point, we can prove an important property of local weak solutions to the operator L; namely, the *local uniform boundedness* of weak solutions to the operator L, also known as a *Moser* estimate. We capture this result in

Theorem 6.5.19. *Assume that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$*,* $A \equiv I$*, and* $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ *with* $V \geq 0$ *a.e. on* $\mathbb{R}^n$*. Let* $B \equiv B(X_0, R) \subset \mathbb{R}^n$*,* $f \in L^\infty(B)$*, and suppose* u *solves* $Lu = f$ *in the weak sense on* $2B := B(X_0, 2R)$*. Then* u *is locally essentially bounded, and*

$$\|u\|_{L^\infty(\frac{1}{4}B)} \leq C\Big\{\Big(\fint_{2B}|u|^2\Big)^{\frac{1}{2}} + R^2\Big(\fint_{2B}|f|^2\Big)^{\frac{1}{2}}\Big\}.$$

Proof. This result is proven in Lemma 1.3 of [She96b] for magnetic Schrödinger operators with potentials $V + \varepsilon$ where $\varepsilon > 0$ is a constant. The proof for the magnetic Schrödinger operators satisfying the hypothesis of the theorem as stated here follows as soon as one establishes a Kato-Simon inequality for such operators. More precisely, we want to prove that

$$|\dot{L}^{-1}f|(X) \leq \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)|f(Y)|\,dy, \qquad \text{for a.e. } X \in \mathbb{R}^n, \tag{6.5.20}$$

is true for each $f \in L^2(\mathbb{R}^n)$ with compact support (note that (6.5.20) should make sense even when $\int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)f(Y)\,dy = +\infty$). We note that if $f \in L^\infty(\mathbb{R}^n)$ with compact support, then we can prove (6.5.20) as follows: Recall that

$$(-\Delta + \varepsilon)^{-1}|f|(X) \leq \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)|f(Y)|\,dy, \qquad \text{for each } \varepsilon > 0, \text{ a.e. } X \in \mathbb{R}^n.$$

Using the Kato-Simon inequality (6.5.4) for operators $L + \varepsilon$ and the previous estimate, we can write that

$$\Big|(L+\varepsilon)^{-1}f(X)\Big| \leq \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)|f(Y)|\,dy, \qquad \text{for each } \varepsilon > 0, \text{ a.e. } X \in \mathbb{R}^n.$$

Consequently, using a) in Theorem 6.5.17 and taking $\liminf$ as $\varepsilon \searrow 0$ in the above inequality yields

$$|\dot{L}^{-1}f(X)| \leq \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)|f(Y)|\,dy, \qquad \text{a.e. } X \in \mathbb{R}^n,$$

which establishes (6.5.20) in this case. Now suppose $f \in L^2(\mathbb{R}^n)$ has compact support, and for each $k \in \mathbb{N}$ let

$$f_k(X) := \begin{cases} f(X), & \text{if } |f(X)| \le k \\ k, & \text{if } |f(X)| > k. \end{cases}$$

We note that for each $k \in \mathbb{N}$, $f \in L^\infty(\mathbb{R}^n)$, $\operatorname{supp} f_k = \operatorname{supp} f$, $f_k \to f$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$, and

$$|f_k(X)| \nearrow |f(X)|, \qquad \text{for a.e. } X \in \mathbb{R}^n \text{ as } k \to \infty.$$

Observe that for each $k \in \mathbb{N}$ and a.e. $X \in \mathbb{R}^n$,

$$|\dot{L}^{-1} f_k(X)| \le \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)|f_k(Y)|\, dy \le \int_{\mathbb{R}^n} \Gamma_{-\Delta}(X,Y)|f(Y)|\, dy. \tag{6.5.21}$$

Since $f_k \to f$ in $L^2(\mathbb{R}^n)$ and $\operatorname{supp} f_k = \operatorname{supp} f$, it follows that $f_k \to f$ in the topology of $\dot{\mathcal{V}}'$. Hence $\dot{L}^{-1} f_k \to \dot{L}^{-1} f$ in $\dot{\mathcal{V}}$ as $k \to \infty$, since $\dot{L}^{-1}$ is a continuous operator. In particular, a subsequence of $\{\dot{L}^{-1} f_k\}$ converges to $\dot{L}^{-1} f$ pointwise a.e. in $\mathbb{R}^n$. Hence, passing to infinity along this subsequence in (6.5.21) implies the desired estimate (6.5.20).

With (6.5.20) at hand, the proof in [She96b] can be retraced to prove the theorem in the desired generality. □

6.5.3 The fundamental solution of the magnetic Schrödinger operator with smooth coefficients

Suppose that $\mathbf{a} \in C_c^\infty(\mathbb{R}^n)$, A is an elliptic matrix with complex, bounded measurable coefficients, and $V \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfies (6.2.7), (6.2.1) with $c_2 \equiv c_4 \equiv 0$. In this case, the elements of $D(\mathfrak{l})$ coincide with those of $W^{1,2}(\mathbb{R}^n)$. To see this, suppose $u \in W^{1,2}(\mathbb{R}^n)$. Then by definition $u \in L^2(\mathbb{R}^n)$, thus $|\Re e\, V|^{\frac{1}{2}} u \in L^2(\mathbb{R}^n)$, and the expression $D_{\mathbf{a}} u = \nabla u - i\mathbf{a}u$ clearly lies in $L^2(\mathbb{R}^n)$, showing $W^{1,2}(\mathbb{R}^n) \subset D(\mathfrak{l})$. On the other hand, if $u \in D(\mathfrak{l})$, then $u \in L^2(\mathbb{R}^n)$ by definition, and it can easily be shown that $D_{\mathbf{a}} u \in L^2(\mathbb{R}^n)$; hence $\nabla u \in L^2(\mathbb{R}^n)$, so that $D(\mathfrak{l}) \subset W^{1,2}(\mathbb{R}^n)$. In a similar way, the elements of $\dot{\mathcal{V}}$ can be shown to coincide with $Y^{1,2}$, which is the space of elements of $\dot{W}^{1,2}(\mathbb{R}^n)$ which lie also in $L^{2^*}(\mathbb{R}^n)$.

Next, further assume that $A \equiv I$ and that $V \geq 0$. Then we recover all the previous results; so, for instance we have the results of Lemma 6.2.15 and Theorem 6.5.17. Moreover, we can prove

Theorem 6.5.22. *Assume $\mathbf{a} \in C_c^\infty(\mathbb{R}^n)$, $A \equiv I$, and $V \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a non-negative function. Then the function $\Gamma_L(X,Y)$ from Theorem 6.5.17 is the fundamental solution to the operator $\dot{L}$, and moreover enjoys the following properties:*

a) For fixed $Y \in \mathbb{R}^n$, $\Gamma_L(\cdot, Y) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{Y\})$, and $D_{\mathbf{a}} \cdot D_{\mathbf{a}} \Gamma_L \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \backslash \{Y\})$.

b) For fixed $Y \in \mathbb{R}^n$, the equation $L\Gamma_L = \delta_y$ holds in the sense of distributions. That is, for each $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \Big[\Gamma_L(X,Y) \overline{D_{\mathbf{a}} \cdot D_{\mathbf{a}} \phi(X)} + V(X) \Gamma_L(X,Y) \overline{\phi(X)} \Big] \, dx = \overline{\phi(Y)}.$$

c) For almost every $Y \in \mathbb{R}^n$, the equation

$$D_{\mathbf{a}} \cdot D_{\mathbf{a}} \Gamma_L(X,Y) + V(X) \Gamma_L(X,Y) = 0 \tag{6.5.23}$$

holds (in particular, in the sense of distributions on $\mathbb{R}^n \backslash \{Y\}$).

d) The identity

$$\Gamma_L(X,Y) \equiv \overline{\Gamma_{L^*}(Y,X)} \tag{6.5.24}$$

holds in the a.e. sense on $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. By Theorem 6.5.17, it follows that

$$u_f = \int_{\mathbb{R}^n} \Gamma_L(\cdot, Y) f(Y) \, dy$$

lies in $Y^{1,2}$ and solves $Lu_f = f$ in the weak sense. That is, for $\phi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \Big[D_{\mathbf{a}} \Big(\int_{\mathbb{R}^n} \Gamma_L(X,Y) f(Y) \, dy \Big) \overline{D_{\mathbf{a}} \phi(X)} + V(X) \Big(\int_{\mathbb{R}^n} \Gamma_L(X,Y) f(Y) \, dy \Big) \overline{\phi(X)} \Big] \, dx = \int_{\mathbb{R}^n} f(X) \overline{\phi(X)} \, dx,$$

which we can write as

$$\int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n}\Gamma_L(X,Y)f(Y)\,dy\overline{D_{\mathbf{a}}\cdot D_{\mathbf{a}}\phi(X)}+\int_{\mathbb{R}^n}V(X)\Gamma_L(X,Y)f(Y)\overline{\phi(X)}\,dy\Big]\,dx$$
$$=\int_{\mathbb{R}^n}f(X)\overline{\phi(X)}\,dx.$$

Using Fubini's Theorem, which is valid due to our assumptions on $\mathbf{a}$ and V, the above equality becomes

$$\int_{\mathbb{R}^n}f(Y)\int_{\mathbb{R}^n}\left[\Gamma_L(X,Y)\overline{D_{\mathbf{a}}\cdot D_{\mathbf{a}}\phi(X)}+V(X)\Gamma_L(X,Y)\overline{\phi(X)}\right]\,dx\,dy$$
$$=\int_{\mathbb{R}^n}f(X)\overline{\phi(X)}\,dx.$$

Since $f\in C_c^\infty(\mathbb{R}^n)$ was arbitrary, the equality

$$\int_{\mathbb{R}^n}\left[\Gamma_L(X,Y)\overline{D_{\mathbf{a}}\cdot D_{\mathbf{a}}\phi(X)}+V(X)\Gamma_L(X,Y)\overline{\phi(X)}\right]\,dx=\overline{\phi(Y)} \tag{6.5.25}$$

holds for almost every $Y\in\mathbb{R}^n$. This proves property b). By letting $h(Y)=\dot{L}^*\phi(Y)$ on $\mathbb{R}^n$, we see that $h\in C_c^\infty(\mathbb{R}^n)$. The invertibility of $\dot{L}^*$ implies that $\phi=(\dot{L}^*)^{-1}h$, and hence we can rewrite (6.5.25) to see that

$$\overline{(\dot{L}^*)^{-1}h(Y)}=\int_{\mathbb{R}^n}\Gamma_L(X,Y)h(X)\,dx.$$

In particular, we have shown that for a.e. $Y\in\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\left[\Gamma_L(X,Y)-\overline{\Gamma_{L^*}(Y,X)}\right]h(X)\,dx=0,\qquad \text{for each } h\in\dot{L}^*(C_c^\infty(\mathbb{R}^n)). \tag{6.5.26}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\dot{\mathcal{V}}$, then the bounded operator $\dot{L}^*:\dot{\mathcal{V}}\to\dot{\mathcal{V}}'$ maps $C_c^\infty(\mathbb{R}^n)$ to a dense set in $\dot{\mathcal{V}}'$. Consequently, (6.5.24) follows.

Now fix $Y\in\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ a bounded open set, $r>0$, call $\mathscr{O}:=\Omega\backslash B(Y,r)$, and let $\phi\in C_c^\infty(\mathscr{O})$. Then, from (6.5.25) it follows that for almost every such Y,

$$\int_{\mathbb{R}^n}\Gamma_L(X,Y)\overline{D_{\mathbf{a}}\cdot D_{\mathbf{a}}\phi(X)}\,dx=-\int_{\mathbb{R}^n}V(X)\Gamma_L(X,Y)\overline{\phi(X)}\,dx, \tag{6.5.27}$$

whence we have

$$\left| \int_{\mathscr{O}} \Gamma_L(X,Y) \overline{D_{\mathbf{a}} \cdot D_{\mathbf{a}} \phi(X)}\, dx \right| \leq C \|\phi\|_{L^1(\mathscr{O})} \tag{6.5.28}$$

where C depends on V and $\mathscr{O}$ but does *not* depend on ϕ. So consider the distribution

$$D_{\mathbf{a}} \cdot D_{\mathbf{a}} \Gamma_L,$$

defined on $C_c^\infty(\mathscr{O})$. Extend its definition to $L^1(\mathscr{O})$ using (6.5.27)-(6.5.28). By (6.5.28) we observe that this distribution is a bounded linear functional on $L^1(\mathscr{O})$. Therefore, it actually is a measurable function living in $L^\infty(\mathscr{O})$ and, per (6.5.27), (6.5.23) follows.

To see that Γ_L satisfies the conditions of Definition 6.5.9, it remains to show that for each $Y \in \mathbb{R}^n$, $D_{\mathbf{a}}\Gamma_L \in L^2_{\text{loc}}(\mathbb{R}^n \backslash \{Y\})^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, and define $\mathscr{O} = \Omega \backslash B(Y,r)$ for some $r > 0$. Let $\mathbf{h} \in L^2(\mathscr{O})^n$. We can use a Hodge Decomposition as in Lemma 4 of Chapter 4 of [AT98], so that there exists $g = g_{\mathbf{h}} \in \dot{W}^{1,2}(\mathscr{O})$ such that $\|\nabla g_{\mathbf{h}}\|_{L^2(\mathscr{O})} \leq \|\mathbf{h}\|_{L^2(\mathscr{O})}$. Moreover, $\mathbf{h} - \nabla g$ is a divergence-free vector in the weak sense; that is,

$$\int_{\mathscr{O}} (\mathbf{h} - \nabla g) \cdot \overline{\nabla \phi} = 0,$$

for each $\phi \in \dot{W}^{1,2}(\mathscr{O})$. We denote by $\mathscr{D}'(\mathscr{O})$ the space of distributions on $\mathscr{O}$. Let $\{f_k\} \subset C^\infty(\mathscr{O})$ satisfy

$$f_k \longrightarrow \Gamma_L \qquad \text{in } D'(\mathscr{O}) \text{ and in } L^2(\mathscr{O}) \text{ as } k \to \infty.$$

We can also choose $\{f_k\}$ so that

$$\|f_k\|_{L^\infty(\mathscr{O})} \leq C.$$

For each $k \in \mathbb{N}$, we can write

$$(D_{\mathbf{a}} f_k, \mathbf{h} - \nabla g) = (\nabla f_k, \mathbf{h} - \nabla g) - (i\mathbf{a} f_k, \mathbf{h} - \nabla g) = -(i\mathbf{a} f_k, \mathbf{h} - \nabla g),$$

and since

$$|(\mathbf{a} f_k, \mathbf{h} - \nabla g)| \leq C \|\mathbf{h} - \nabla g\|_{L^2(\mathscr{O})},$$

it follows that for any $\mathbf{h} \in L^2(\mathscr{O})^n$,

$$|(D_{\mathbf{a}} f_k, \mathbf{h} - \nabla g_{\mathbf{h}})| \leq C\|\mathbf{h}\|_{L^2(\mathscr{O})},$$

where C does not depend on $\mathbf{h}$. Consequently, in order to prove that $D_{\mathbf{a}}\Gamma_L$ is a bounded linear functional on $L^2(\mathscr{O})^n$ (hence, it lies in $L^2(\mathscr{O})^n$), it suffices to check that

$$\Big|(D_{\mathbf{a}}\Gamma_L, \nabla g)\Big| \leq C\|\nabla g\|_{L^2(\mathscr{O})}, \qquad \text{for each } g \in C_c^\infty(\mathscr{O}). \tag{6.5.29}$$

Since

$$(D_{\mathbf{a}}\Gamma_L, \nabla g) = (D_{\mathbf{a}}\Gamma_L, D_{\mathbf{a}} g) + (D_{\mathbf{a}}\Gamma_L, i\mathbf{a} g),$$

and

$$\Big|(D_{\mathbf{a}}\Gamma_L, D_{\mathbf{a}} g)\Big| \leq \Big|(D_{\mathbf{a}} \cdot D_{\mathbf{a}}\Gamma_L, g)\Big| \leq C\|g\|_{L^1(\mathscr{O})} \leq C\|g\|_{L^{2^*}(\mathbb{R}^n)} \leq C\|\nabla g\|_{L^2(\mathscr{O})},$$

$$\begin{aligned}\Big|(D_{\mathbf{a}}\Gamma_L, i\mathbf{a} g)\Big| &= \Big|(\Gamma_L, iD_{\mathbf{a}} \cdot (\mathbf{a} g))\Big| \\ &\leq C\|g\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla g - i|\mathbf{a}|^2 g\|_{L^1(\mathscr{O})} \leq C\|\nabla g\|_{L^2(\mathscr{O})},\end{aligned}$$

then it is clear that (6.5.29) follows. Hence $D_{\mathbf{a}}\Gamma_L \in L^2(\mathscr{O})$. This ends the proof of the theorem. □

Remark 6.5.30. The above method of proof instantly generalizes the results of this theorem to the case where $\mathbf{a} \in L^4_{\text{loc}}(\mathbb{R}^n)$, div $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$.

6.5.4 The fundamental solution of the magnetic Schrödinger operator, part II

It is our intent to approximate the magnetic Schrödinger operators with rough coefficients by those with smooth coefficients. Take $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $V \geq 0$ a.e. on $\mathbb{R}^n$. We can construct sequences $\{\mathbf{a}_k\} \subset C_c^\infty(\mathbb{R}^n)$, $\{V_k\} \subset C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ respectively, which converge to $\mathbf{a}, V$ in the topology of $L^2_{\text{loc}}(\mathbb{R}^n), L^1_{\text{loc}}(\mathbb{R}^n)$, respectively. Moreover, the V_k's can be chosen to be non-negative. We denote by L_k the operator associated to $\mathbf{a}_k$ and V_k. For each $k \in \mathbb{N}$, Theorem 6.5.22 applies, giving the existence of a fundamental solution Γ_k to the operator L_k.

We will obtain our desired result in two main steps: First, we show weak convergence of $\dot{L}_k^{-1}$ to $\dot{L}^{-1}$ in $\dot{\mathcal{V}}_{\mathbf{a},V}$. Second, we show convergence of the fundamental solutions Γ_k to Γ_L in the local topology associated to $\mathcal{V}_{\mathbf{a},V}(\mathbb{R}^n\backslash\{Y\})$.

Lemma 6.5.31. *Assume that $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, $A \equiv I$, and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $V \geq 0$ a.e. on $\mathbb{R}^n$. Fix $f \in L^\infty(\mathbb{R}^n)$ with compact support. Then*

a) $\{D_{\mathbf{a}_k}\dot{L}_k^{-1}f\}$ converges to $D_{\mathbf{a}}\dot{L}^{-1}f$ strongly in $L^2(\mathbb{R}^n)$.

b) $\{V_k^{\frac{1}{2}}\dot{L}_k^{-1}f\}$ converges to $V^{\frac{1}{2}}\dot{L}^{-1}f$ strongly in $L^2(\mathbb{R}^n)$.

c) $\{\dot{L}_k^{-1}f\}$ converges weakly to $\dot{L}^{-1}f$ in $L^{2^*}(\mathbb{R}^n)$, and strongly in the $L^2_{\text{loc}}(\mathbb{R}^n)$ sense.

Proof. By Theorem 6.5.17, the sequence $u_k = \dot{L}_k^{-1}f$ is well-defined, and for each $k \in \mathbb{N}$, $u_k \in \dot{W}^{1,2}(\mathbb{R}^n)$. By construction, we have

$$\int_{\mathbb{R}^n} D_{\mathbf{a}_k}u_k\overline{D_{\mathbf{a}_k}\phi} + V_k u_k\overline{\phi} = (f, \phi), \tag{6.5.32}$$

for each $\phi \in \dot{W}^{1,2}(\mathbb{R}^n)$ and each $k \in \mathbb{N}$. Plugging in $\phi = u_k$, we see that

$$\|u_k\|^2_{\dot{\mathcal{V}}_{\mathbf{a}_k,V_k}} = \int_{\mathbb{R}^n} |D_{\mathbf{a}_k}u_k|^2 + V_k|u_k|^2 = (f, u_k) \leq C\|f\|_{L^2(\operatorname{supp} f)}\|u_k\|_{L^2(\operatorname{supp} f)},$$

and since by the Sobolev inequality and the diamagnetic inequality we have

$$\|u_k\|_{L^2(\operatorname{supp} f)} \leq C\|u_k\|_{L^{2^*}(\mathbb{R}^n)} \leq C\|D_{\mathbf{a_k}}u_k\|_{L^2(\mathbb{R}^n)} \leq C\|u_k\|_{\dot{\mathcal{V}}_{\mathbf{a}_k,V_k}},$$

where C depends on $\operatorname{supp} f$ but not on k, then it follows that

$$\|u_k\|_{\dot{\mathcal{V}}_{\mathbf{a}_k,V_k}} \leq C$$

where C depends on f. It follows that the sequences $\{D_{\mathbf{a_k}}u_k\}, \{V_k^{\frac{1}{2}}u_k\}$ are uniformly bounded in $L^2(\mathbb{R}^n)$. Applying the diamagnetic inequality for each $k \in \mathbb{N}$, it follows that $\{u_k\}$ is a uniformly bounded sequence in $L^{2^*}(\mathbb{R}^n)$, so that in particular $\{u_k\}$ is uniformly bounded in the $L^2_{\text{loc}}(\mathbb{R}^n)$ sense (by this we mean that the sequence is uniformly bounded in $L^2(\Omega)$ for each bounded set $\Omega \subset \mathbb{R}^n$). Actually, using the Moser estimate, Theorem 6.5.19, for each $k \in \mathbb{N}$, we can see that $\{u_k\}$ is a uniformly bounded sequence in the $L^\infty_{\text{loc}}(\mathbb{R}^n)$ sense. Let g, h, u be weak limits in $L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)$ and $L^{2^*}(\mathbb{R}^n)$ of $\{D_{\mathbf{a_k}}u_k\}, \{V_k^{\frac{1}{2}}u_k\}, \{u_k\}$, respectively. Now fix $\psi \in C_c^\infty(\mathbb{R}^n)$. Diagonalizing, we pass to an indexing set where all three sequences achieve the aforementioned weak limits, and for ease of notation we say that the entire sequences converge. Observe that for each $k \in \mathbb{N}$ we have

$$(D_{\mathbf{a}}u, \psi) = \int_{\mathbb{R}^n} u\overline{D_{\mathbf{a}}\psi} = \int_{\mathbb{R}^n} u_k\overline{D_{\mathbf{a}_k}\psi} + \int_{\mathbb{R}^n} u_k\overline{[D_{\mathbf{a}}\psi - D_{\mathbf{a}_k}\psi]} + \int_{\mathbb{R}^n} [u-u_k]\overline{D_{\mathbf{a}}\psi}, \tag{6.5.33}$$

and since

$$\Big|\int_{\mathbb{R}^n} [u-u_k]\overline{D_{\mathbf{a}}\psi}\Big| \leq C\|u-u_k\|_{L^2(\operatorname{supp}\psi)} \longrightarrow 0,$$

$$\Big|\int_{\mathbb{R}^n} u_k\overline{[D_{\mathbf{a}}\psi - D_{\mathbf{a}_k}\psi]}\Big| \leq C\|u_k\|_{L^2(\operatorname{supp}\psi)}\|\mathbf{a}-\mathbf{a}_k\|_{L^2(\operatorname{supp}\psi)} \longrightarrow 0,$$

as $k \to \infty$, then from (6.5.33) we have

$$(D_{\mathbf{a}}u, \psi) = \lim_{k\to\infty}\int_{\mathbb{R}^n} u_k\overline{D_{\mathbf{a}_k}\psi} = \lim_{k\to\infty}(D_{\mathbf{a}_k}u_k, \psi) = (g, \psi),$$

whence by varying over $\psi \in C_c^\infty(\mathbb{R}^n)$ we conclude that $D_{\mathbf{a}}u \in L^2(\mathbb{R}^n)$ and $D_{\mathbf{a}}u \equiv g$ in $L^2(\mathbb{R}^n)$. Likewise, since $\{V_k^{\frac{1}{2}}\}$ is a uniformly bounded sequence in the $L^2_{\text{loc}}(\mathbb{R}^n)$ sense, and

$$\Big|\int_{\mathbb{R}^n} V^{\frac{1}{2}}[u-u_k]\psi\Big| \leq \|\psi\|_{L^\infty(\mathbb{R}^n)}\|V\|_{L^1(\operatorname{supp}\psi)}\|u-u_k\|_{L^2(\operatorname{supp}\psi)} \longrightarrow 0,$$

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} [V^{\frac{1}{2}} - V_k^{\frac{1}{2}}]u_k\psi\Big| &\leq \|\psi\|_{L^\infty(\mathbb{R}^n)}\|u_k\|_{L^2(\operatorname{supp}\psi)}\Big(\int_{\operatorname{supp}\psi} |V^{\frac{1}{2}} - V_k^{\frac{1}{2}}|^2\Big)^{\frac{1}{2}} \\ &\leq C\|V-V_k\|_{L^1(\operatorname{supp}\psi)}^{\frac{1}{2}} \longrightarrow 0,\end{aligned}$$

as $k \to \infty$, a similar argument to the one above concludes that $V^{\frac{1}{2}}u \in L^2(\mathbb{R}^n)$ and $V^{\frac{1}{2}}u \equiv h$ in $L^2(\mathbb{R}^n)$. Hence it follows that $u \in \dot{\mathcal{V}}_{\mathbf{a},V}$. Taking limit in (6.5.32) as $k \to \infty$ for fixed $\phi \in C_c^\infty(\mathbb{R}^n)$, we get

$$\int_{\mathbb{R}^n} D_{\mathbf{a}}u\overline{D_{\mathbf{a}}\phi} + Vu\overline{\phi} = (f, \phi),$$

which is valid by the aforementioned observations and the fact that $\phi \in C_c^\infty(\mathbb{R}^n)$. By the uniqueness of such a function u, we have proven that $u = \dot{L}^{-1}f$. Since we defined u to be a weak limit of $\{u_k\}$ in $L^{2^*}(\mathbb{R}^n)$, then the first part of c) is proven.

To prove the last part of c), fix $\Omega \subset \mathbb{R}^n$ a bounded open set. By the Moser estimate, Theorem 6.5.19, it follows that $\{u_k\}$ is a uniformly bounded sequence in $L^\infty(\Omega)$. Since $\{a_k\}$ is a uniformly bounded sequence in $L^2(\Omega)$ and $\{D_{\mathbf{a}_k} u_k\}$ is uniformly bounded in $L^2(\Omega)$, it therefore follows that $\{\nabla u_k\}$ is a uniformly bounded sequence in $L^2(\Omega)$. By the Rellich-Kondrakov Theorem, a subsequence of $\{u_k\}$ must be strongly convergent in $L^2(\Omega)$, and this limit has no choice but to be u, hence the limit is unique and the strong convergence occurs along the whole sequence.

Finally, to prove the strong convergences in a) and b), first observe that

$$\left| \int_{\mathbb{R}^n} f(u_k - u) \right| \leq \|f\|_{L^2(\operatorname{supp} f)} \|u_k - u\|_{L^2(\operatorname{supp} f)} \longrightarrow 0 \quad \text{as } k \to \infty,$$

and so by an argument analogous to (6.2.18), we deduce that

$$\|D_{\mathbf{a}_k} u_k\|^2_{L^2(\mathbb{R}^n)} + \|V_k^{\frac{1}{2}} u_k\|^2_{L^2(\mathbb{R}^n)} \longrightarrow \|D_{\mathbf{a}} u\|^2_{L^2(\mathbb{R}^n)} + \|V^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^n)} \tag{6.5.34}$$

as $k \to \infty$. Since the sequence of numbers on the left-hand side of (6.5.34) is non-negative, it follows that both terms converge independently. Suppose that there exists $\varepsilon > 0$ such that

$$\|D_{\mathbf{a}_k} u_k\|^2_{L^2(\mathbb{R}^n)} \longrightarrow \|D_{\mathbf{a}} u\|^2_{L^2(\mathbb{R}^n)} + \varepsilon.$$

Then, owing to (6.5.34), it must be the case that

$$\|V_k^{\frac{1}{2}} u_k\|^2_{L^2(\mathbb{R}^n)} \longrightarrow \|V^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^n)} - \varepsilon,$$

but this is absurd, since by the weak convergence of the sequence $\{V_k^{\frac{1}{2}} u_k\}$, we have

$$\|V^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^n)} \leq \liminf_{k \to \infty} \|V_k^{\frac{1}{2}} u_k\|^2_{L^2(\mathbb{R}^n)}.$$

Therefore, there cannot exist such $\varepsilon > 0$. Since $\varepsilon < 0$ is also absurd by the weak convergence of the sequence $\{D_{\mathbf{a}_k} u_k\}$, it follows that

$$\|D_{\mathbf{a}_k} u_k\|^2_{L^2(\mathbb{R}^n)} \longrightarrow \|D_{\mathbf{a}} u\|^2_{L^2(\mathbb{R}^n)}, \qquad \|V_k^{\frac{1}{2}} u_k\|^2_{L^2(\mathbb{R}^n)} \longrightarrow \|V^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R}^n)}$$

which, together with the weak convergences already shown, imply the respective strong convergences. $\square$

Theorem 6.5.35. *Assume that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$*,* $A \equiv I$*, and* $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ *with* $V \geq 0$ *a.e. on* $\mathbb{R}^n$*. Then for each* $Y \in \mathbb{R}^n$*, the fundamental solutions* Γ_k *to the operators* L_k *converge in the weak topology of* $\mathcal{V}_{\mathbf{a},V}(\mathbb{R}^n \backslash \{Y\})$*, and locally in the strong* $L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n \backslash \{X = Y\})$ *sense, to* Γ_L*, the measurable function of Theorem 6.5.17. In particular,* $\Gamma_L(\cdot, Y) \in \mathcal{V}_{\mathbf{a},V,loc}(\mathbb{R}^n \backslash \{Y\})$*, and* $L\Gamma_L = 0$ *in the weak sense on* $\mathbb{R}^n \backslash \{Y\}$*, so that* Γ_L *is the fundamental solution of the operator* $\dot{L}$*. Moreover,*

$$\Gamma_L(X, Y) \equiv \overline{\Gamma_{L^*}(Y, X)} \tag{6.5.36}$$

is true in the $a.e.$ *sense on* $\mathbb{R}^n \times \mathbb{R}^n$*.*

Proof. Let U_1, U_2 be arbitrary open bounded subsets of $\mathbb{R}^n$ such that

$$\operatorname{dist}(U_1, U_2) = 3r, \quad r > 0, \tag{6.5.37}$$

and let $\phi \in C_c^\infty(U_1)$, $f \in C_c^\infty(U_2)$. On the set $\Omega := U_1 \times U_2$, $\{\Gamma_k\}$ is a uniformly bounded sequence in $L^p(\Omega)$ for $p \in [1, \infty]$ owing to property b) in Theorem 6.5.17 (with a norm depending on U_1, U_2, r). By Theorem 6.5.22, for each $Y \in U_2$, $\Gamma_k(\cdot, Y)$ solves $L\Gamma_k(\cdot, Y) = 0$ on $\mathbb{R}^n \backslash \{Y\}$ in the weak sense, and the analogous statement holds for the $Y-$variable because of (6.5.24). Cover U_1 by a family of balls $\{B_r^m\}$ such that the balls intersect a uniformly finite number of times depending only on dimension (recall that r is given by (6.5.37)). Applying the Caccioppoli inequality (6.2.21) with $R = 2r$ on each ball B_r^m, it is straightforward that

$$\|D_{\mathbf{a}_k} \Gamma_k(\cdot, Y)\|^2_{L^2(B_r^m)} \leq C(r) \int_{B_{2r}^m} |\Gamma_k(\cdot, Y)|^2.$$

Consequently,

$$\begin{aligned} \|D_{\mathbf{a}_k} \Gamma_k(\cdot, Y)\|^2_{L^2(U_1)} &\leq \sum_{m=1}^\infty \|D_{\mathbf{a}_k} \Gamma_k(\cdot, Y)\|^2_{L^2(B_r^m)} \leq \sum_{m=1}^\infty C(r) \int_{B_{2r}^m} |\Gamma_k(\cdot, Y)|^2 \\ &\leq C(r, n) \int_{U_1 + r} |\Gamma_k(\cdot, Y)|^2 \leq C(r, n, U_1), \end{aligned}$$

whence we see that for each $Y \in U_2$, the sequence $\{D_{\mathbf{a}_k} \Gamma_k(\cdot, Y)\}$ is uniformly bounded in $L^2(U_1)$. Similarly, thanks to (6.5.24), it is proven that for each $X \in U_1$, $\{D_{\mathbf{a}_k}\Gamma_k(X,\cdot)\}$ is uniformly bounded in $L^2(U_2)$. Combining these results, we obtain that $\{D_{\mathbf{a}_k}\Gamma_k\}$ is uniformly bounded in $L^2(\Omega)$. Therefore, since $\{\Gamma_k\}$ is a uniformly bounded sequence in $L^\infty(\Omega)$ (due to (6.5.18)), then from the fact that

$$D_{\mathbf{a}_k}\Gamma_k = \nabla\Gamma_k - i\mathbf{a}_k\Gamma_k,$$

it actually follows that $\{\nabla\Gamma_k\}$ is a uniformly bounded sequence in $L^2(\Omega)$. By the Rellich-Kondrachov Theorem, it follows that a subsequence of $\{\Gamma_k\}$ converges strongly in $L^2(\Omega)$. Hence we pass to such an $L^2(\Omega)-$convergent subsequence (which we denote as the whole sequence).

On the other hand, from Lemma 6.5.31, it follows that $\{\dot{L}_k^{-1}f\}$ is a uniformly bounded sequence in $L^{2^*}(\mathbb{R}^n)$, so that in particular we can write

$$\int_{\mathbb{R}^n}\Big[\dot{L}_k^{-1}f - \dot{L}^{-1}f\Big]\phi(X)\,dx \longrightarrow 0$$

as $k \to \infty$. Using the kernel representation, we have that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big[\Gamma_k(X,Y) - \Gamma_L(X,Y)\Big]f(Y)\,dy\phi(X)\,dx \longrightarrow 0 \tag{6.5.38}$$

as $k \to \infty$. Therefore, $\Gamma_L(X,Y)$ must be the unique strong limit in $L^2(\Omega)$ of the whole sequence $\{\Gamma_k\}$. After passing to a subsequence, we can assume that $\{\Gamma_k\}$ converges pointwise a.e. on Ω to Γ_L. In particular, for a.e. $Y \in U_2$, there is a subsequence of $\{\Gamma_k(\cdot,Y)\}$, with the indexing set independent of Y (but depending on $\Omega = U_1 \times U_2$), which converges pointwise a.e. on U_1 to $\{\Gamma_L(\cdot,Y)\}$.

By the aforementioned discussion and similar argumentation to that in Lemma 6.5.31, it must be the case that $\{D_{\mathbf{a}_k}\Gamma_k(\cdot,Y)\}$ converges weakly in $L^2(U_1)$ to $D_{\mathbf{a}}\Gamma_L(\cdot,Y)$, and moreover by the weak* convergence in $L^\infty(U_1)$ of a subsequence of $\{\Gamma_k(\cdot,Y)\}$, it therefore follows that $\{V_k\Gamma_k(\cdot,Y)\}$ converges to $\{V\Gamma_L(\cdot,Y)\}$ on U_1 in the sense of distributions. Then, since $\Gamma_L(\cdot,Y) \in L^\infty(U_1)$, we have shown that $\Gamma_L(\cdot,Y) \in \mathcal{V}_{\mathbf{a},V}(U_1)$. For fixed $\phi \in C_c^\infty(U_1)$, taking the limit as $k \to \infty$ on each identity (6.2.19) satisfied by $\Gamma_k(\cdot,Y)$, we arrive at the fact that Γ_L solves $L\Gamma_L(\cdot,Y) = 0$ on U_1 in the weak sense. By (6.5.24), all results in the $X-$variable are also true in the $Y-$variable, and the pointwise a.e. convergence property of $\{\Gamma_k\}$ to Γ_L, coupled with (6.5.24), gives (6.5.36) immediately on $U_1 \times U_2$. Varying over admissible U_1, U_2 finishes the proof of the theorem. $\square$

6.6 Upper bound on the exponential decay of the fundamental solution

In the following definition, $\mathscr{L}$ is either $L + \varepsilon$ for $\varepsilon > 0$ or $\dot{L}$, with $\mathscr{H}, \mathscr{R}$ the domain and range respectively of these operators. Specifically, when $\mathscr{L} = L + \varepsilon$, we write $\mathscr{H} = D(L)$, $\mathscr{R} = L^2(\mathbb{R}^n)$. When $\mathscr{L} = \dot{L}$, we write $\mathscr{H} = \dot{\mathcal{V}}$, $\mathscr{R} = \dot{\mathcal{V}}'$.

Definition 6.6.1. We say that the operator $\mathscr{L}$ has the *zero-source local uniform boundedness property* if for each ball $B \subset \mathbb{R}^n$ and each function u which solves $\mathscr{L}u = 0$ in the weak sense on $2B$, it follows that $u \in L^\infty(B)$ and

$$\|u\|_{L^\infty(cB)} \leq C\Big(\fint_{2B} |u|^2\Big)^{\frac{1}{2}}, \tag{6.6.2}$$

where c, C are independent of B and u.

We briefly remark that for the operators under consideration, the diamagnetic inequality guarantees that $u \in L^2_{\text{loc}}(\mathbb{R}^n)$.

For the fundamental solution Γ associated to the operator $\mathscr{L}$, as given in Definition 6.5.9, it will be useful to consider the conditions

$$|\Gamma(X,Y)| \leq \frac{C}{|X-Y|^{n-2}}, \qquad \text{for a.e. } X, Y \in \mathbb{R}^n,\ X \neq Y \tag{6.6.3}$$

and

$$\frac{C_1}{|X-Y|^{n-2}} \leq |\Gamma(X,Y)| \leq \frac{C_2}{|X-Y|^{n-2}}, \qquad \text{for a.e. } X, Y \in \mathbb{R}^n,\ X \neq Y, \tag{6.6.4}$$

where the constants C, C_1, C_2 depend on the dimension only. In some situations (strictly speaking, not necessarily including ours), the condition (6.6.3) is equivalent to a Moser-type bound [KK10], but we do not explore this direction.

Let us first state a result analogous to Proposition 6.4.3 for the operator $\dot{L}$. The proof is virtually identical to that of Proposition 6.4.3, and is thus omitted.

Proposition 6.6.5. *Assume that* $\mathbf{a} \in L^2_{\rm loc}(\mathbb{R}^n)$*, A is an elliptic matrix with complex, bounded, measurable coefficients, and* $V \in L^1_{\rm loc}(\mathbb{R}^n)$ *is real-valued with* $V \geq 0$ *a.e. on* $\mathbb{R}^n$*. If* $\mathbf{a} \equiv 0$*, assume* $V \in RH_{\frac{n}{2}}$*, otherwise take assumptions (6.1.5). Suppose* $U \subset \mathbb{R}^n$ *is a bounded open set. Let* $u \in \mathcal{V}_{\rm loc}(\mathbb{R}^n \backslash U)$ *be a solution to* $Lu = 0$ *in the weak sense on* $\mathbb{R}^n \backslash U$*. Suppose* $\phi \in C^\infty_c(\mathbb{R}^n)$ *is such that* $\phi \equiv 0$ *on* $2U$*. Let* $g \in C^\infty(\mathbb{R}^n)$ *be a non-negative function satisfying* $|\nabla g(X)| \leq C_2 m(X, V + |\mathbf{B}|)$ *for every* $X \in \mathbb{R}^n$*. Then*

$$\int_{\mathbb{R}^n} m(X, V + |\mathbf{B}|)^2 |u\phi|^2 e^{2\varepsilon g}\, dx \leq C \int_{\mathbb{R}^n} |u|^2 |\nabla \phi|^2 e^{2\varepsilon g}\, dx, \qquad (6.6.6)$$

for any $\varepsilon \in (0, \varepsilon_0)$*, where* ε_0 *and* C *depend only on* $C_A, C_2, n, \|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$*, and the constants from (6.1.5).*

The next results follow the lines of the argument in [She99]:

Theorem 6.6.7. *Suppose* $\mathbf{a} \in L^2_{\rm loc}(\mathbb{R}^n)$*, A is an elliptic matrix with complex, bounded coefficients,* $V \in L^1_{\rm loc}(\mathbb{R}^n)$*, and that L is an operator for which there exists a fundamental solution in the sense of Definition 6.5.9 satisfying (6.6.3). Moreover, if* $\mathbf{a} \equiv 0$*, assume* $V \in RH_{\frac{n}{2}}$*; otherwise assume (6.1.5). Then there exists* $\varepsilon > 0$ *and a constant* $C > 0$*, depending on L, such that*

$$\left(\fint_{B(X, \frac{1}{m(X, V + |\mathbf{B}|)})} |\Gamma(Z, Y)|^2\, dz \right)^{\frac{1}{2}} \leq \frac{C e^{-\varepsilon d(X, Y, V + |\mathbf{B}|)}}{|X - Y|^{n-2}} \text{ for all } X, Y \in \mathbb{R}^n. \qquad (6.6.8)$$

If L satisfies the zero-source local uniform boundedness property, then

$$|\Gamma(X, Y)| \leq \frac{C e^{-\varepsilon d(X, Y, V + |\mathbf{B}|)}}{|X - Y|^{n-2}} \text{ for all } X, Y \in \mathbb{R}^n. \qquad (6.6.9)$$

Proof. Fix $X_0, Y_0 \in \mathbb{R}^n$, $X_0 \neq Y_0$. Since for each $X, Y \in \mathbb{R}^n$ we have

$$d(X, Y) \leq C \quad \text{if} \quad |X - Y| < \frac{C}{m(X, V + |\mathbf{B}|)},$$

then in particular we may also assume that $|X_0 - Y_0| \geq \frac{C}{m(Y_0, V)}$. Indeed, otherwise $d(X_0, Y_0) \leq C$, and so from (6.6.3), we observe

$$|\Gamma(X_0, Y_0)| \leq \frac{C}{|X - Y|^{n-2}} \leq \frac{C e^{\varepsilon C} e^{-\varepsilon d(X_0, Y_0)}}{|X - Y|^{n-2}}$$

which gives (6.6.9). Furthermore, we can assume that

$$B\Big(X_0, \frac{4}{m(X_0, V+|\mathbf{B}|)}\Big) \cap B\Big(Y_0, \frac{4}{m(Y_0, V+|\mathbf{B}|)}\Big) = \varnothing. \tag{6.6.10}$$

Indeed, suppose there exists $Z \in B\Big(X_0, \frac{4}{m(X_0,V+|\mathbf{B}|)}\Big) \cap B\Big(Y_0, \frac{4}{m(Y_0,V+|\mathbf{B}|)}\Big)$. Then we note

$$\begin{aligned}|X_0 - Y_0| \le |X_0 - Z| + |Z - Y_0| &\le \frac{4}{m(X_0, V+|\mathbf{B}|)} + \frac{4}{m(Y_0, V+|\mathbf{B}|)} \\ &\le 8\max\Big\{\frac{1}{m(X_0, V+|\mathbf{B}|)}\,,\ \frac{1}{m(Y_0, V+|\mathbf{B}|)}\Big\}\end{aligned}$$

which once again implies $d(X_0, Y_0) \le C$.

Per our assumptions, $L\Gamma(\cdot, Y_0) = 0$ in the weak sense on any ball centered around X_0 which does not contain Y_0. Let $r = \frac{1}{m(Y_0,V+|\mathbf{B}|)}$. Applying Proposition 6.6.5 with $u = \Gamma(\cdot, Y_0)$ and $U = B(Y_0, 4r)$, we can carry out the argument of Theorem 6.4.16 up to proving (6.4.18) to establish that

$$\begin{aligned}&\int_{B_M\setminus 4U} m(\cdot, V+|\mathbf{B}|)^2 |u|^2 e^{2\varepsilon\varphi_j} \\ &\qquad \le C\Big\{\int_{4U\setminus 2U} |u|^2 \frac{1}{|\operatorname{diam} U|^2} e^{2\varepsilon\varphi_j} + \int_{2B_M\setminus B_M} |u|^2 \frac{1}{M^2} e^{2\varepsilon\varphi_j}\Big\}.\end{aligned} \tag{6.6.11}$$

where φ_j is as in Proposition 6.4.13 with $w = V + |\mathbf{B}|$. Owing to (6.6.3) and the fact that ϕ_j is uniformly bounded in $2B_M\setminus B_M$, it follows that the second term in the right-hand side of (6.6.11) drops to 0 as $M \to \infty$. Therefore, we may conclude that

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(Y_0,4r)} m(X, V+|\mathbf{B}|)^2 |\Gamma(X, Y_0)|^2 e^{2\varepsilon\varphi_j(X,Y_0)}\, dx \\ &\qquad \le \frac{C}{r^2}\int_{B(Y_0,4r)\setminus B(Y_0,2r)} |\Gamma(X, Y_0)|^2 e^{2\varepsilon\varphi_j(X,Y_0)}\, dx.\end{aligned} \tag{6.6.12}$$

We note that for $X \in B(Y_0, 4r)\setminus B(Y_0, 2r)$, by Proposition 6.3.24 we have $d(X, Y_0, V + |\mathbf{B}|) \le C$, which by (6.4.11) implies $\varphi(X, Y_0) \le C$, where φ is as in Proposition 6.4.10. But of course, by the construction of the φ_j, we see that $\varphi_j(X, Y_0) \le \varphi(X, Y_0) \le C$ for all j and $X \in B(Y_0, 4r)\backslash B(Y_0, 2r)$. It follows that

$$\sup_{X \in B(Y_0,4r)\backslash B(Y_0,2r)} e^{2\varepsilon\varphi_j(X,Y_0)} \le C, \qquad \text{for each } j \in \mathbb{N},$$

so that, using (6.6.3),

$$\int_{\mathbb{R}^n\backslash B(Y_0,4r)} m(X, V + |\mathbf{B}|)^2 |\Gamma(X, Y_0)|^2 e^{2\varepsilon\varphi_j(X,Y_0)}\, dx \le \frac{C}{r^{n-2}}.$$

By Fatou's Lemma we have

$$\int_{\mathbb{R}^n\backslash B(Y_0,4r)} m(X, V + |\mathbf{B}|)^2 |\Gamma(X, Y_0)|^2 e^{2\varepsilon\varphi(X,Y_0)}\, dx \le \frac{C}{r^{n-2}}. \tag{6.6.13}$$

Let $R = \frac{1}{m(X_0, V+|\mathbf{B}|)}$. We claim that $B(X_0, R) \subset \mathbb{R}^n\backslash B(Y_0, 4r)$. Suppose not. Then there is a point $Z \in B(X_0, R) \cap \Big(\mathbb{R}^n\backslash B(Y_0, 4r)\Big)^c = B(X_0, R) \cap B(Y_0, 4r)$, which contradicts (6.6.10). Now, using (6.4.11) and (6.6.13), we get

$$\int_{B(X_0,R)} m(X, V + |\mathbf{B}|)^2 |\Gamma(X, Y_0)|^2 e^{2\varepsilon d(X,Y_0)}\, dx \le \frac{C}{r^{n-2}}.$$

By the Triangle Inequality we observe

$$d(X_0, Y_0) \le d(X_0, X) + d(X, Y_0), \quad \text{for each } X \in B(X_0, R).$$

Recall that we have by Proposition 6.3.23 that $d(X_0, X) \le L$, L a constant depending on $\|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$ but independent of X_0 and X. It follows that

$$e^{2\varepsilon d(X,Y_0)} \ge e^{-2\varepsilon L} e^{2\varepsilon d(X_0,Y_0)} = C e^{2\varepsilon d(X_0,Y_0)}, \quad \text{for each } X \in B(X_0, R).$$

From this fact and the fact that $m(X, V + |\mathbf{B}|) \sim m(X_0, V + |\mathbf{B}|) = R^{-1}$ for every $X \in B(X_0, R)$ owing to (6.3.15), we can conclude

$$\frac{1}{R^2} \int_{B(X_0,R)} |\Gamma(X, Y_0)|^2 e^{2\varepsilon d(X_0,Y_0)}\, dx \le \frac{C}{r^{n-2}}.$$

Dividing out by R^{n-2} and taking square root we see that

$$\Big(\frac{1}{R^n}\int_{B(X_0,R)}|\Gamma(X,Y_0)|^2\,dx\Big)^{1/2} \leq \quad \frac{Ce^{-\varepsilon d(X_0,Y_0)}}{(Rr)^{(n-2)/2}}$$
$$\leq C\Big[m(X_0,V+|\mathbf{B}|)m(Y_0,V+|\mathbf{B}|)\Big]^{(n-2)/2}e^{-\varepsilon d(X_0,Y_0)}. \quad (6.6.14)$$

We claim that if $|X-Y|m(X,V+|\mathbf{B}|)\geq 2$ then

$$d(X,Y)\geq c\Big[1+|X-Y|m(X,V+|\mathbf{B}|)\Big]^{1/(k_0+1)}, \qquad (6.6.15)$$

for some $k_0>0$. To see this, choose $\gamma:[0,1]\to\mathbb{R}^n$ such that $\gamma(0)=X$, $\gamma(1)=Y$, and

$$2d(X,Y)\geq\int_0^1 m(\gamma(t),V+|\mathbf{B}|)|\gamma'(t)|\,dt$$

which can be done by the definition of d and the fact that $X\neq Y$ necessarily in this case. It follows from (6.3.20) that

$$2d(X,Y)\geq c\int_0^1\frac{m(X,V+|\mathbf{B}|)|\gamma'(t)|\,dt}{\big[1+|\gamma(t)-X|m(X,V+|\mathbf{B}|)\big]^{\frac{k_0}{k_0+1}}}.$$

The integral on the right-hand side of the above inequality is greater than or equal to the geodesic distance from X to Y in the metric

$$\frac{m(X,V+|\mathbf{B}|)dz}{\big[1+|Z-X|m(X,V+|\mathbf{B}|)\big]^{\frac{k_0}{k_0+1}}},$$

which is

$$\int_0^1\frac{m(X,V+|\mathbf{B}|)|Y-X|\,dt}{\big[1+t|Y-X|m(X,V+|\mathbf{B}|)\big]^{\frac{k_0}{k_0+1}}}\geq c'\Big[1+|X-Y|m(X,V+|\mathbf{B}|)\Big]^{1/(k_0+1)}$$

and so (6.6.15) follows.

Hence, owing to (6.6.10) and (6.6.15), we have for each $\varepsilon'>0$,

$$|X_0-Y_0|m(X_0,V+|\mathbf{B}|)\leq 1+|X_0-Y_0|m(X_0,V+|\mathbf{B}|)$$
$$\leq\frac{1}{c^{k_0+1}}d(X_0,Y_0)^{k_0+1}\leq\frac{1}{c^{k_0+1}}C_{\varepsilon'/2}e^{(\varepsilon'/2)d(X_0,Y_0)}.$$

Likewise,

$$|X_0 - Y_0| m(Y_0, V + |\mathbf{B}|) \le \frac{1}{c^{k_0+1}} C_{\varepsilon'/2} e^{(\varepsilon'/2) d(X_0,Y_0)}.$$

Adding the last two inequalities we see

$$|X_0 - Y_0| m(Y_0, V + |\mathbf{B}|) + |X_0 - Y_0| m(X_0, V + |\mathbf{B}|) \le \frac{2}{c^{k_0+1}} C_{\varepsilon'/2} e^{(\varepsilon'/2) d(X_0,Y_0)},$$

for any $\varepsilon' > 0$. Squaring the above inequality we have

$$2|X_0 - Y_0|^2 m(X_0, V + |\mathbf{B}|) m(Y_0, V + |\mathbf{B}|) \le 4c^{-2(k_0+1)} C^2_{\varepsilon'/2} e^{\varepsilon' d(X_0,Y_0)}.$$

Now, given ε small enough, choose $\varepsilon' = \frac{1}{n-2}\varepsilon$. Then in view of (6.6.14), we get

$$\Big(\frac{1}{R^n} \int_{B(X_0,R)} |\Gamma(X, Y_0)|^2 \, dx \Big)^{1/2} \le \frac{C e^{-(\varepsilon/2) d(X_0,Y_0)}}{|X_0 - Y_0|^{n-2}},$$

which is (6.6.8). Since $\Gamma(\cdot, Y_0)$ is a weak solution to $Lu = 0$ on $B(X_0, R)$, then if the operator L has the zero-source local uniform boundedness property, by (6.6.2) and (6.6.8), we immediately achieve (6.6.9). $\square$

The exponential decay result of the last theorem holds in the pointwise a.e. sense for the fundamental solutions of some of the operators previously considered. In particular, we have

Corollary 6.6.16. *Let L_1 be a generalized magnetic Schrödinger operator formally given by (1.1.8) where $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$, $A \equiv I$, $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $V \ge 0$ a.e. on $\mathbb{R}^n$, and assumptions (6.1.5) are satisfied. Then there exists $\varepsilon > 0$ and a constant $C > 0$, depending on $\|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, n, and the constants from (6.1.5), such that*

$$|\Gamma_{L_1}(X, Y)| \le \frac{C e^{-\varepsilon d(X,Y,V+|\mathbf{B}|)}}{|X - Y|^{n-2}} \text{ for a.e. } X, Y \in \mathbb{R}^n. \tag{6.6.17}$$

Let L_2 be a generalized magnetic Schrödinger operator formally given by (1.1.8) where $\mathbf{a} \equiv 0$, A is an elliptic matrix with real, bounded coefficients, and $V \in RH_{\frac{n}{2}}$. Then there *exists $\varepsilon > 0$ and a constant $C > 0$, depending on $\|V\|_{RH_{\frac{n}{2}}}$, n, and C_A such that*

$$\Gamma_{L_2}(X,Y) \le \frac{Ce^{-\varepsilon d(X,Y,V)}}{|X-Y|^{n-2}} \text{ for all } X,Y \in \mathbb{R}^n \text{ with } X \neq Y. \tag{6.6.18}$$

Proof. In the first setting, the results of Section 6.5 are true, and so the hypothesis of Theorem 6.6.7 hold. In the second setting, the theory of fundamental solutions set forth in [DHM18] applies, whence the hypothesis of Theorem 6.6.7 hold. Furthermore, in this case the fundamental solution is actually positive and continuous (see Section 6.7.1), so (6.6.9) holds pointwise on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{X = Y\}$.

6.7 Lower bound on the exponential decay of the fundamental solution

6.7.1 Properties of the generalized Schrödinger operator, without the magnetic potential.

Recall the definition of the operator L_E,

$$L_E = -\text{div}\, A\nabla \ + \ V,$$

which is the operator L with $\mathbf{a} \equiv 0$. For this operator with A an elliptic matrix with real, bounded coefficients, and $V \in RH_{\frac{n}{2}}$, the theory set forth in [DHM18] applies, so that the fundamental solution in the sense of Definition 6.5.9 exists. Actually, its fundamental solution is known to be continuous and positive. Below we present a few lemmas that apply to this operator; the most critical for us in obtaining the lower bound estimate is the scale-invariant Harnack Inequality.

Lemma 6.7.1. [DHM18] *(Moser-type Estimate) Assume that $\mathbf{a} \equiv 0$, A is an elliptic matrix with real, bounded coefficients, and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $V \geq 0$ a.e. on $\mathbb{R}^n$. Let $B_R \subset \mathbb{R}^n$ be a ball, and let $u \in \mathcal{V}_{0,V}(B_R)$ solve $L_E u = f$ in the weak sense on B_R, where $f \in L^q(B_R)$ for some $q > \frac{n}{2}$. Then for any r, $0 < r < R$,*

$$\|u\|_{L^\infty(B_{R/2})} \le C\Big[\Big(\fint_{B_R} |u|^r\Big)^{1/r} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\Big]$$

where C depends on n, p, q, and C_A.

Lemma 6.7.2. [DHM18] *(Hölder Continuity Estimate) Assume that $\mathbf{a} \equiv 0$, A is an elliptic matrix with real, bounded coefficients, and $V \in RH_{\frac{n}{2}}$. Let u solve $L_E u = 0$ in the weak sense on a ball $B_{R_0} \subset \mathbb{R}^n, R_0 > 0$. Then there exists $\eta \in (0,1)$ depending on R_0, and $C_{R_0} > 0$ such that whenever $0 < R \leq R_0$,*

$$\sup_{X,Y \in B_{R/2},\, X \neq Y} \frac{|u(X) - u(Y)|}{|X-Y|^\eta} \leq C_{R_0} R^{-\eta} \Big[\Big(\fint_{B_R} |u|^{2^*} \Big)^{1/2^*} + R^{2-\frac{n}{p}} \|V\|_{L^p(B_R)} \Big].$$

Lemma 6.7.3. [CFG86] *(Scale-Invariant Harnack Inequality) Assume $\mathbf{a} \equiv 0$, A is an elliptic matrix with real, bounded coefficients, and $V \in RH_{\frac{n}{2}}$. There exists a small constant $c = c(n, C_A)$ such that whenever $B = B(X_0, r)$, $r < \frac{c}{m(X_0,V)}$, $X_0 \in \mathbb{R}^n$, the following property holds. For any u which solves $L_E u = 0$ in the weak sense on $2B$,*

$$\sup_{X \in B} u(X) \leq C \inf_{X \in B} u(X), \tag{6.7.4}$$

with the constant $C > 0$ depending on n, C_A and V only.

Remark 6.7.5. We remark that the Harnack inequality, of course, holds for any $0 < r < r_0$, $r_0 > 0$, but typically with the constant growing exponentially in r_0 and $\|V\|_{L^p(B(X_0,r_0))}$. The important feature of (6.7.4) is that the constant depends on n, C_A and C_V only.

The following lemma is a trivial extension of a particular result of Theorem 1.1 in [GW82]:

Lemma 6.7.6. *Suppose that A is an elliptic matrix with real, bounded coefficients. Let Γ_0 be the unique fundamental solution to the operator $L_0 = -\text{div}\, A\nabla$. Then there exist constants c_n, C_n greater than 0 and depending on n, C_A, such that*

$$\frac{c_n}{|X-Y|^{n-2}} \leq \Gamma_0(X,Y) \leq \frac{C_n}{|X-Y|^{n-2}}. \tag{6.7.7}$$

It will be useful to consider the conclusion of Lemma 6.7.3 in greater generality:

Definition 6.7.8. We say that the operator L satisfying assumptions (6.1.5) has the *m-scale invariant Harnack Inequality* if whenever $B = B(X_0, r)$, $r < \frac{c}{m(X_0, V + |\mathbf{B}|)}$, $X_0 \in \mathbb{R}^n$, the following property holds. For any u which solves $Lu = 0$ in the weak sense on $2B$,

$$\sup_{X\in B}|u(X)| \le C \inf_{X\in B}|u(X)|, \tag{6.7.9}$$

with the constant $C > 0$ independent of B.

6.7.2 Proof of the lower bound estimate

First, we establish two auxiliary propositions.

Proposition 6.7.10. *Let* $\mathbf{a} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$*, and let* A *be an elliptic matrix with complex, bounded coefficients. Assume that* $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *satisfies (6.2.7), (6.2.1) with* $c_2 \equiv c_4 \equiv 0$*. Let*

$$L_V = -(\nabla - i\mathbf{a})^T A(\nabla - i\mathbf{a}) + V, \qquad L_0 = -(\nabla - i\mathbf{a})^T A(\nabla - i\mathbf{a}).$$

Then for each $f \in (\dot{\mathcal{V}}_{\mathbf{a},0})'$, $\dot{L}_V^{-1} f$ *is well-defined, belongs to* $\dot{\mathcal{V}}_{\mathbf{a},V}$*, and the identity*

$$\dot{L}_0^{-1} f = \dot{L}_V^{-1} f + \dot{L}_0^{-1} V \dot{L}_V^{-1} f \tag{6.7.11}$$

holds in $\dot{\mathcal{V}}_{\mathbf{a},0}$*. Moreover, if* L_0*, its adjoint* L_0^* *and* L_V *are operators whose inverses have fundamental solutions* $\Gamma_0, \Gamma_0^*, \Gamma_V$ *respectively which satisfy*

$$\Gamma_V, \Gamma_0, \Gamma_0^* \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n \backslash \{X = Y\}), \tag{6.7.12}$$

then the identity

$$\Gamma_0(X,Y) = \Gamma_V(X,Y) + \int_{\mathbb{R}^n} \overline{\Gamma_0^*(Z,X)}\Gamma_V(Z,Y)V(Z)\,dz \tag{6.7.13}$$

holds a.e. on $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let $f \in (\dot{\mathcal{V}}_{\mathbf{a},0})'$. Since

$$\|u\|_{\dot{\mathcal{V}}_{\mathbf{a},0}} \le C\|D_{\mathbf{a}}u\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{\dot{\mathcal{V}}_{\mathbf{a},V}},$$

it follows that $\dot{\mathcal{V}}_{\mathbf{a},0}$ continuously embeds into $\dot{\mathcal{V}}_{\mathbf{a},V}$, which implies in particular that

$f \in (\dot{\mathcal{V}}_{\mathbf{a},V})'$, and $\dot{L}_V^{-1} f \in \dot{\mathcal{V}}_{\mathbf{a},V} \subset \dot{\mathcal{V}}_{\mathbf{a},0}$. Therefore $\dot{L}_0 \dot{L}_V^{-1} f \in (\dot{\mathcal{V}}_{\mathbf{a},0})'$. Hence

$$(\dot{\mathcal{V}}_{\mathbf{a},0})' \ni \; f - \dot{L}_0 \dot{L}_V^{-1} f = (\dot{L}_V - \dot{L}_0)\dot{L}_V^{-1} f = V \dot{L}_V^{-1} f,$$

so that $\dot{L}_0^{-1} V \dot{L}_V^{-1} f \in \dot{\mathcal{V}}_{\mathbf{a},0}$. Now, since

$$\dot{L}_0\Big(\dot{L}_V^{-1} f + \dot{L}_0^{-1} V \dot{L}_V^{-1} f\Big) = \dot{L}_0 \dot{L}_V^{-1} f + V \dot{L}_V^{-1} f = \dot{L}_V \dot{L}_V^{-1} f = f,$$

by the invertibility of $\dot{L}_0$, we obtain (6.7.11). Now let $\phi \in C_c^\infty(\mathbb{R}^n)$. Multiplying (6.7.11) by $\overline{\phi}$ and integrating over $\mathbb{R}^n$ we observe

$$\begin{aligned} \int_{\mathbb{R}^n} \Big[\dot{L}_0^{-1} f - \dot{L}_V^{-1} f\Big]\overline{\phi} &= \int_{\mathbb{R}^n} \Big((\dot{L}_0)^{-1} V \dot{L}_V^{-1} f\Big)\overline{\phi} \\ &= \int_{\mathbb{R}^n} \Big(V \dot{L}_V^{-1} f\Big)\overline{\Big((\dot{L}_0^*)^{-1}\phi\Big)}. \end{aligned} \tag{6.7.14}$$

Now let $U_1, U_2 \subset \mathbb{R}^n$ be open sets such that $\operatorname{dist}(U_1, U_2) > 0$, and $f \in C_c^\infty(U_1), \phi \in C_c^\infty(U_2)$. Writing out the integral representations of the operators in (6.7.14) and using Fubini's Theorem (justified since (6.7.12), $V \in L^1_{\text{loc}}(\mathbb{R}^n)$, and f, ϕ are smooth, compactly supported, satisfying $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g) > 0$), we have that

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[\Gamma_0(X,Y) - \Gamma_V(X,Y)\Big] f(Y)\overline{\phi(X)}\, dx\, dy \\ &\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} V(Z)\Gamma_V(Z,Y)\overline{\Gamma_0^*(Z,X)} f(Y)\overline{\phi(X)}\, dx\, dy\, dz. \end{aligned} \tag{6.7.15}$$

We may rewrite (6.7.15) as

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[&\Gamma_0(X,Y) - \Gamma_V(X,Y) \\ &- \int_{\mathbb{R}^n} V(Z)\Gamma_V(Z,Y)\overline{\Gamma_0^*(Z,X)}\, dz\Big] f(Y)\overline{\phi(Z)}\, dy\, dx = 0. \end{aligned}$$

Since the above identity holds for arbitrary $f, \phi \in C_c^\infty(\mathbb{R}^n)$ with disjoint supports, then (6.7.13) holds for a.e. $X, Y \in \mathbb{R}^n$. $\square$

Lemma 6.7.16. *Suppose that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$*, A is an elliptic matrix with complex, bounded coefficients,* $V \in L^1_{\text{loc}}(\mathbb{R}^n)$*, and that* L*,* $L_0 := L - V$*,* L_0^* *are operators for which there exist fundamental solutions* $\Gamma \equiv \Gamma_V$*,* Γ_0*,* Γ_0^* *in the sense of Definition 6.5.9. Assume that*

L has the zero-source local uniform boundedness property and that $\Gamma_V, \Gamma_0, \Gamma_0^*$ satisfy (6.6.3). Moreover for $p > \frac{n}{2}$, if $\mathbf{a} \equiv 0$, assume $V \in RH_p$; otherwise assume (6.1.5) with $\frac{n}{2}$ replaced by p. Let $\tilde{p} = p$ if $n \geq 4$ and $\tilde{p} \in \left(\frac{3}{2}, \min\{3, p\}\right)$ if $n = 3$, and denote $\delta := 2 - \frac{n}{\tilde{p}}$. Then

$$|\Gamma(Y, X) - \Gamma_0(Y, X)| \leq \frac{C[|X - Y| m(Y, V + |\mathbf{B}|)]^{\delta}}{|X - Y|^{n-2}}, \tag{6.7.17}$$

for a.e. $X, Y \in \mathbb{R}^n$ such that $|X - Y| < \frac{1}{m(Y, V+|\mathbf{B}|)}$. Here C depends on $\|V + |\mathbf{B}|\|_{RH_p}, p, C_A, n$ and the constants from (6.1.5).

Proof. Since $\Gamma_V, \Gamma_0, \Gamma_0^*$ satisfy (6.6.3), then they also satisfy (6.7.12). It follows from Proposition 6.7.10, the upper bound of $\Gamma(X, Y)$ in Theorem 6.6.7 and (6.6.3) that

$$|\Gamma(Y, X) - \Gamma_0(Y, X)| \leq C \int_{\mathbb{R}^n} \frac{e^{-\varepsilon d(Z, Y, V+|\mathbf{B}|)}}{|Z - X|^{n-2} |Z - Y|^{n-2}} V(Z)\, dz \leq C(I_1 + I_2 + I_3), \tag{6.7.18}$$

where I_1, I_2, and I_3 denote the integrals over $B(X, r/2)$, $B(Y, r/2)$, and

$$\Omega := \left\{ Z \in \mathbb{R}^n : \ |Z - X| \geq r/2, \ \ |Z - Y| \geq r/2 \right\},$$

respectively, with $r = |X - Y|$. Recall that $|X - Y| < \frac{1}{m(Y, V+|\mathbf{B}|)}$. Write $R = \frac{1}{m(Y, V+|\mathbf{B}|)} \sim \frac{1}{m(X, V+|\mathbf{B}|)}$. For $Z \in B(X, r/2)$, we have $|Z - Y| > r/2$, so

$$\begin{aligned} I_1 &\leq \frac{C}{r^{n-2}} \int_{B(X, r/2)} \frac{V(Z)\, dz}{|Z - X|^{n-2}} \leq \frac{C}{r^{n-2}} \left[\frac{1}{r^{n-2}} \int_{B(X, r/2)} V \right] \\ &\leq \frac{C}{r^{n-2}} \left(\frac{r}{R} \right)^{2 - \frac{n}{p}} \left[\frac{1}{R^{n-2}} \int_{B(X, R)} V + |\mathbf{B}| \right] \leq \frac{C}{r^{n-2}} \left(\frac{r}{R} \right)^{2 - \frac{n}{p}}, \end{aligned} \tag{6.7.19}$$

where the second inequality is due to (6.3.22), the third one is due to Lemma 6.3.9, and the last one is due to Proposition 6.3.12. Similarly we achieve

$$I_2 \leq C \left(\frac{r}{R} \right)^{2 - \frac{n}{p}} \frac{1}{r^{n-2}}. \tag{6.7.20}$$

To estimate I_3, we note that for all $Z \in \Omega$,

$$|Z - Y| \leq |Z - X| + |X - Y| = |Z - X| + r \leq |Z - X| + 2|Z - X| = 3|Z - X|,$$

and so

$$\begin{aligned}
I_3 &\leq C\int_{|Z-Y|\geq \frac{r}{2}} \frac{e^{-\varepsilon d(Z,Y,V+|\mathbf{B}|)}V(Z)\,dz}{|Z-Y|^{2n-4}} \\
&\leq C\int_{2R>|Z-Y|\geq r/2} \frac{V(Z)\,dz}{|Z-Y|^{2n-4}} \;+\; C\int_{|Z-Y|\geq 2R} \frac{e^{-\varepsilon d(Z,Y,V+|\mathbf{B}|)}V(Z)\,dz}{|Z-Y|^{2n-4}} \\
&= C(I_{31}+I_{32}). \qquad (6.7.21)
\end{aligned}$$

To estimate I_{31}, we proceed similarly to the proof of Proposition 6.3.21. Recall $\tilde{p}$ from the statement of the Lemma. We note that $\tilde{p} \leq p$ in any dimension and hence $V + |\mathbf{B}| \in RH_{\tilde{p}}(\mathbb{R}^n)$. Let q be the Hölder conjugate of $\tilde{p}$. Since $\{Z \mid |Z-Y| \in [r/2, 2R]\} \subset B(Y, 2R)$, by Hölder's Inequality we have

$$I_{31} \leq \Big(\int_{B(Y,2R)} V^{\tilde{p}}\Big)^{\frac{1}{\tilde{p}}}\Big(\int_{2R>|Z-Y|\geq r/2} \frac{1}{|Z-Y|^{2(n-2)q}}\,dz\Big)^{\frac{1}{q}}. \qquad (6.7.22)$$

Next,

$$\begin{aligned}
\Big(\int_{B(Y,2R)} V^{\tilde{p}}\Big)^{\frac{1}{\tilde{p}}} &= |B(Y,2R)|^{\frac{1}{\tilde{p}}}\Big(\frac{1}{|B(Y,2R)|}\int_{B(Y,2R)} V^{\tilde{p}}\Big)^{\frac{1}{\tilde{p}}} \\
&\leq \|V+|\mathbf{B}|\|_{RH_{\tilde{p}}}\big(C(n)\big)^{\frac{1}{\tilde{p}}-1}(2R)^{\frac{n}{\tilde{p}}-2}\Big(\frac{1}{(2R)^{n-2}}\int_{B(Y,2R)} V+|\mathbf{B}|\Big).
\end{aligned}$$

Hence, by the definition of R and Proposition 6.3.12 we have

$$\Big(\int_{B(Y,2R)} V^{\tilde{p}}\Big)^{\frac{1}{\tilde{p}}} \leq CR^{\frac{n}{\tilde{p}}-2}. \qquad (6.7.23)$$

Now, observe that $n + \frac{n}{\tilde{p}} > 4$ and hence $n - 2q(n-2) < 0$ for our choice of $\tilde{p}$. Therefore,

$$\int_{2R>|Z-Y|\geq r/2} \frac{1}{|Z-Y|^{2(n-2)q}}\,dz \leq C(n,q)\, r^{n-2q(n-2)}. \qquad (6.7.24)$$

From (6.7.22), (6.7.23), and (6.7.24), we obtain

$$I_{31} \leq CR^{\frac{n}{\tilde{p}}-2} r^{\frac{n}{q}-2(n-2)} = C\Big(\frac{r}{R}\Big)^{2-\frac{n}{\tilde{p}}}\frac{1}{r^{n-2}}.$$

As for I_{32}, first observe that $n + \frac{n}{\tilde{p}} > 4$ can be written as

$$n - 2 - \left(2 - \frac{n}{\tilde{p}}\right) > 0. \tag{6.7.25}$$

With this in mind, we split up $\{|Z - Y| \geq 2R\}$ into annuli of radius $2^j R$ for each $j \in \mathbb{N}$:

$$I_{32} = \sum_{j=1}^{\infty} \int_{2^j R \leq |Z-Y| \leq 2^{j+1} R} \frac{e^{-\varepsilon d(Z,Y,V+|\mathbf{B}|)}\, V(Z)\, dz}{|Z - Y|^{2n-4}}.$$

For $Z \in \{2^j R \leq |Z - Y| \leq 2^{j+1} R\}$, we observe that

$$|Z - Y| m(Y, V + |\mathbf{B}|) \geq 2^j R m(Y, V + |\mathbf{B}|) = 2^j.$$

Hence (6.6.15) implies that there exists $\ell > 1$ such that

$$d(Z, Y, V + |\mathbf{B}|) \geq c\ell^j, \qquad \text{for each } Z \in \{2^j R \leq |Z - Y| \leq 2^{j+1} R\}.$$

Therefore,

$$\begin{aligned} I_{32} &\leq \sum_{j=1}^{\infty} \frac{e^{-\varepsilon c \ell^j}}{(2^j R)^{2n-4}} \int_{B(Y, 2^{j+1} R)} V \\ &\leq \sum_{j=1}^{\infty} \frac{e^{-\varepsilon c \ell^j}}{(2^j R)^{2n-4}} C_0^{j+1} \int_{B(Y,R)} V + |\mathbf{B}| = \frac{C_0}{R^{n-2}} \sum_{j=1}^{\infty} e^{-\varepsilon c \ell^j} 2^{-(2n-4)j} C_0^j \leq \frac{C}{R^{n-2}}, \end{aligned}$$

where on the second inequality we invoked Proposition 6.3.3 for $V + |\mathbf{B}|$, and C_0 is the constant in (6.3.4). Here, C depends on C_A, ℓ, n, and C_0, hence, on C_A, n and $\|V + |\mathbf{B}|\|_{RH_{\tilde{p}}}$ only. As a result,

$$I_{32} \leq \frac{C}{R^{n-2}} = C\left(\frac{r}{R}\right)^{2-\frac{n}{\tilde{p}}} \left(\frac{r}{R}\right)^{n-2-\left(2-\frac{n}{\tilde{p}}\right)} \frac{1}{r^{n-2}} \leq C\left(\frac{r}{R}\right)^{2-\frac{n}{\tilde{p}}} \frac{1}{r^{n-2}},$$

using (6.7.25).

Since $2 - \frac{n}{\tilde{p}} \leq 2 - \frac{n}{p}$ for any $n \geq 3$ and $r < R$ by assumption, the above estimations on I_1, I_2, I_3 together with (6.7.18) imply

$$|\Gamma(Y,X) - \Gamma_0(Y,X)| \leq C\left(\frac{r}{R}\right)^{2-\frac{n}{\tilde{p}}} \frac{1}{r^{n-2}}$$

which translates to (6.7.17). $\square$

Theorem 6.7.26. *Suppose that* $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)$*, A is an elliptic matrix with complex, bounded coefficients,* $V \in L^1_{\text{loc}}(\mathbb{R}^n)$*, and that L, $L_0 := L - V$, L_0^* are operators for which there exist fundamental solutions $\Gamma \equiv \Gamma_V$, Γ_0 Γ_0^* in the sense of Definition 6.5.9. Assume that Γ_0 satisfies (6.6.4), that Γ_0^*, Γ_V satisfy (6.6.3), that L has the zero-source local uniform boundedness property, and that L satisfies the m-scale invariant Harnack Inequality. Moreover, if $\mathbf{a} \equiv 0$, assume that $V \in RH_{\frac{n}{2}}$; otherwise assume (6.1.5). Then there exist constants c and ε_2 depending on C_A, $\|V + |\mathbf{B}|\|_{RH_{\frac{n}{2}}}$, n and the constants from (6.1.5) such that*

$$|\Gamma(X,Y)| \geq \frac{ce^{-\varepsilon_2 d(X,Y,V+|\mathbf{B}|)}}{|X-Y|^{n-2}}. \tag{6.7.27}$$

Proof. Fix $X, Y \in \mathbb{R}^n$. If $|X - Y|m(X, V + |\mathbf{B}|) \leq c$ for c small enough, then by Lemma 6.7.16 and (6.6.4),

$$|\Gamma(Y,X)| \geq |\Gamma_0(Y,X)| - |\Gamma(Y,X) - \Gamma_0(Y,X)| \geq \frac{c'}{|X-Y|^{n-2}}. \tag{6.7.28}$$

Fix $\mathcal{C}$ as the constant from Proposition 6.3.25. By the $m-$scale invariant Harnack Inequality (6.7.9), inequality (6.7.28) implies that for any $\mathcal{C} \geq c$,

$$|\Gamma(Y,X)| \geq \frac{\tilde{c}}{|X-Y|^{n-2}}, \qquad \text{if } |X-Y|m(X,V+|\mathbf{B}|) \leq \mathcal{C} \tag{6.7.29}$$

where $\tilde{c}$ depends on $\mathcal{C}$, and hence for such X, Y the estimate (6.7.27) trivially holds, since $\varepsilon_2 d(X, Y, V + |\mathbf{B}|) \geq 0$. So it suffices to show (6.7.27) in the case that $X, Y \in \mathbb{R}^n, X \neq Y$ satisfy $|X - Y|m(X, V + |\mathbf{B}|) > \mathcal{C}$. To this end, choose $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma(0) = X, \gamma(1) = Y$, and

$$\int_0^1 m(\gamma(t), V + |\mathbf{B}|)|\gamma'(t)|\, dt \leq 2d(X, Y, V + |\mathbf{B}|)$$

which can be done by the definition of d. Let

$$t_0 = \sup\Big\{t \in [0,1] \;\Big|\; |X - \gamma(t)| \le \frac{\mathcal{C}}{m(X, V + |\mathbf{B}|)}\Big\}$$

Thus t_0 satisfies $|X - \gamma(t_0)| \le \frac{\mathcal{C}}{m(X,V+|\mathbf{B}|)}$, and hence by Lemma 6.3.14 we have $m(\gamma(t_0), V + |\mathbf{B}|) \sim m(X, V + |\mathbf{B}|)$. In the case that

$$|Y - \gamma(t_0)| \le \frac{1}{m(\gamma(t_0), V + |\mathbf{B}|)}$$

we note that

$$\begin{aligned} |X - Y| &\le |X - \gamma(t_0)| + |Y - \gamma(t_0)| \\ &\qquad \le \frac{\mathcal{C}}{m(X, V + |\mathbf{B}|)} + \frac{1}{m(\gamma(t_0), V + |\mathbf{B}|)} \le \frac{\mathcal{C} + C}{m(X, V + |\mathbf{B}|)} \end{aligned}$$

which establishes (6.7.27) due to (6.7.29) (used with $A + C$ in place of A).

Now suppose that $|Y - \gamma(t_0)| > \frac{1}{m(\gamma(t_0),V+|\mathbf{B}|)}$. Since $m(\cdot, V + |\mathbf{B}|)$ is locally bounded, we can define a finite sequence $t_0 < t_1 < \cdots < t_m \le 1$ such that

$$t_j = \inf\Big\{t \in [t_{j-1}, 1] \;\Big|\; |\gamma(t) - \gamma(t_{j-1})| \ge \frac{1}{m(\gamma(t_{j-1}), V + |\mathbf{B}|)}\Big\}.$$

Respectively, it has the following properties

$$\gamma(t) \in B\Big(\gamma(t_{j-1}), \frac{1}{m(\gamma(t_{j-1}), V + |\mathbf{B}|)}\Big) \quad \text{for } t \in [t_{j-1}, t_j),\;\; j = 1, 2, \ldots, m, \tag{6.7.30}$$

$$|\gamma(t_j) - \gamma(t_{j-1})| = \frac{1}{m(\gamma(t_{j-1}), V + |\mathbf{B}|)}, \qquad j = 1, \ldots, m \tag{6.7.31}$$

and

$$\gamma(t) \in B\Big(\gamma(t_m), \frac{1}{m(\gamma(t_m), V + |\mathbf{B}|)}\Big) \quad \text{for } t \in [t_m, 1]. \tag{6.7.32}$$

It follows that

$$\int_0^1 m(\gamma(t), V + |\mathbf{B}|)|\gamma'(t)|\, dt \ge \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} m(\gamma(t), V + |\mathbf{B}|)|\gamma'(t)|\, dt$$

$$\geq c\sum_{j=1}^{m-1} m(\gamma(t_{j-1}), V+|\mathbf{B}|)|\gamma(t_j)-\gamma(t_{j-1})| = cm$$

where the second inequality is due to (6.3.15) and the last equality is due to (6.7.31). Hence

$$m \leq \frac{2}{c}\, d(X,Y,V+|\mathbf{B}|). \tag{6.7.33}$$

By definition of t_0, we have that

$$|X-\gamma(t_j)| \geq \frac{\mathcal{C}}{m(X,V+|\mathbf{B}|)}$$

for each $j=0,\dots,m$. It follows by (6.3.26) that

$$X \notin B\Big(\gamma(t_j), \frac{2}{m(\gamma(t_j),V+|\mathbf{B}|)}\Big)$$

for each $j=0,1,\dots,m$. Let $u(Z):=\Gamma(Z,X)$. Then we have that u is a weak solution of $Lu=0$ in $B\Big(\gamma(t_j), \frac{2}{m(\gamma(t_j),V+|\mathbf{B}|)}\Big)$. By (6.7.31) and the Harnack Inequality, we can deduce the following chain of inequalities:

$$|u(\gamma(t_0))| \leq C|u(\gamma(t_1))| \leq \cdots \leq C^m|u(\gamma(t_m))| \leq C^{m+1}|u(Y)|,$$

with the constant C depending on n, C_A and $\|V+|\mathbf{B}|\|_{RH_{\frac{n}{2}}}$ and the constants from (6.1.5) only. This implies

$$\begin{aligned}|\Gamma(Y,X)| &\geq C^{-m-1}|\Gamma(\gamma(t_0),X)|\\ &\geq C^{-m-2}\frac{1}{|\gamma(t_0)-X|^{n-2}} \geq C^{-m-3}\Big(m(X,V+|\mathbf{B}|)\Big)^{n-2},\end{aligned}$$

possibly enlarging the value of C which still depends on the same parameters. Here, the second inequality holds due to the fact that $\gamma(t_0)$ satisfies the hypothesis of (6.7.29). From (6.7.33) we deduce

$$|\Gamma(Y,X)| \geq C^{-\frac{2}{c}\, d(X,Y,V+|\mathbf{B}|)}\Big(m(X,V+|\mathbf{B}|)\Big)^{n-2}.$$

We can choose $\varepsilon_2 > 0$ large enough so that

$$\varepsilon_2 - \frac{2}{c} \ln C \geq 0$$

in which case we can write

$$|\Gamma(Y, X)| \geq Ce^{-\varepsilon_2 d(X,Y,V+|\mathbf{B}|)} \Big(m(X, V + |\mathbf{B}|)\Big)^{n-2}.$$

Finally, from the hypothesis that $m(X, V + |\mathbf{B}|) \geq \frac{\mathcal{C}}{|X-Y|^{n-2}}$ we obtain

$$|\Gamma(Y, X)| \geq \frac{ce^{-\varepsilon_2 d(X,Y,V+|\mathbf{B}|)}}{|X - Y|^{n-2}} \tag{6.7.34}$$

for a.e. $X, Y \in \mathbb{R}^n$, $X \neq Y$.

It is immediate that (6.7.34) implies (6.7.27). Indeed, fix $X, Y \in \mathbb{R}^n$ with $X \neq Y$. Since the right-hand side of (6.7.34) is symmetric with respect to X, Y, it follows that

$$|\Gamma(X, Y)| \geq \frac{ce^{-\varepsilon_2 d(Y,X,V+|\mathbf{B}|)}}{|Y - X|^{n-2}} = \frac{ce^{-\varepsilon_2 d(X,Y,V+|\mathbf{B}|)}}{|X - Y|^{n-2}},$$

for almost every such X, Y, as desired. □

Corollary 6.7.35. *Let L_2 be a generalized magnetic Schrödinger operator formally given by (1.1.8) where $\mathbf{a} \equiv 0$, A is a real, bounded, elliptic matrix, and $V \in RH_{\frac{n}{2}}$. Then there exist constants c and ε_2 depending on C_A, n, and $\|V\|_{RH_{\frac{n}{2}}}$ such that its fundamental solution, Γ_{L_2}, satisfies*

$$\Gamma_{L_2}(X, Y) \geq \frac{ce^{-\varepsilon_2 d(X,Y,V)}}{|X - Y|^{n-2}}. \tag{6.7.36}$$

Proof. Per the results given in Section 6.7.1, the operator L_2 satisfies the hypothesis of Theorem 6.7.26. The result follows. □

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